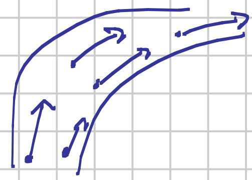
F&W Übung 1 25.10.2010

Wiederholung zu HM

Skalarfelder: ordnen jedem Punkt im Raum einen skalaren Wert zu. Bsp: Temperatur, el. Potential

$f: \mathbb{R}^n \to \mathbb{R}$, in F&W (oft) $f: \mathbb{R}^3 \to \mathbb{R}$

Vektorfelder: ordnen jedem Punkt im Raum einen Vektor zu Bsp: Strömungen , $\vec{E}, \vec{D}, \vec{B}, \vec{H}$

$f: \mathbb{R}^n \to \mathbb{R}^m$

in F&W oft: $\mathbb{R}^3 \to \mathbb{R}^3$

Partielle Ableitungen: geg. Vektorfeld $\vec{V} = \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix}$

PA: $\dfrac{\partial v_x}{\partial x}, \dfrac{\partial v_y}{\partial y}, \dfrac{\partial v_z}{\partial z}$

Differentialoperatoren:

Gradient: f sei Skalarfeld ($f: \mathbb{R}^3 \to \mathbb{R}$)

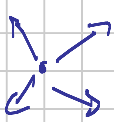
dann $\operatorname{grad}(f) := \begin{pmatrix} \partial f/\partial x \\ \partial f/\partial y \\ \partial f/\partial z \end{pmatrix}$

Vektorfeld
Richtung: steilster Anstieg
Stärke: Änderungsrate

Divergenz: $\vec{f}$ sei Vektorfeld $\mathbb{R}^3 \to \mathbb{R}^3$

dann $\operatorname{div}(\vec{f}) := \underbrace{\dfrac{\partial f_x}{\partial x} + \dfrac{\partial f_y}{\partial y} + \dfrac{\partial f_z}{\partial z}}_{\text{Skalarfeld}}$
(kart)

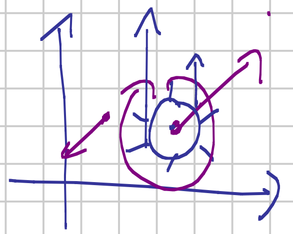
Skalarfeld: gibt an, wie viel Strömung im jeweiligen Punkt entsteht / endet

 Quelle $\operatorname{div} > 0$

 Senke $\operatorname{div} < 0$

Rotation: $\vec{f}: \mathbb{R}^3 \to \mathbb{R}^3$

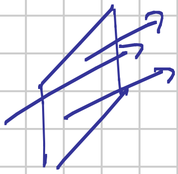
$$\text{rot } \vec{f} = \begin{pmatrix} \partial f_z/\partial y - \partial f_y/\partial z \\ \partial f_x/\partial z - \partial f_z/\partial x \\ \partial f_y/\partial x - \partial f_x/\partial y \end{pmatrix}$$



Drehrichtung / -Stärke eines imaginären Rädchens in der Strömung

Flächenintegrale über Vektorfelder

$$\iint_F \vec{A}\, d\vec{f}$$



„Wie viel $\vec{A}$ fließt durch F"?
(nur Anteil $\perp$ F)

$d\vec{f}$: infinitesimales Flächenelement, Richtung: Normalenvektor der Fläche F

Volumenintegrale über Skalarfelder

$$\iiint_V A\, dV$$

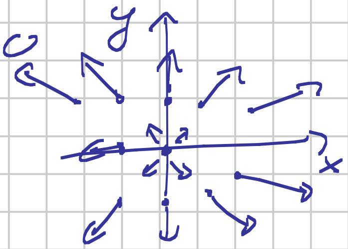
„Wie viel A ist in V"

dV: Volumenelement

A1

geg $\vec{V} = \frac{4}{3}\left(x\vec{e}_x + y\vec{e}_y + z\vec{e}_z\right) = \frac{4}{3}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

