Particle Physics 1

Lecture 3: Antiparticles & discrete symmetries

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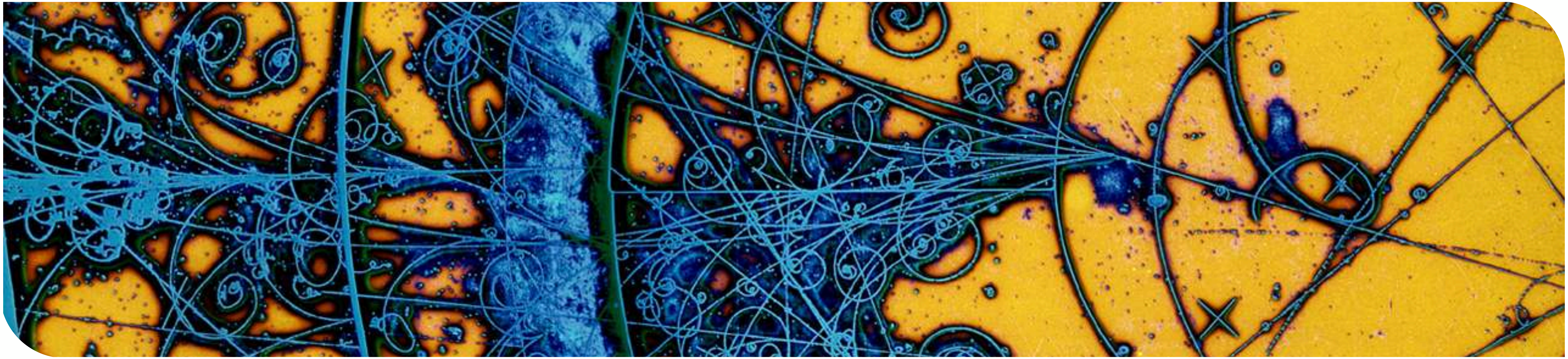

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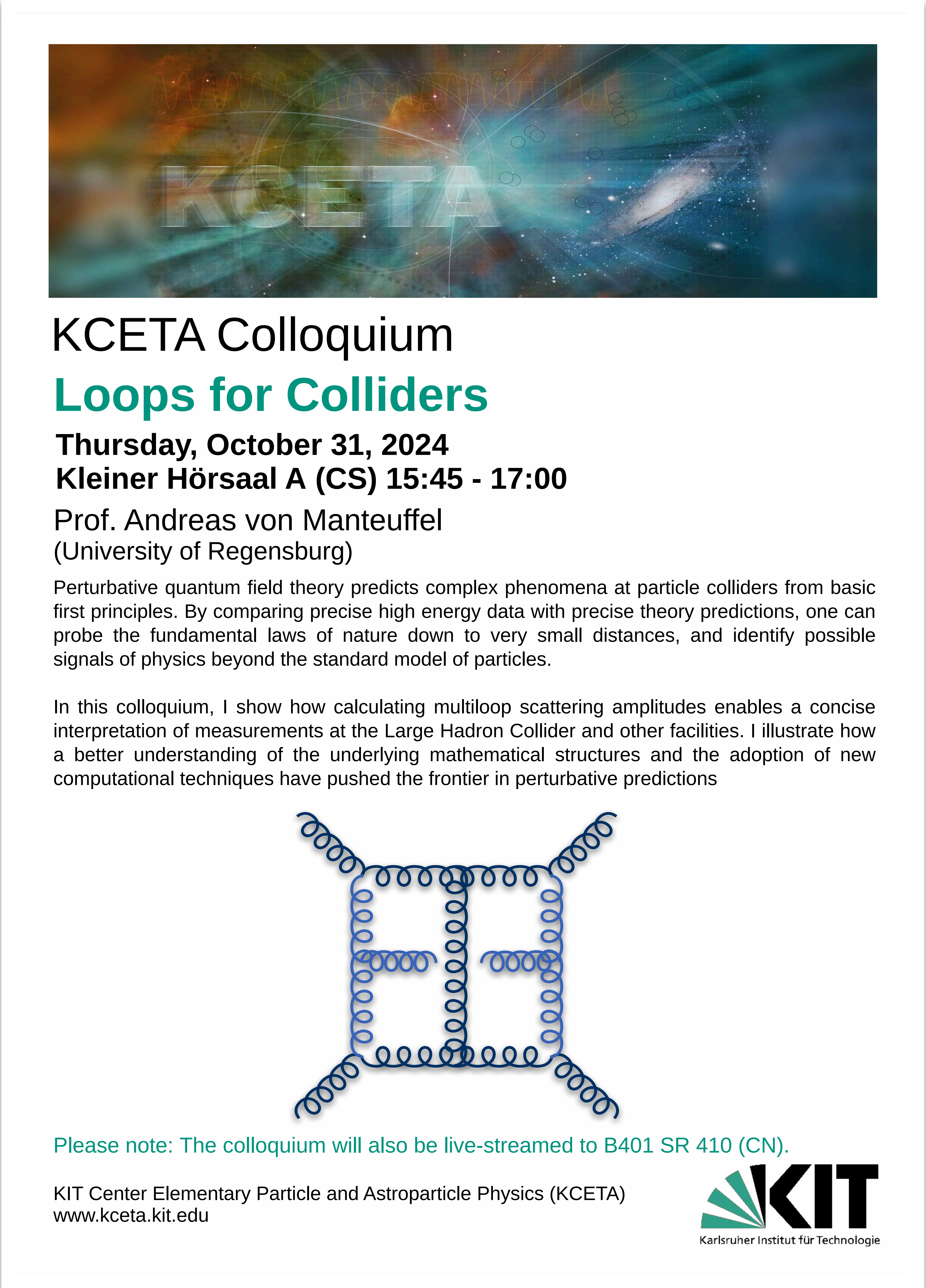

KCETA Colloquium

Loops for Colliders

Thursday, October 31, 2024
Kleiner Hörsaal A (CS) 15:45 - 17:00

Prof. Andreas von Manteuffel
(University of Regensburg)

Perturbative quantum field theory predicts complex phenomena at particle colliders from basic first principles. By comparing precise high energy data with precise theory predictions, one can probe the fundamental laws of nature down to very small distances, and identify possible signals of physics beyond the standard model of particles.

In this colloquium, I show how calculating multiloop scattering amplitudes enables a concise interpretation of measurements at the Large Hadron Collider and other facilities. I illustrate how a better understanding of the underlying mathematical structures and the adoption of new computational techniques have pushed the frontier in perturbative predictions

Please note: The colloquium will also be live-streamed to B401 SR 410 (CN).

KIT Center Elementary Particle and Astroparticle Physics (KCETA)
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Questions from first lecture?

