EMF Übung 2

Freitag, 10. Mai 2019 10:34

Erfasster Bildschirmausschnitt: 10.05.2019 10:35

Wiederholung: Satz von Gauß: $\oint\!\!\oint_{\partial V} \vec{v}\, d\vec{f} = \iiint_V \mathrm{div}(\vec{v})\, dv$

Einleitung:

wichtige Größen

$\vec{E}$: elektrische Feld $\left(\vec{E} := \frac{\vec{F}}{q}\right)$

$\vec{D}$: elektrische Verschiebungsdichte $\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$

in isotropen Materialien: $\vec{D} = \varepsilon_0 \varepsilon_r \vec{E}$

$\vec{P} \sim \vec{E}$

ε_r: Dielektrizitätszahl des Materials

$\varepsilon_r = 1$ (Vakuum (und Luft))

$\varepsilon_r = 80$ (für Wasser)

$\varepsilon_r = 10000$ (für spezielle Keramiken)

ρ: Raumladungsdichte $Q = \iiint \rho\, dv$ (Ladung pro Volumen)

σ: Flächenladungsdichte $Q = \iint \sigma\, df$ (Ladung pro Fläche)

ϕ: el. Potential $-\int_{\vec{P}_1}^{\vec{P}_2} \vec{E}\, d\vec{l} = \phi(\vec{P}_2) - \phi(\vec{P}_1)$

Linienintegral (wegunabhängig)

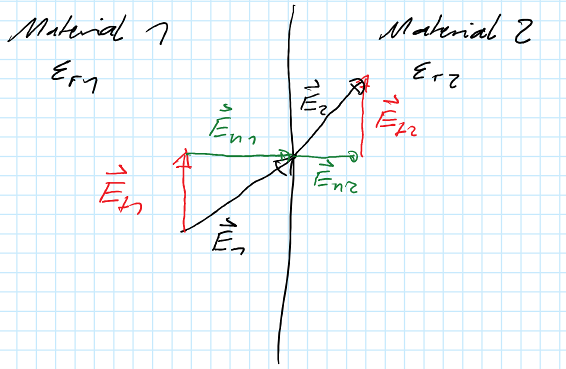
oft $\phi(\infty) = 0$ gesetzt (ist aber beliebig)

nur Potentialdifferenz hat Bedeutung

Maxwell: $\mathrm{div}(\vec{D}) = \rho \Rightarrow \oint\!\!\oint \vec{D}\, d\vec{f} = \iiint \rho\, dv$

per Definition: $\vec{E} = -\mathrm{grad}\, \phi$

Grenzübergänge



FS: $E_{t1} = E_{t2}$

$\sigma = D_{n2} - D_{n1}$

$\Rightarrow$ Vorgehen: Felder zerlegen in Tangentialund Normalkomponente

① $D_{n2} = \sigma + D_{n1}$

$\Rightarrow \varepsilon_0 \varepsilon_{r2} E_{n2} = \sigma + \varepsilon_0 \varepsilon_{r1} E_{n1}$

$\Rightarrow E_{n2} = \dfrac{\sigma}{\varepsilon_0 \varepsilon_{r2}} + \dfrac{\varepsilon_{r1}}{\varepsilon_{r2}} E_{n1}$

② $E_{t2} = E_{t1}$

$\Rightarrow \dfrac{D_{t2}}{\varepsilon_0 \varepsilon_{r2}} = \dfrac{D_{t1}}{\varepsilon_0 \varepsilon_{r1}}$