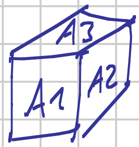
betrachte $z=0$



a) ges: $\operatorname{div}\vec{V} = ?$

$$\overset{FS}{=} \underbrace{\frac{\partial}{\partial x}\left(\frac{4}{3}x\right)}_{=4/3} + \underbrace{\frac{\partial}{\partial y}\left(\frac{4}{3}y\right)}_{4/3} + \underbrace{\frac{\partial}{\partial z}\left(\frac{4}{3}z\right)}_{4/3} = 4 \text{ (const)}$$

b) $$\iint_{\partial W} \vec{V}\,d\vec{f} = \iint_{A1} \vec{V}\,d\vec{f} + \dots + \iint_{A6} \vec{V}\,d\vec{f}$$



Rand des Würfels W

A1: $$\iint_{A1} \vec{V}\,d\vec{f} = \int_{z=-a}^{a}\int_{x=-a}^{a} \underbrace{-\vec{V}\vec{e}_y}\,dx\,dz$$

$d\vec{f} = -\vec{e}_y\,dx\,dz$

$$\boxed{\begin{array}{l} \vec{e}_x \cdot \vec{e}_x = 1 \\ \vec{e}_y \cdot \vec{e}_x = 0 \\ \vdots \end{array}}$$

$$\vec{V}\cdot\vec{e}_y = \frac{4}{3}\left(x\vec{e}_x + y\vec{e}_y + z\vec{e}_z\right)\cdot\vec{e}_y = \frac{4}{3}\cdot y$$

$$\iint_{A1} \vec{V}\,d\vec{f} = \int_{-a}^{a}\int_{-a}^{a} \frac{4}{3}y\Big|_{y=a}\,dx\,dz = \frac{4}{3}a\int_{-a}^{a}\underbrace{\int_{-a}^{a} 1\,dx}_{[x]_{-a}^{a}=2a}\,dz = \frac{4}{3}a\cdot 2a\cdot 2a$$

$$= \frac{4}{3}4a^3$$

$$\iint_{A2} \vec{V}\,d\vec{f} = \int_{-a}^{a}\int_{-a}^{a} \underbrace{\vec{V}\,\vec{e}_x}\,dy\,dz = \int_{-a}^{a}\int_{-a}^{a} \frac{4}{3}a\,dy\,dz = \frac{4}{3}a\int_{-a}^{a}\int_{-a}^{a} 1\,dy\,dz$$

$$= \frac{4}{3}a\cdot 2a\cdot 2a = \frac{4}{3}4a^3$$

A2: $x = +a$

$d\vec{f} = \vec{e}_x\,dy\,dz$

$$\iint_{\partial W} \vec V\, d\vec f = \sum_{i=1}^{6} A_i = 6 \cdot \frac{4}{3} k a^3 = 8 k a^3$$

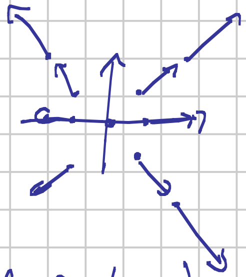
c) ges: $\iiint_W \operatorname{div} \vec V\, dv = \int_{-a}^{a}\int_{-a}^{a}\int_{-a}^{a} k \cdot dx\, dy\, dz = k \underbrace{\iiint_W dv}_{(2a)^3} = 8 k a^3$

→ identisches Ergebnis

A2 |

$\vec V = \frac{k}{3} r \vec e_r$

$= k/3\, r\vec e_r + 0\vec e_\varphi + 0\vec e_\vartheta$



$V_r = \frac{k}{3} r$

a) ges: $\operatorname{div} \vec V = \frac{1}{r^2} \underbrace{\frac{\partial}{\partial r}\left(r^2 \frac{k}{3} r\right)}_{k r^2} + \frac{1}{r \sin\vartheta} \frac{\partial}{\partial \vartheta}\left(\overset{=0}{V_\vartheta} \sin\vartheta\right) + \frac{1}{r\sin\vartheta} \frac{\partial \overset{=0}{V_\varphi}}{\partial \varphi}$