L01 Summary

Review

- Relativistic quantum mechanics incorporates relativistic energy- momentum relation: $p^\mu p_\mu = E^2 - \vec{p}^2 = m^2$
- Canonical operator replacement $p_\mu = i\partial_\mu$ ($E \rightarrow i\frac{\partial}{\partial t}$ and $p \rightarrow -i\vec{\nabla}$)
- Klein-Gordon equation of motion $\left(\partial^\mu \partial_\mu + m^2\right)\phi = 0$ for spin-0 particles (scalars)
- Dirac equation $\left(i\gamma^\mu \partial_\mu - m\right)\psi = 0$: equation of motion for relativistic spin-1/2 particles
  - ***Today*** *we'll see that the 4D spinor ψ simultaneously describes particles and anti-particles with spin up/down*

γ Matrices (Key points)

Review

- $\gamma^\mu = \left(\gamma^0, \gamma^1, \gamma^2, \gamma^3\right)$ is **not a four-vector**, but the same in each coordinate system
- Usually use the Dirac-Pauli (or chiral) representation:

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \text{ and } \gamma^a = \begin{pmatrix} 0 & \sigma_a \\ -\sigma_a & 0 \end{pmatrix}$$

with Pauli matrices $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \left[\sigma_a, \sigma_b\right] = 2i\epsilon_{a,b,c}$

- Special combination $\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$ with $\{\gamma^5, \gamma^0\} = 0$ and $\left(\gamma^5\right)^2 = 1$

Adjoint spinor & covariant current

Review

- The probability density and probability current can be written compactly as

$$j^{\mu} = (\rho, \mathbf{j}) = \psi^{\dagger}\gamma^{0}\gamma^{\mu}\psi,$$

4-vector (See Appendix B.3)

- Introduce the adjoint spinor: $\overline{\psi} \equiv \psi^{\dagger}\gamma^{0}.$

$$\overline{\psi} = \psi^{\dagger}\gamma^{0} = (\psi^{*})^{T}\gamma^{0} = (\psi_1^{*}, \psi_2^{*}, \psi_3^{*}, \psi_4^{*})\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = (\psi_1^{*}, \psi_2^{*}, -\psi_3^{*}, -\psi_4^{*})$$

- Which allows the 4-vector current to be written as: $j^{\mu} = \overline{\psi}\gamma^{\mu}\psi.$

Solutions to the Dirac equation

- **Goal**: Identify explicit forms for the wavefunctions of spin-half particles
- Recall the plane wave solutions to the K-G equation
  $\psi(\vec{x}, t) = Ne^{i(\vec{p}\cdot\vec{x}-Et)}$
- We now know that Dirac hamiltonian $\hat{H}_D = \overrightarrow{\alpha} \cdot \hat{p} + \beta m$ is a 4×4 ***matrix of operators*** *that* ***must*** *act on a 4-component wave function which we've identified as a*

Dirac spinor $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$

- Now look for a free-particle plane wave solution of the form
  $\psi(\vec{x}, t) = u(E, \vec{p})e^{i(\vec{p}\cdot\vec{x}-Et)}$

Solutions to the Dirac equation $\left(i\gamma^\mu\partial_\mu - m\right)\psi = 0$

- $\psi(\vec{x},t) = u(E,\vec{p})\underbrace{e^{i(\vec{p}\cdot\vec{x}-Et)}}_{\text{position \& time dependencies, } \therefore\ \partial_\mu\psi \text{ only acts on the exponent}}$

$$\partial_0\psi = \frac{\partial\psi}{\partial t} = -iE\psi\ , \quad \partial_1\psi = \frac{\partial\psi}{\partial x} = ip_x\psi\ ,$$

Substitute into Dirac eqn: $\left(\gamma^0 E - \gamma^1 p_x - \gamma^2 p_y - \gamma^3 p_z - m\right) u(E,\vec{p})e^{i(\vec{p}\cdot\vec{x}-Et)} = 0$

$\therefore$ $u(E,\vec{p})$ satisfies $\underbrace{\left(\gamma^\mu p_\mu - m\right)u = 0}_{\text{Free particle Dirac eqn. for the spinor } u \text{ written in terms of } p_\mu.}$

Particle at rest $\vec{p} = 0$

- $\psi = u(E,0)e^{-iEt}$

$$(\gamma^\mu p_\mu - m)u = 0$$

$$\Rightarrow \quad E\gamma^0 u = mu$$

Eigenvalue eqn. for the components of the spinor u.

$$E\begin{pmatrix}1 & & & \\ & 1 & & 0 \\ & & -1 & \\ 0 & & & -1\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\\ \phi_4\end{pmatrix} = m\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\\ \phi_4\end{pmatrix}$$

Diagonalize $\Rightarrow$ 4 orthogonal solutions

	Spin ↑	Spin ↓
$E>0$	$u_1(E,0) = N\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$	$u_2(E,0) = N\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$
$E<0$	$u_3(E,0) = N\begin{pmatrix}0\\0\\1\\0\end{pmatrix}$	$u_4(E,0) = N\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$

4 states are also eigenstates of the $\hat{S}_z$ operator

Particle at rest $\vec{p} = 0$

- Now including time dependence

$$\psi_1 = N \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} e^{-imt}, \ \psi_2 = N \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} e^{-imt}, \ \psi_3 = N \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} e^{+imt} \ \text{and} \ \psi_4 = N \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} e^{+imt}.$$

General free particle solution $\vec{p} \neq 0$

- Can be derived from the solution for $\vec{p} = 0$
- Better to directly solve the Dirac equation for $\psi(\vec{x}, t) = u(E, \vec{p})e^{i(\vec{p}\cdot\vec{x} - Et)}$
- Start with the free-particle Dirac equation for the spinor u (slide 8)

$$(\gamma^{\mu} p_{\mu} - m)\, u = 0,$$

$$\Rightarrow \left(E\gamma^0 - p_x\gamma^1 - p_y\gamma^2 - p_z\gamma^3 - m\right) u = 0$$

- Express in matrix form using the Pauli-Dirac representation of the γ -matrices

$$\left[\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} E - \begin{pmatrix} 0 & \boldsymbol{\sigma}\cdot\mathbf{p} \\ -\boldsymbol{\sigma}\cdot\mathbf{p} & 0 \end{pmatrix} - m \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}\right] u = 0,$$

$$u = \begin{pmatrix} u_A \\ u_B \end{pmatrix} = \begin{pmatrix} / \\ / \\ / \\ / \end{pmatrix}$$

4×4 matrix in
2×2 block form:

$$\boldsymbol{\sigma}\cdot\mathbf{p} \equiv \sigma_x p_x + \sigma_x p_y + \sigma_x p_z = \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix}$$

General free particle solution $\vec{p} \neq 0$

- Allows

$$\left[\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} E - \begin{pmatrix} 0 & \boldsymbol{\sigma}\cdot\mathbf{p} \\ -\boldsymbol{\sigma}\cdot\mathbf{p} & 0 \end{pmatrix} - m \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}\right] u = 0,$$

- To be written as

$$\begin{pmatrix} (E-m)I & -\boldsymbol{\sigma}\cdot\mathbf{p} \\ \boldsymbol{\sigma}\cdot\mathbf{p} & -(E+m)I \end{pmatrix} \begin{pmatrix} u_A \\ u_B \end{pmatrix} = 0,$$

- Which gives the coupled equations

$$u_A = \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E-m} u_B,$$

$$u_B = \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E+m} u_A.$$

2 simplest choices for u_A ?