$\Rightarrow D_{t2} = \dfrac{\varepsilon_{r2}}{\varepsilon_{r1}} D_{t1}$

① & ② liefert $\underbrace{E_{n2}, E_{t2}}_{\vec{E}_2}$

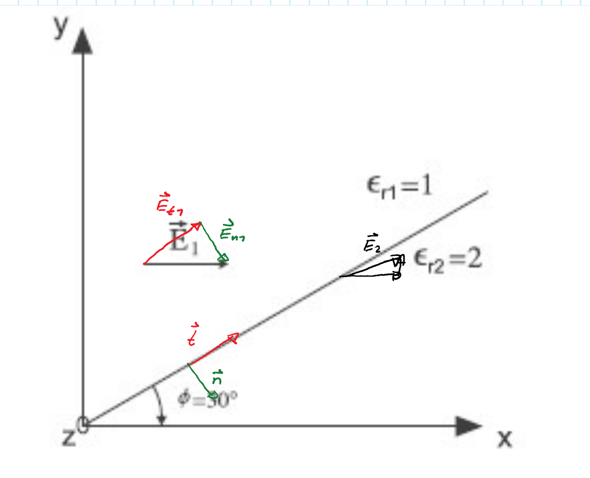
und $\underbrace{D_{n2}, D_{t2}}_{\vec{D}_2}$

A4) geg.: $\vec{E}_1 = E_0 \vec{e}_x$, $\sigma = 0$, $\varepsilon_{r1} = 1$, $\varepsilon_{r2} = 2$

ges.: $\vec{E}_2$, $\vec{D}_2$, $\vec{D}_1$

$\vec{D}_1 = \varepsilon_0 \varepsilon_{r1} \vec{E}_1 = \varepsilon_0 E_0 \vec{e}_x$

$D_{n2} \overset{FS}{=} \sigma + D_{n1} \Rightarrow E_{n2} = \dfrac{\sigma}{\varepsilon_0 \varepsilon_{r2}} + \dfrac{\varepsilon_{r1}}{\varepsilon_{r2}} E_{n1} \Big|_{\sigma = 0}$



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$E_{n2} = \dfrac{\varepsilon_{r1}}{\varepsilon_{r2}} E_{n1}$

$E_{t1} = E_{t2}$

$E_{t1} = E_0 \cos(\phi)$

$E_{n1} = E_0 \sin(\phi)$

$\vec{E}_2 = E_{n2} \vec{n} + E_{t2} \vec{t}$

$$\vec{t} = \begin{pmatrix} \cos(\varphi) \\ \sin(\varphi) \\ 0 \end{pmatrix} \qquad \vec{n} = \begin{pmatrix} \sin(\varphi) \\ -\cos(\varphi) \\ 0 \end{pmatrix}$$

$$\vec{E}_2 = \frac{\varepsilon_{r1}}{\varepsilon_{r2}} E_0 \sin(\varphi) \begin{pmatrix} \sin(\varphi) \\ -\cos(\varphi) \\ 0 \end{pmatrix} + E_0 \cos(\varphi) \begin{pmatrix} \cos(\varphi) \\ \sin(\varphi) \\ 0 \end{pmatrix}$$

$$= \frac{1}{2} E_0 \frac{1}{2} \begin{pmatrix} 1/2 \\ -\sqrt{3}/2 \\ 0 \end{pmatrix} + E_0 \frac{\sqrt{3}}{2} \begin{pmatrix} \sqrt{3}/2 \\ 1/2 \\ 0 \end{pmatrix}$$

$$= E_0 \left[\begin{pmatrix} 1/8 \\ -\sqrt{3}/8 \\ 0 \end{pmatrix} + \begin{pmatrix} 3/4 \\ \sqrt{3}/4 \\ 0 \end{pmatrix} \right]$$

$$= E_0 \begin{pmatrix} 7/8 \\ \sqrt{3}/8 \\ 0 \end{pmatrix} = \underline{E_0 \begin{pmatrix} 0,875 \\ 0,22 \\ 0 \end{pmatrix}}$$

$$\vec{D}_2 = \varepsilon_0 \varepsilon_{r2} \vec{E}_2 = \underline{\varepsilon_0 2 \vec{E}_2}$$

