Felder und Wellen Übung 2 29.10.2010

Einleitung

wichtige Größen

Vektorfelder:

- $\vec{E}$: elektrisches Feld $\left(\vec{E} = \frac{\vec{F}}{q}\right)$
- $\vec{D}$: elektrische Verschiebungsdichte $\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$ allg.

in isotropen Materialien $\vec{D} = \varepsilon_0 \underbrace{\varepsilon_r}_{\text{Dielektrizitätszahl des Materials}} \vec{E}$

$\varepsilon_r = 1$ für Vakuum
$\varepsilon_r = 80$ für Wasser
$\varepsilon_r > 2000$ für spezielle Materialien

Skalarfelder:

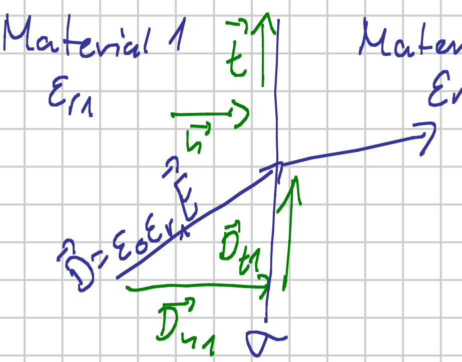
- ϱ : Raumladungsdichte $Q = \iiint_V \varrho \, dv$ („Ladung pro Volumen")
- σ : Flächenladungsdichte $Q = \iint_F \sigma \, dF$ („Ladung pro Fläche")
- Φ : elektrisches Potential : $-\underbrace{\int_{\vec{p}_1}^{\vec{p}_2} \vec{E} \, d\vec{l}}_{\text{Linienintegral}} = \Phi(\vec{p}_2) - \Phi(\vec{p}_1)$ (wegunabh.)

oft $\Phi(\infty) = 0$ Nur Potentialdifferenzen haben Bedeutung

Maxwell: $\boxed{\operatorname{div} \vec{D} = \varrho}$ $\xRightarrow{\text{Satz von Gauß}}$ $\boxed{\oiint \vec{D} \, d\vec{f} = \iiint \varrho \, dv}$

per Def: $\boxed{\vec{E} = -\operatorname{grad} \Phi}$

Grenzübergänge



$\vec{D} = \varepsilon_0 \varepsilon_r \vec{E}$ $\Longrightarrow$ (isotrop)

FS: $E_{t1} = E_{t2}$ (Tangentialkomponente)

$D_{n2} - D_{n1} = \sigma$ (Normalkomp.)

$D_{t1} = \varepsilon_0 \varepsilon_{r1} E_{t1}$, $D_{t2} = \varepsilon_0 \varepsilon_{r2} E_{t2}$

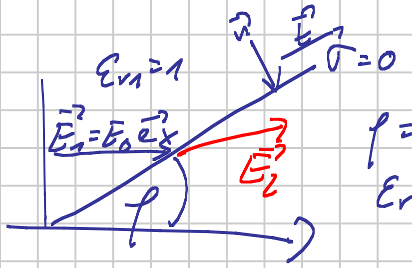
$\Rightarrow E_{t2} = \frac{D_{t2}}{\varepsilon_0 \varepsilon_{r2}}$

$E_{t1} = E_{t2}$

$D_{t1} = \frac{\cancel{\varepsilon_0} \varepsilon_{r1}}{\cancel{\varepsilon_0} \varepsilon_{r2}} D_{t2}$ $D_{t2} = \frac{\varepsilon_{r2}}{\varepsilon_{r1}} D_{t1}$

mit $\sigma = 0$ $\Rightarrow E_{n2} = \frac{\varepsilon_{r1}}{\varepsilon_{r2}} E_{n1}$

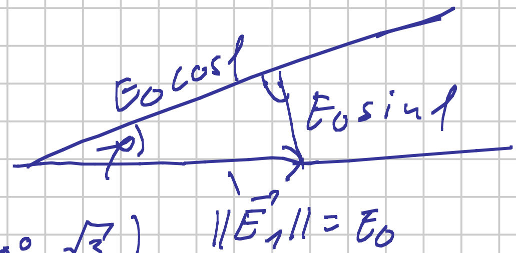
A4

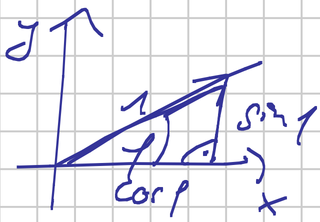


$\varphi = 30°$
$\varepsilon_{r2}=2$

RS: $E_{t1} = E_{t2}$ (Tangentialkomponente)
$D_{n2} - D_{n1} = \sigma$ (Normalkomp.)