$u_A = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $u_A = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

same for u_B.

✪ Think of analogy to spin

$\Rightarrow$ z-axis is chosen to label the states

General free particle solution $\vec{p} \neq 0$

- The 4 orthogonal plane wave solutions to the free-particle Dirac equation of the form $\psi_i(\vec{x},t) = u_i(E,\vec{p})e^{i(\vec{p}\cdot\vec{x}-Et)}$

$$\underbrace{u_1 = N_1 \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{pmatrix}, \quad u_2 = N_2 \begin{pmatrix} 0 \\ 1 \\ \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix}}_{u_A}, \quad \underbrace{u_3 = N_3 \begin{pmatrix} \frac{p_z}{E-m} \\ \frac{p_x+ip_y}{E-m} \\ 1 \\ 0 \end{pmatrix} \quad u_4 = N_4 \begin{pmatrix} \frac{p_x-ip_y}{E-m} \\ \frac{-p_z}{E-m} \\ 0 \\ 1 \end{pmatrix}}_{u_B}$$

pos. energy spinors with $E=+|\sqrt{p^2+m^2}|$

neg. energy spinors with $E=-|\sqrt{p^2+m^2}|$

substitute any of these into $(\gamma^\mu p_\mu - m)u = 0$ & you'll get

$E^2 = p^2+m^2$

General free particle solution $\vec{p} \neq 0$

- Is it possible to interpret all 4 solutions as having $E > 0$?

No! If all have $E>0$, the exponent of $\psi(\vec{x},t) = u(E,\vec{p})\underbrace{e^{i(\vec{p}\cdot\vec{x}-Et)}}_{}$

Would be the same for all 4 solutions.

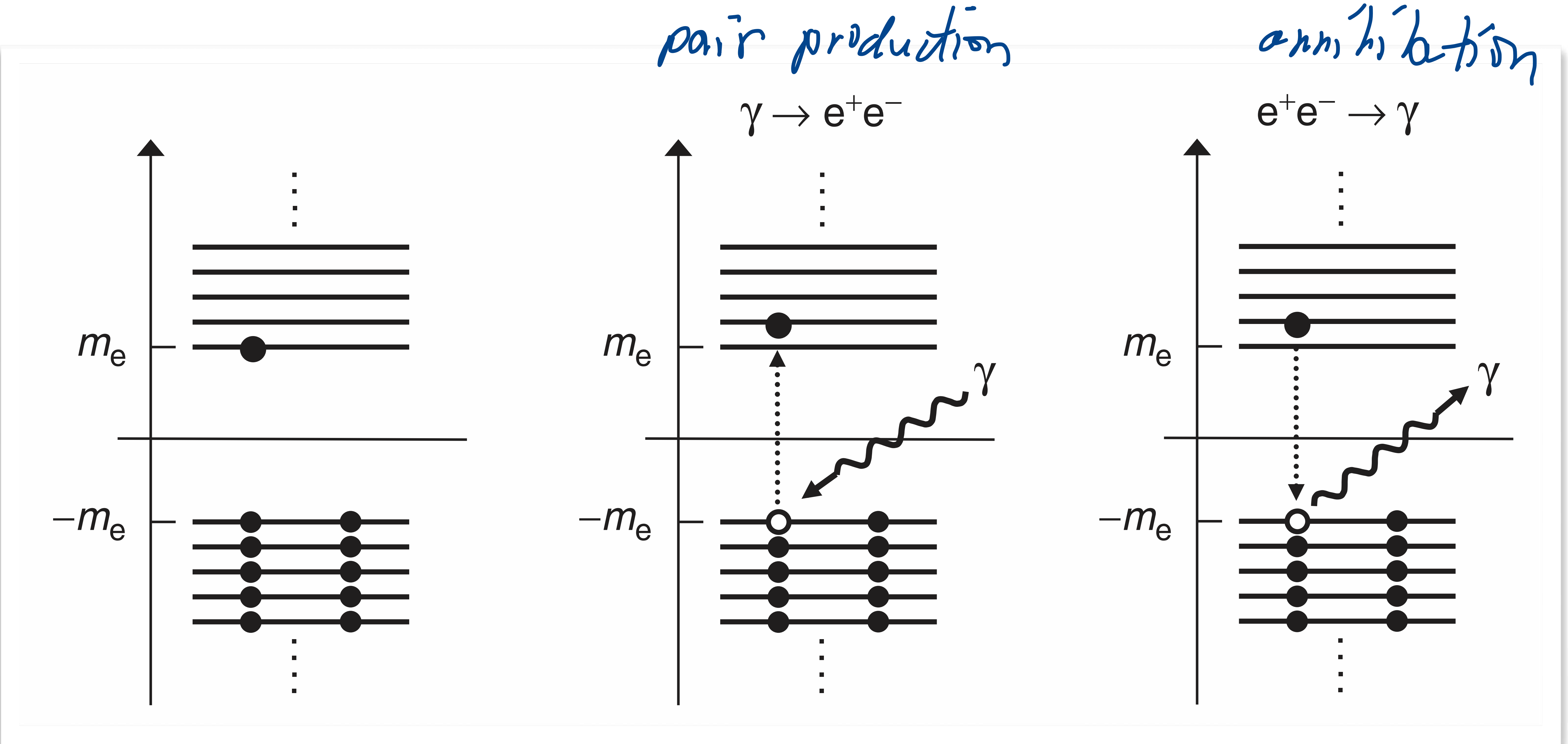
$\therefore$ No longer 4 independent solutions.

Could express, e.g., $u_1 = \frac{p_z}{E+m} u_3 + \frac{p_x + ip_y}{E+m} u_4$

$\Rightarrow$ Only 4 independent solutions if 2 have $E<0$.

Dirac equation: Original interpretation

- Interpret as a single-particle wave function
  - *Problem*: electrons can emit infinite amount of energy via photons by changing into $-E$ states
- Proposed solution: “Dirac sea”
  - Ground states (“vacuum”): all $-E$ states filled with electrons following Pauli exclusion principle
    - No transitions from $+E$ states to $-E$ state
    - But electrons can be elevated from $-E$ states to $+E$ states
  - Hole in Dirac sea: corresponds $+E$ antiparticles with the opposite charge to the particle states