$= k$

b) $\iint_{\partial K} \vec V\, d\vec f = \int_0^{2\pi}\int_0^{\pi} \vec V \vec e_r\, r^2 \sin\vartheta \Big|_{r=a} d\vartheta\, d\varphi = \int_0^{2\pi}\int_0^{\pi} \overset{\vec V \vec e_r}{\left(\frac{k}{3} a\right)} a^2 \sin\vartheta\, d\vartheta\, d\varphi$

$\vec e_r \cdot \vec e_r$

FS: $d\vec f = \vec e_r r^2 \sin\vartheta\, d\vartheta\, d\varphi$

$= \frac{k}{3} a^3 \underbrace{\int_0^{2\pi}\int_0^{\pi} \sin\vartheta\, d\vartheta\, d\varphi}_{\left[-\cos\vartheta\right]_0^{\pi} = 1-(-1) = 2} = \frac{k}{3} a^3 \cdot 2 \int_0^{2\pi} d\varphi = \frac{2k}{3} a^3\, 2\pi = \frac{4\pi}{3} k a^3$

c) $\iiint_K \operatorname{div} \vec V\, dv = \iiint_K k\, dv = k \cdot \int_0^{2\pi}\int_0^{\pi}\int_0^{a} r^2 \sin\vartheta\, dr\, d\vartheta\, d\varphi$

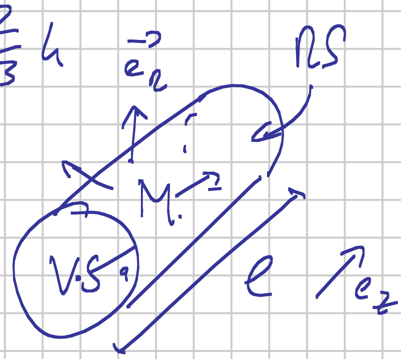
$= k \iiint_K dv = k \cdot \frac{4\pi}{3} a^3$

A3) ges: $\vec V = \underbrace{\frac{k}{3} R \vec e_R}_{\text{vgl. A2}} + \underbrace{k \vec e_z}_{\text{const. Anteil in z-Richtung}}$ (Zylinderkoord.)

a) ges: $\operatorname{div} \vec V \overset{FS}{=} \frac{1}{R} \frac{\partial}{\partial R} \Big(\underbrace{R \cdot \frac{k}{3} R}_{R^2 \cdot k/3} \Big) + \frac{1}{R} \frac{\partial}{\partial \varphi} \overset{=0}{V_\varphi} + \frac{\partial}{\partial z} k$

$= \frac{1}{\cancel{R}} \frac{2}{3} k \cdot \cancel{R} + 0 + 0 = \frac{2}{3} k$

b) $\iint_{\partial Z} \vec V d\vec f = \iint_{VS} \vec V d\vec f + \iint_{RS} \vec V d\vec f + \iint_{M} \vec V d\vec f$



VS: $d\vec f = -\vec e_z R\, dR\, d\varphi$
$z = -\ell/2$

$\iint_{VS} \vec V d\vec f = \int_0^{2\pi} \int_0^a -k \cdot R\, dR\, d\varphi = -k \int_0^{2\pi} \underbrace{\int_0^a R\, dR}_{= [\frac{1}{2} R^2]_0^a = \frac{1}{2} a^2} d\varphi = -k \cdot \frac{1}{\cancel{2}} a^2 \cdot \cancel{2}\pi$
$= -k\pi a^2$

RS: $d\vec f = \vec e_z R\, dR\, d\varphi$, $z = +\ell/2$

M: $d\vec f = \vec e_R R \cdot d\varphi\, dz$, $\varphi = 0 .. 2\pi$, $z = -\ell/2 .. \ell/2$
$R = a$

c) ges: $\iiint_Z \operatorname{div} \vec V dv = \frac{2}{3} k \underline{\iiint_Z dv} = 2/3\, k\, \underline{\pi a^2 \ell}$