A 5) geg.: $\rho(r) = \begin{cases} \frac{\rho_1}{R_0^2} r^2 & 0 \le r < R_0 \\ \rho_2 R_0^4 \frac{1}{r^4} & r \ge R_0 \end{cases}$

$\varepsilon_r = 1$

ges.: $\vec{E}(r, \vartheta, \varphi) = \vec{E}(r) = E(r)\, \vec{e}_r$ (wegen Kugelsymm.)

$$\vec{E}(r) = \frac{1}{\varepsilon_0} \vec{D}(r)$$

$$\underbrace{\oint\!\!\oint_{\partial K(r)} \vec{D}\, d\vec{f}} = \iiint_{K(r)} \rho \, dV$$

$$\hookrightarrow \oint\!\!\oint \vec{D}(r)\, \underline{d\vec{f}} = \int_{\varphi=0}^{2\pi} \int_{\vartheta=0}^{\pi} D(r) \overbrace{\vec{e}_r \vec{e}_r}^{=1} \underbrace{r^2 \sin\vartheta \, d\vartheta \, d\varphi}_{= d\vec{f}}$$

$$= D(r)\, r^2 \int_{\varphi=0}^{2\pi} \int_{\vartheta=0}^{\pi} \sin\vartheta \, d\vartheta \, d\varphi = D(r)\, 4\pi r^2$$

$$= D(r)\, r^2 \int_{\varphi=0} \int_{\vartheta=0} \underbrace{\sin\vartheta \; d\vartheta}_{\left[-\cos\vartheta\right]_0^{\pi} = 2} d\varphi = D(r)\, 4\pi r^2$$

$$= 4\pi r^2 \varepsilon_0 E(r)$$

$$\iiint \rho \, dv = \int_{\varphi=0}^{2\pi} \int_{\vartheta=0}^{\pi} \int_{r'=0}^{r} \rho(r')\, r'^2 \sin\vartheta \; dr' \, d\vartheta \, d\varphi$$

$$= 2 \cdot 2\pi \int_0^r \rho(r')\, r'^2 \, dr'$$

1. Fall $\quad r < R_0$

$$4\pi \int_0^r \rho(r')\, r'^2 \, dr' = 4\pi \int_0^r \frac{\rho_1}{R_0^2} r'^2 \, r'^2 \, dr'$$

$$= \frac{4\pi \rho_1}{R_0^2} \int_0^r r'^4 \, dr' = \frac{4\pi\rho_1}{R_0^2} \left[\frac{1}{5} r'^5\right]_0^r = \frac{4\pi \rho_1}{5 R_0^2} r^5$$

$$\Rightarrow \quad 4\pi \varepsilon_0 E(r)\, r^2 = \frac{4\pi \rho_1}{5 R_0^2} r^5$$

$$E(r) = \frac{\rho_1}{5 \varepsilon_0 R_0^2} r^3 \, , \quad r < R_0$$

2. Fall $\quad r \geq R_0$

$$4\pi \int_{r'=0}^{r} \rho(r')\, r'^2 \, dr' = 4\pi \int_{r'=0}^{R_0} \frac{\rho_1}{R_0^2} r'^2 r'^2 \, dr' + \int_{r'=R_0}^{r} \rho_2 \frac{R_0^4}{r'^4} r'^2 \, dr'$$

$$= \frac{4\pi \rho_1}{5 R_0^2} R_0^5 + 4\pi \rho_2 R_0^4 \underbrace{\int_{R_0}^{r} \frac{1}{r'^2} \, dr'}_{\left[-\frac{1}{r'}\right]_{R_0}^{r}}$$