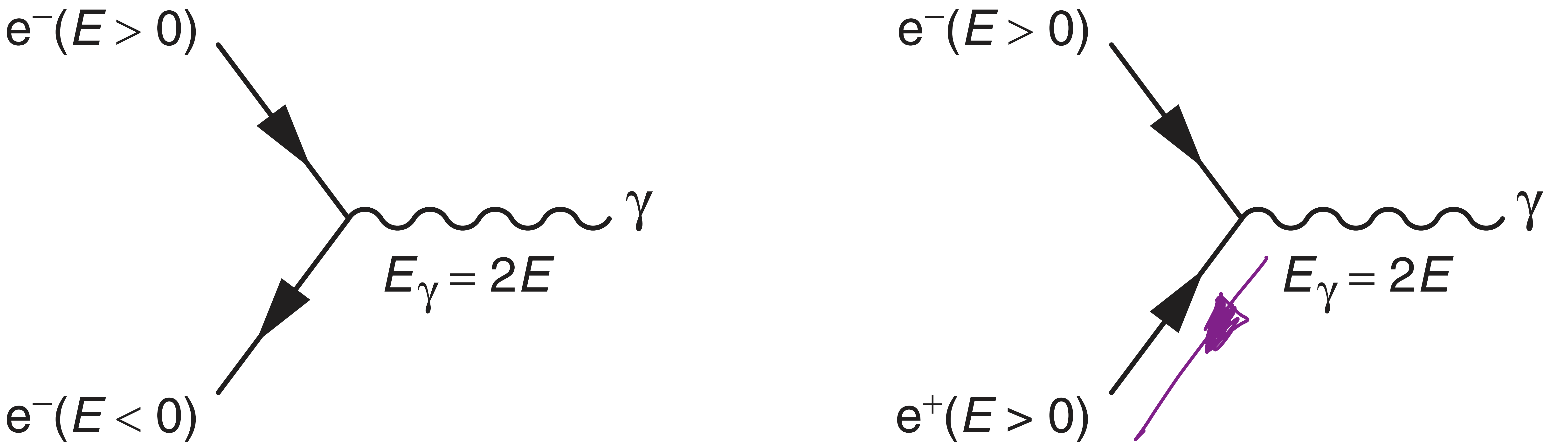
Thomson Fig. 4.1

Problems: ① Antiparticles for bosons, where the Pauli exclusion principle doesn't apply

② Fully occupied sea implies that the vacuum has ∞ E.

Dirac equation: Actual solution

- Multi-particle system (Feynman, Stückelberg)
  - Requires quantized fields. Particles with $-E$ backwards in time = anti-particle with $+E$ forward in time



Thomson Fig. 4.2

convention to write this way

Both are equivalent since

$$e^{-iEt} = e^{-i(-E)(-t)}$$

Antiparticle spinors

- Nothing stopping us from performing calculations with the $-E$ particle spinors u_3 & u_4

$$u_1 = N_1 \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{pmatrix}, \quad u_2 = N_2 \begin{pmatrix} 0 \\ 1 \\ \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix}, \quad u_3 = N_3 \begin{pmatrix} \frac{p_z}{E-m} \\ \frac{p_x+ip_y}{E-m} \\ 1 \\ 0 \end{pmatrix} \quad u_4 = N_4 \begin{pmatrix} \frac{p_x-ip_y}{E-m} \\ \frac{-p_z}{E-m} \\ 0 \\ 1 \end{pmatrix}$$

- … but always need to remember
  - E is the negative of the *physical* energy
  - p is the negative of the *physical* momentum (since u_3 & u_4 are interpreted as propagating backwards in time)

Antiparticle spinors

- Rewrite the $-E$ particle spinors u_3 & u_4 in terms of the physical $+E$ antiparticle spinors ν_1 & ν_2

$$v_1(E,\mathbf{p})e^{-i(\mathbf{p}\cdot\mathbf{x}-Et)} = u_4(-E,-\mathbf{p})e^{i[-\mathbf{p}\cdot\mathbf{x}-(-E)t]}$$

$$v_2(E,\mathbf{p})e^{-i(\mathbf{p}\cdot\mathbf{x}-Et)} = u_3(-E,-\mathbf{p})e^{i[-\mathbf{p}\cdot\mathbf{x}-(-E)t]}.$$

- Can be formally derived (as done in slides 11-13) by looking for solutions of the Dirac equation of the form

$$\psi(\vec{x},t) = \nu(E,\vec{p})e^{-i(\vec{p}\cdot\vec{x}-Et)}$$

Summary: particle & antiparticle spinors

- In terms of the physical energy:

Two particle solutions to the Dirac equation $\psi_i = u_i e^{+i(\mathbf{p}\cdot\mathbf{x}-Et)}$

$$u_1(p) = \sqrt{E+m}\begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{pmatrix} \quad \text{and} \quad u_2(p) = \sqrt{E+m}\begin{pmatrix} 0 \\ 1 \\ \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix},$$

Two antiparticle solutions to the Dirac equation $\psi_i = v_i e^{-i(\mathbf{p}\cdot\mathbf{x}-Et)}$

$$v_1(p) = \sqrt{E+m}\begin{pmatrix} \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad v_2(p) = \sqrt{E+m}\begin{pmatrix} \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \\ 1 \\ 0 \end{pmatrix}.$$

Wavefunction normalization

$$\psi = u_1(p)\, e^{i(\vec{p}\cdot\vec{x} - Et)}$$

$$\rho = \psi^\dagger \psi = (\psi^*)^T \psi = u_1^\dagger u$$

$$= |N|^2 \left(1 + \frac{p_z^2}{(E+m)^2} + \frac{p_x^2 + p_y^2}{(E+m)^2} \right)$$

$$= |N|^2 \frac{2E}{E+m}$$

To normalize the wavefunction to the conventional $2E$ particles / unit V

Need $N = \sqrt{E+m}$

(see chap. 3)

Take 5

Dirac equation relation to spin

Read at home: Thomson Section 4.4

Review

- The Dirac equation provides a natural description of spin-half particles.

  Spin emerges as a direct consequence of requiring the wavefunction to satisfy the Dirac equation

QM basics: Time dependence of an observable corresponding to an operator $\hat{O}$ is given by

$$\frac{\mathrm{d}O}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{O}\rangle = i\langle\psi|[\hat{H},\hat{O}]|\psi\rangle$$

→

If the operator for an observable commutes with the H, it is a constant of the motion

$$[\hat{H}_D, \hat{\mathbf{L}}] = -i\boldsymbol{\alpha}\times\hat{\mathbf{p}}.$$

✗

$$[\hat{H}_D, \hat{\mathbf{S}}] = i\,\boldsymbol{\alpha}\times\hat{\mathbf{p}}.$$

✗

$$\left[\hat{H}_D, \hat{\mathbf{J}}\right] \equiv \left[\hat{H}_D, \hat{\mathbf{L}}+\hat{\mathbf{S}}\right] = 0.$$

✓

Spin and helicity

- **Subtle but important point regarding the *physical* spin of the antiparticle spinors v**
- For $\left[\hat{H}_D, \hat{\mathbf{J}}\right] \equiv \left[\hat{H}_D, \hat{\mathbf{L}} + \hat{\mathbf{S}}\right] = 0.$ to hold for the antiparticles spinors,

  the operator giving the physical spin states of the v spinors must be

$$\hat{\mathbf{S}}^{(v)} = -\hat{\mathbf{S}}$$

4 × 4 matrix operator formed from the Pauli spin-matrices

$$\hat{\mathbf{S}} \equiv \tfrac{1}{2}\hat{\boldsymbol{\Sigma}} \equiv \frac{1}{2}\begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}$$

Do you see why?

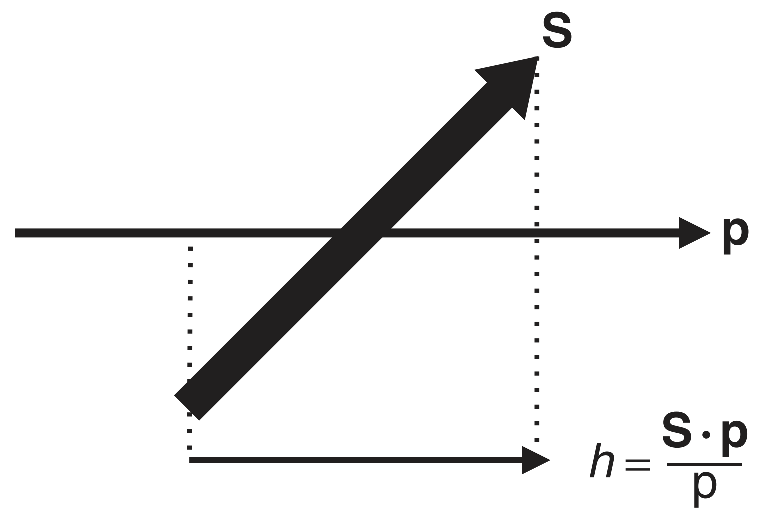
For antiparticles $\vec{p} \to -\vec{p}$ $\therefore$ $L = \vec{r} \times \vec{p} \Rightarrow -L$

$\therefore$ need $S \to -S$

Spin and helicity

- Interaction cross sections will be analyzed in terms of the spin states of the involved particles.
- Problem: $\hat{S}_z$ does not commute with $\hat{H}_D$, *so cannot define a basis of simultaneous eigenstates of* $\hat{S}_z$ & $\hat{H}_D$
- Solution: Introduce **helicity**, *the normalized component of a particles spin along its direction of flight*

$$h \equiv \frac{\mathbf{S}\cdot\mathbf{p}}{\mathrm{p}}.$$

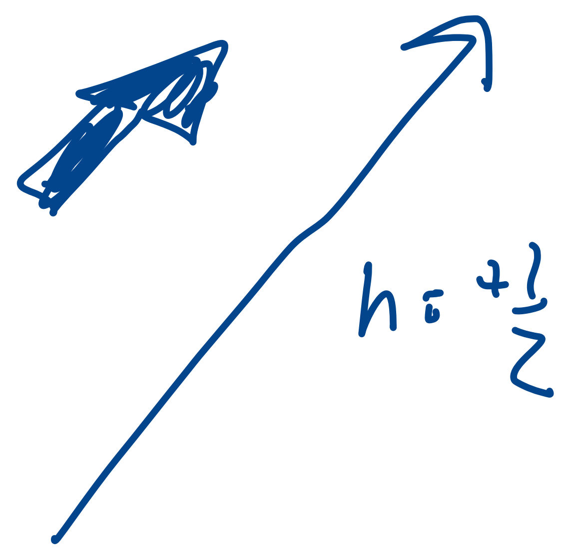


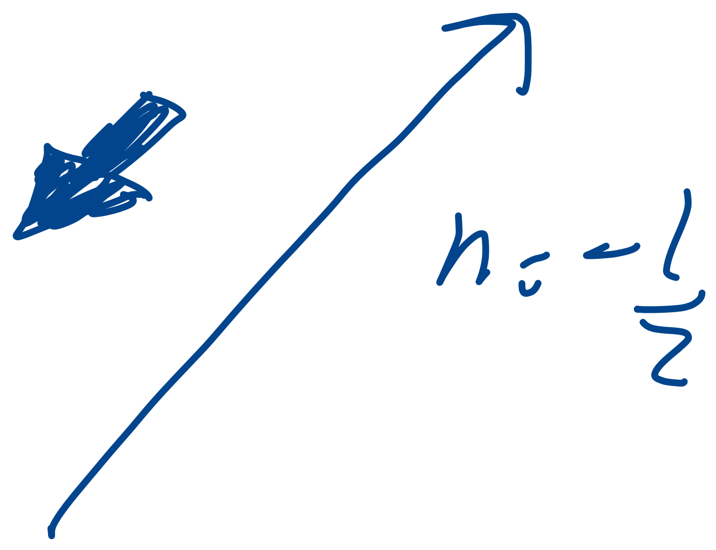
For a 4-component Dirac spinor

$$\hat{h} = \frac{\hat{\boldsymbol{\Sigma}}\cdot\hat{\mathbf{p}}}{2\mathrm{p}} = \frac{1}{2\mathrm{p}}\begin{pmatrix} \boldsymbol{\sigma}\cdot\hat{\mathbf{p}} & 0 \\ 0 & \boldsymbol{\sigma}\cdot\hat{\mathbf{p}} \end{pmatrix} \quad \Rightarrow \quad [\hat{H}_D, \hat{\boldsymbol{\Sigma}}\cdot\hat{\mathbf{p}}] = 0$$

Spin and helicity

- Two possible helicity states for a spin-half fermion.





Problem: Helicity not Lorentz invariant.

Why?

For particles with mass, can always transform to a frame in which the direction of the particle is reversed.

Spin and helicity

- The simultaneous eigenstates $\hat{H}_D$ and $\hat{h}$ are solutions to the Dirac equation which satisfy

$$\hat{h}u = \lambda u.$$

Solve the eigenvalue equation and express the helicity states in terms of spherical polar coordinates

$$\mathbf{p} = (\mathrm{p}\sin\theta\cos\phi,\ \mathrm{p}\sin\theta\sin\phi,\ \mathrm{p}\cos\theta)$$

RH helicity
particle spinor

$$u_\uparrow = \sqrt{E+m}\begin{pmatrix} c \\ se^{i\phi} \\ \frac{\mathrm{p}}{E+m}c \\ \frac{\mathrm{p}}{E+m}se^{i\phi} \end{pmatrix}$$

LH helicity
particle spinor

$$u_\downarrow = \sqrt{E+m}\begin{pmatrix} -s \\ ce^{i\phi} \\ \frac{\mathrm{p}}{E+m}s \\ -\frac{\mathrm{p}}{E+m}ce^{i\phi} \end{pmatrix},$$