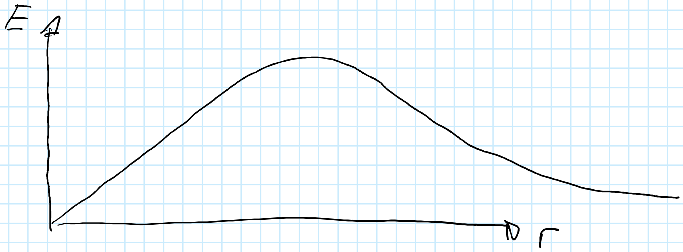
$$= \frac{4\pi \rho_1}{5} R_0^3 + 4\pi \rho_2 R_0^4 \left(\frac{1}{R_0} - \frac{1}{r}\right)$$

$$4\pi \varepsilon_0 E(r) r^2 = 4\pi R_0^3 \left(\frac{\rho_1}{5} + \rho_2 R_0 \left(\frac{1}{R_0} - \frac{1}{r}\right)\right)$$

$$\Rightarrow E(r) = \frac{R_0^3}{\varepsilon_0 r^2}\left(\frac{\rho_1}{5} + \rho_2 R_0 \left(\frac{1}{R_0} - \frac{1}{r}\right)\right) \quad \text{für } r \geq R_0$$

A6) a) geg.: $\vec{E} = \frac{E_0}{r_0} r e^{-\frac{r}{r_0}} \vec{e}_r$, $\varepsilon = \varepsilon_0$, $\phi(\infty) = 0$

ges.: $\phi(r)$



FS: $-\int_{r_1}^{r_2} \vec{E}\, d\vec{s} = \phi(r_2) - \phi(r_1)$

Integration entlang $\vec{e}_r$

$$-\int_{r_1}^{r_2} \frac{E_0}{r_0} r' e^{-\frac{r'}{r_0}} \overbrace{\vec{e}_r \vec{e}_r}^{=1} dr'$$

$$= -\frac{E_0}{r_0} \int_{r_1}^{r_2} r' e^{-\frac{r'}{r_0}} dr' = \phi(r_2) - \phi(r_1)$$

$$\overset{r_2 \to \infty}{=} 0 - \phi(r_1) = -\phi(r_1)$$

$$\phi(r) = \frac{E_0}{r_0} \int_{r'=r}^{\infty} r' e^{-\frac{r'}{r_0}} dr' = \frac{E_0}{r_0}\left[\frac{-\frac{1}{r_0} r' - 1}{1/r_0^2} e^{-\frac{r'}{r_0}}\right]_r^{\infty}$$

$$\boxed{\text{math FS} \quad \int x e^{ax} dx = \frac{ax - 1}{a^2} e^{ax}}$$

$$= \frac{E_0}{r_0}\left[(-r_0 r' - r_0^2) e^{-\frac{r'}{r_0}}\right]_r^{\infty} = \frac{E_0}{r_0}\left(0 - (-r_0 r - r_0^2) e^{-\frac{r}{r_0}}\right)$$

$$= \frac{E_0}{r_0}\left(r_0 r + r_0^2\right) e^{-\frac{r}{r_0}} = E_0 (r + r_0) e^{-\frac{r}{r_0}}$$

b) geg. : $E_r(r)$

ges. : $\varrho(r)$

$$\operatorname{div}(\vec{D}) = \varrho$$

$$\Rightarrow \operatorname{div}(\vec{E}) = \frac{\varrho}{\varepsilon_0}$$

FS: $$\operatorname{div}(\vec{E}) = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 E_r \right)$$

$$\Rightarrow \quad \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 E_r \right) = \frac{\varrho}{\varepsilon_0}$$

$$\frac{E_0}{r_0} \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^3 e^{-\frac{r}{r_0}} \right) = \frac{\varrho}{\varepsilon_0}$$

$$\frac{E_0}{r_0} \frac{1}{r^2} \left(3 r^2 e^{-\frac{r}{r_0}} - \frac{r^3}{r_0} e^{-\frac{r}{r_0}} \right) = \frac{\varrho}{\varepsilon_0}$$

$$\Rightarrow \varrho = \frac{E_0 \varepsilon_0}{r_0} \left(3 - \frac{r}{r_0} \right) e^{-\frac{r}{r_0}}$$