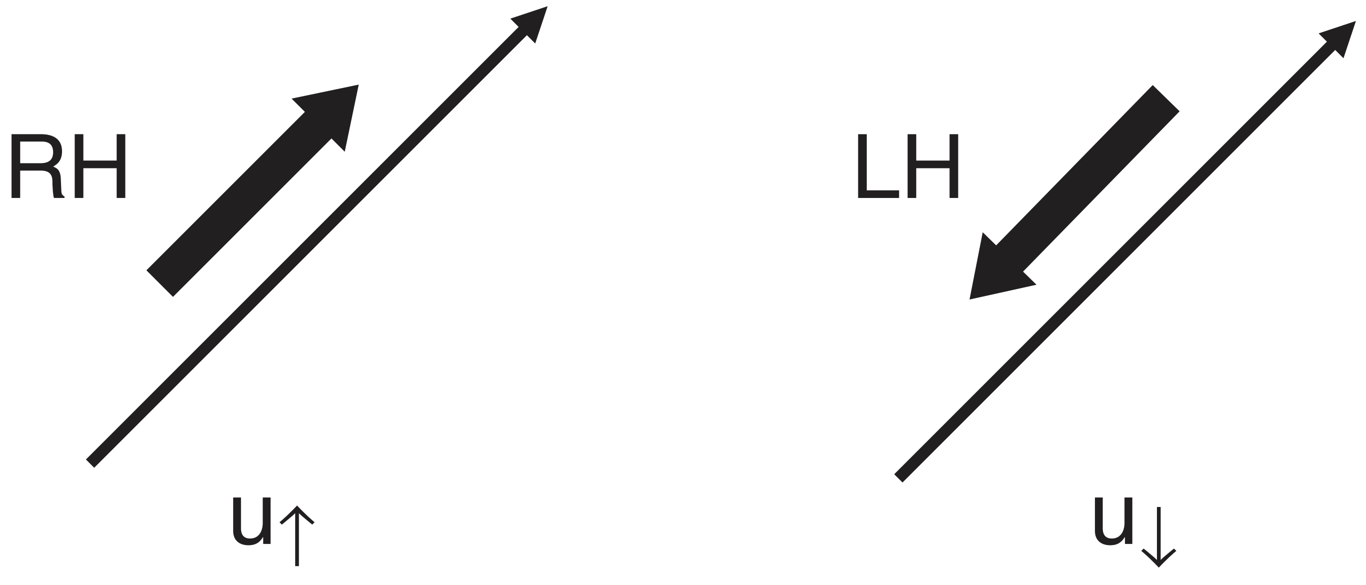
RH helicity
antiparticle spinor

$$v_\uparrow = \sqrt{E+m}\begin{pmatrix} \frac{\mathrm{p}}{E+m}s \\ -\frac{\mathrm{p}}{E+m}ce^{i\phi} \\ -s \\ ce^{i\phi} \end{pmatrix}$$

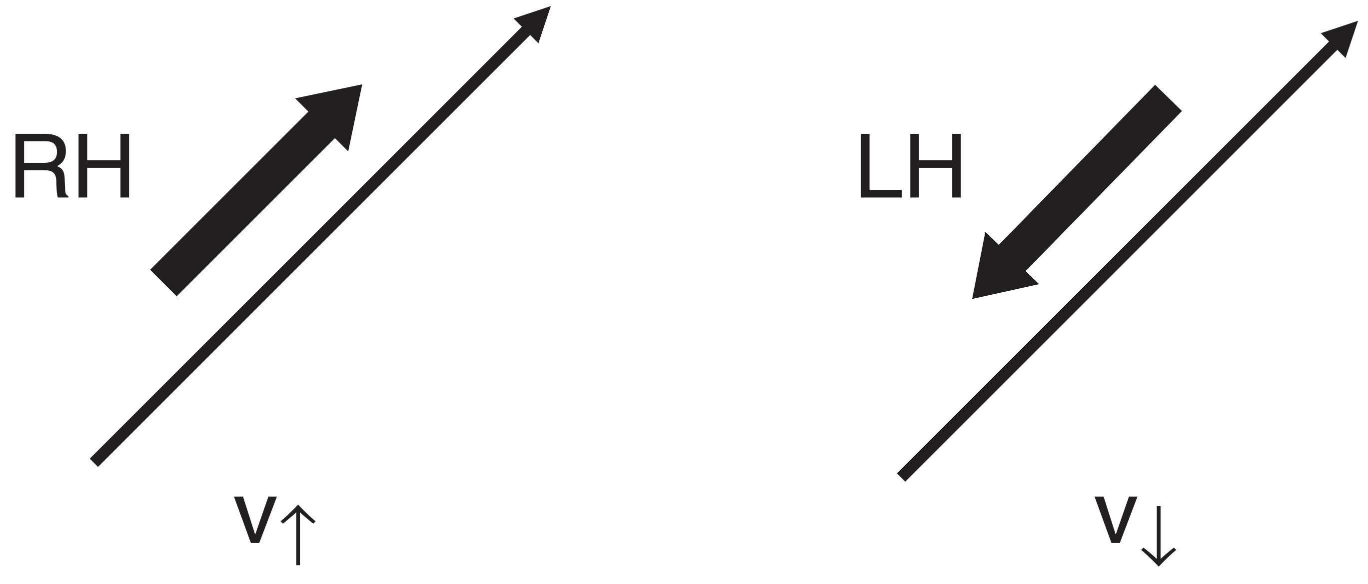
LH helicity
antiparticle spinor

$$v_\downarrow = \sqrt{E+m}\begin{pmatrix} \frac{\mathrm{p}}{E+m}c \\ \frac{\mathrm{p}}{E+m}se^{i\phi} \\ c \\ se^{i\phi} \end{pmatrix}.$$

Particles

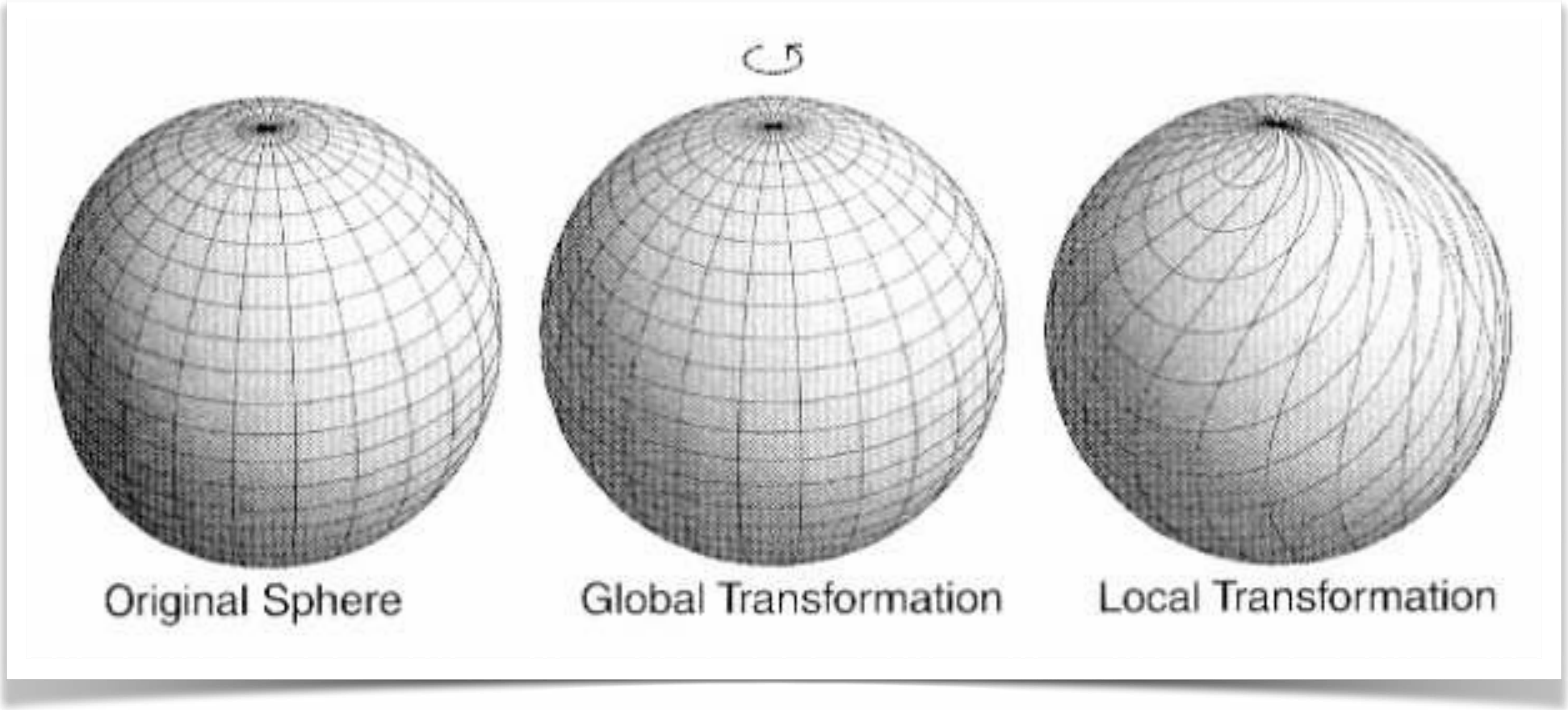


Antiparticles



Symmetries

- Symmetries are operations performed on a system that leaves it invariant
  - **Global** symmetries are the **same** at all space-time points
  - **Local** symmetries are **different** at different space-time points



Source: https://universe-review.ca/I15-04-gauge2.jpg

This is just a primer:
We will discuss symmetries in detail later when we introduce the Lagrange density and the Higgs mechanism

Discrete symmetries

- Discrete symmetries are symmetries that describe non-continuous changes in a system
  - Example: Rotation of a square is a discrete symmetry as only rotations by multiples of 90° yield a symmetry
- Transformation of (Dirac) spinors under 3 discrete symmetry operations C, P, T are very important in particle physics

Discrete symmetries: Charge conjugation

- Charge conjugation operator $\hat{C}$: particle ↔ anti-particle

- What is the form of $\hat{C}$?

- Recall your classical dynamics: The motion of a charged particle in an EM field $A^\mu = (\phi, \overrightarrow{A})$ can be obtained via the minimal substitution:

$$E \to E - q\phi \quad \text{and} \quad \mathbf{p} \to \mathbf{p} - q\mathbf{A}$$

$$\Big\downarrow \quad \text{4-vector notation}$$

$$p_\mu \to p_\mu - qA_\mu$$

Discrete symmetries: Charge conjugation

$$p_\mu \to p_\mu - qA_\mu$$

Operators $\hat{E} \to i\frac{\partial}{\partial t}$ & $\hat{p} \to -i\vec{\nabla}$

$$i\partial_\mu \to i\partial_\mu - qA_\mu$$

Substitute into $\left(i\gamma^\mu\partial_\mu - m\right)\psi = 0$

$$\gamma^\mu(\partial_\mu - ieA_\mu)\psi + im\psi = 0$$

Take the * and pre-multiply by $-i\gamma^2$

$$\gamma^\mu(\partial_\mu + ieA_\mu)i\gamma^2\psi^* + im\, i\gamma^2\psi^* = 0.$$

Define $\psi' = i\gamma^2\psi^*$

$$\gamma^\mu(\partial_\mu + ieA_\mu)\psi' + im\psi' = 0$$

Conclusions:

1. ψ' can be interpreted as the antiparticle wavefunction
2. The charge conjugation operator $\hat{C}$ can be identified as $\psi' = \hat{C}\psi = i\gamma^2\psi^*$

Discrete symmetries: Charge conjugation

- If you still need convincing:

Consider the effect of $\hat{C}$ on the particle spinor $\psi = u_1 e^{i(\vec{p}\cdot\vec{x}-Et)}$

$$\psi' = \hat{C}\psi = i\gamma^2\psi^*$$

$$= \underbrace{i\gamma^2 u_1^*}_{\text{spinor part of } \psi'} e^{-i(\vec{p}\cdot\vec{x}-Et)}$$

$$i\gamma^2 u_1^* = i\begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}\sqrt{E+m}\begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{pmatrix}$$

$$= \sqrt{E+m}\begin{pmatrix} \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \\ 0 \\ 1 \end{pmatrix} = v_1(p) \quad ✓$$

$\therefore$ $\hat{C}$ transforms particle into antiparticle spinors

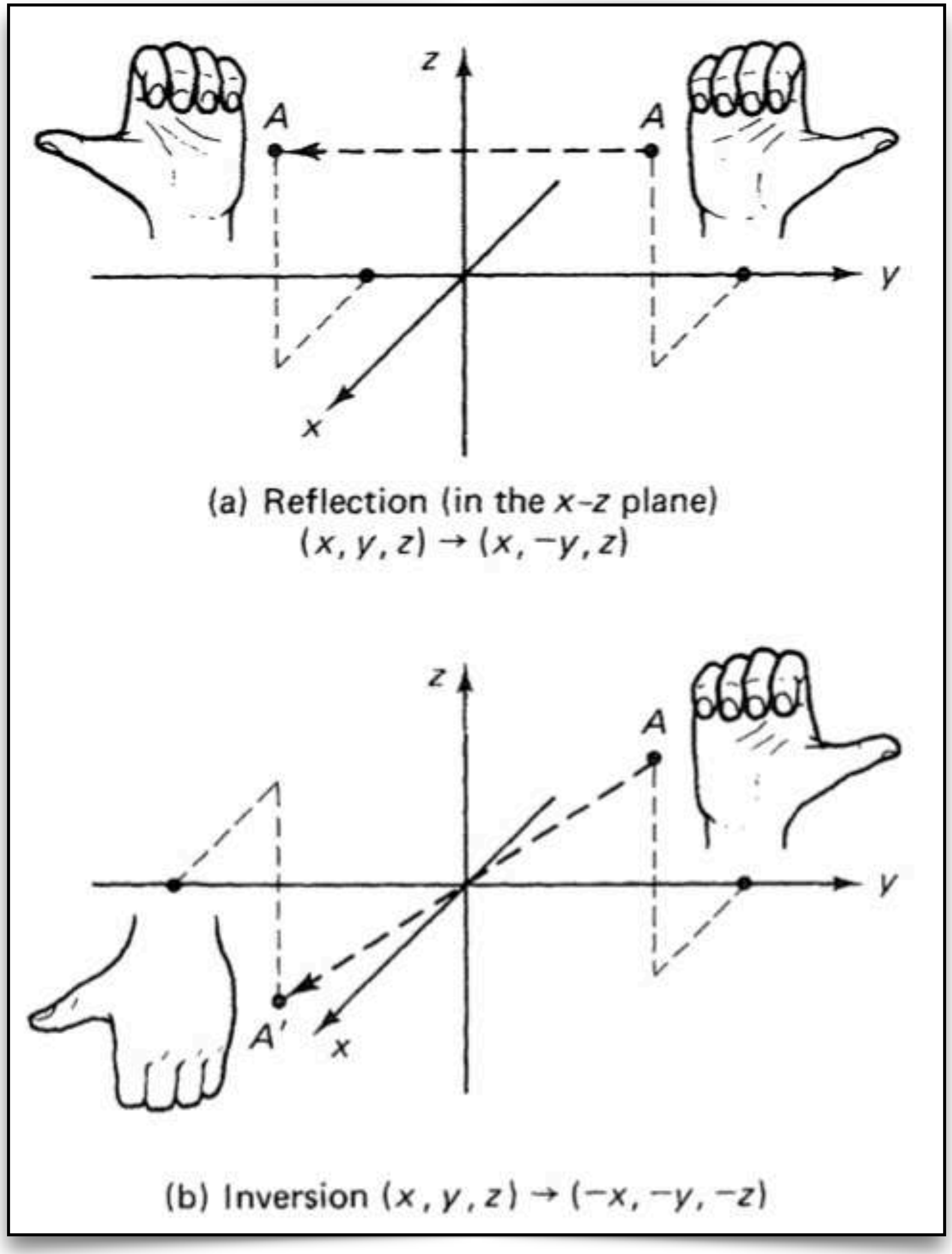
$$\psi = u_1 e^{i(\vec{p}\cdot\vec{x}-Et)} \xrightarrow{\hat{C}} v_1 e^{-i(\vec{p}\cdot\vec{x}-Et)}$$

$$\psi = u_2 e^{i(\vec{p}\cdot\vec{x}-Et)} \xrightarrow{\hat{C}} v_2 e^{-i(\vec{p}\cdot\vec{x}-Et)}$$

Discrete symmetries: Parity

- Parity operator $\hat{P}$ corresponds to a mirroring through the origin:
  $x = (t, \vec{x}) \ \rightarrow \ x^P = (t, -\vec{x})$
- Two questions we need to answer:
  - What is the form of $\hat{P}$?
  - What is the intrinsic parity of Dirac fermions?

“Parity mirror”



Credit: Griffith

Parity operator

- ψ is a solution of the Dirac equation
- ψ' is the solution in the parity mirror obtained from the action of $\hat{P}$ s.t.

$$\psi \to \psi' = \hat{P}\psi$$

- By definition, applying $\hat{P}$ twice recovers the original wavefunction

$$\psi' = \hat{P}\psi \quad \Rightarrow \quad \hat{P}\psi' = \psi.$$

$$\hat{P}^2 = I$$

Parity operator

Start with a WF $\psi(x,y,z,t)$ which satisfies the Dirac eq.:

$$i\gamma^1\frac{\partial\psi}{\partial x}+i\gamma^2\frac{\partial\psi}{\partial y}+i\gamma^3\frac{\partial\psi}{\partial z}-m\psi=-i\gamma^0\frac{\partial\psi}{\partial t}$$

Substitute
$\psi=\hat{P}\psi'$

$$i\gamma^1\hat{P}\frac{\partial\psi'}{\partial x}+i\gamma^2\hat{P}\frac{\partial\psi'}{\partial y}+i\gamma^3\hat{P}\frac{\partial\psi'}{\partial z}-m\hat{P}\psi'=-i\gamma^0\hat{P}\frac{\partial\psi'}{\partial t}$$

1. Premultiply by γ^0 and express the derivatives in terms of the $'$ system (add a “-“ for x, y, z).
2. Use $\gamma^0\gamma^k=-\gamma^k\gamma^0$

$$i\gamma^1\gamma^0\hat{P}\frac{\partial\psi'}{\partial x'}+i\gamma^2\gamma^0\hat{P}\frac{\partial\psi'}{\partial y'}+i\gamma^3\gamma^0\hat{P}\frac{\partial\psi'}{\partial z'}-m\gamma^0\hat{P}\psi'=-i\gamma^0\gamma^0\hat{P}\frac{\partial\psi'}{\partial t'}$$

The P transformed WF $\psi'(x',y',z',t')=\hat{P}\psi(x,y,z,t)$ must satisfy the Dirac eq. in the new coordinate system

$$i\gamma^1\frac{\partial\psi'}{\partial x'}+i\gamma^2\frac{\partial\psi'}{\partial y'}+i\gamma^3\frac{\partial\psi'}{\partial z'}-m\psi'=-i\gamma^0\frac{\partial\psi'}{\partial t'}$$

Note the ′ in the derivative

What has to happen for these to match?

$\gamma\hat{P}\propto I$

We know $\hat{P}^2=I$, so

$$\underbrace{\hat{P}=+\gamma^0}_{\text{convention s.t. } \psi\to\hat{P}\psi=\gamma^0\psi} \quad \text{or} \quad \hat{P}=-\gamma^0$$

Intrinsic parity of Dirac fermions

- Defined by the action of the parity operator $\hat{P} = \gamma^0$ on a spinor for a particle at rest (derived on S9)

$$\hat{P} u_1 = \gamma^0 u_1 = \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix} \sqrt{2m} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = +u_1$$

$$\hat{P} u_2 = +u_2$$

$$\hat{P} v_1 = -v_1$$

$$\hat{P} v_2 = -v_2$$

∴ Intrinsic parity of a spin-$\frac{1}{2}$ particle is opposite to that of a spin-$\frac{1}{2}$ antiparticle

For a particle with momentum $\vec{p}$:

$\hat{P} u_1(E, \vec{p}) = +u_1(E, -\vec{p})$ Stays u_1, so does not change the spin state

Discrete symmetries: T and CPT

- Time reversal operator $\hat{T}$ switches the sign of t:
  - $x = (t, \vec{x}) \rightarrow x^T = (-t, \vec{x})$
  - $\hat{T} = i\gamma^1\gamma^3$
- CPT Theorem (Pauli, Lüders 1957):

  “Every locally Lorentz-invariant quantum field theory is invariant under CPT symmetry”
- Experimentally one can test Lorentz invariance violation (e.g. measuring velocity of neutrinos faster than speed of light) or violations of CPT symmetry (e.g. comparing emission spectra of hydrogen and anti-hydrogen atoms)
  - If physics is described by QFTs, any observation of CPT violation equals Lorentz violation
- Up to today, we have not measured any CPT violation or any Lorentz violation

Reading assignment

- Modern particle physics (Mark Thomson)
  - Chap. 4
    - 4.6-4.9
  - Chap. 17 (suggestion for now; will be required later)
    - 17.2-17.3

What questions do you have?