ges: $\vec{D}_1, \vec{E}_2, \vec{D}_2$





$$\vec{D}_1 = \varepsilon_0\varepsilon_{r1}\vec{E}_1 = \varepsilon_0 E_0 \vec{e}_x$$

$$E_{t1} = E_0\cos\varphi,\quad E_{n1} = E_0\sin\varphi$$

$$E_{t2} = E_{t1} = E_0\cos\varphi = \frac{\sqrt{3}}{2}E_0 \quad \left(\cos 30° = \frac{\sqrt{3}}{2}\right)$$

$$E_{n2} = \frac{\varepsilon_{r1}}{\varepsilon_{r2}}E_{n1} = \frac{1}{2}E_{n1} = \frac{1}{2}E_0\sin\varphi = \frac{1}{4}E_0 \quad \left(\sin 30° = \frac{1}{2}\right)$$

$$\vec{E}_2 = E_{n2}\vec{n} + E_{t2}\vec{t} \qquad \vec{t} = \begin{pmatrix}\cos\varphi\\ \sin\varphi\\ 0\end{pmatrix} \qquad \vec{n} = \begin{pmatrix}\cos(\varphi-90°)\\ \sin(\varphi-90°)\\ 0\end{pmatrix} = \begin{pmatrix}\sin\varphi\\ -\cos\varphi\\ 0\end{pmatrix}$$

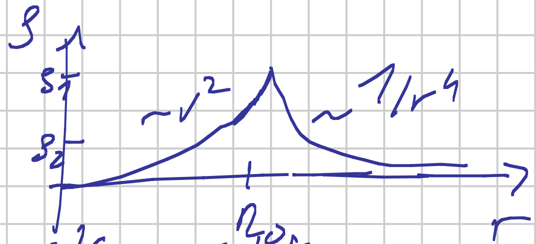
$$= \begin{pmatrix}1/4\,E_0\sin\varphi\\ -1/4\,E_0\cos\varphi\\ 0\end{pmatrix} + \begin{pmatrix}\sqrt{3}/2\,E_0\cos\varphi\\ \sqrt{3}/2\,E_0\sin\varphi\\ 0\end{pmatrix} = \begin{pmatrix}1/8\,E_0 + 3/4\,E_0\\ -\sqrt{3}/8\,E_0 + \sqrt{3}/4\,E_0\\ 0\end{pmatrix} = E_0\begin{pmatrix}7/8\\ \sqrt{3}/8\\ 0\end{pmatrix} \begin{matrix}0{,}875\\ 0{,}22\\ \end{matrix}$$

$$\vec{D}_2 = \varepsilon_0\varepsilon_{r2}\vec{E}_2 = \varepsilon_0\cdot 2\cdot E_0\begin{pmatrix}7/8\\ \sqrt{3}/8\\ 0\end{pmatrix} = \varepsilon_0 E_0\begin{pmatrix}7/4\\ \sqrt{3}/4\\ 0\end{pmatrix}$$

A5

$\varepsilon_r = 1$, Kugelkoordinaten

$$\varrho(r) = \begin{cases}\varrho_1/R_0^2\cdot r^2 & 0\le r\le R_0\\ \varrho_2\cdot R_0^4\cdot 1/r^4 & r > R_0\end{cases}$$



ges: $\vec{E}(r,\vartheta,\varphi) = \vec{E}(r) = E(r)\vec{e}_r$ wegen Kugelsymmetrie

$$\vec{E} = \frac{1}{\varepsilon_0}\vec{D} \qquad \underbrace{\oint\!\!\oint_{\partial K(r)}\vec{D}\,d\vec{f}}_{(*)} = \underbrace{\iiint_{K(r)}\varrho\,dv}_{(**)}$$

$K(r)$: Kugel mit Radius r

$$\underbrace{\oint\!\!\oint\vec{D}(r)\,d\vec{f}}_{(*)} = \int_0^{2\pi}\!\!\int_0^{\pi}\underbrace{D(r)\vec{e}_r}_{\vec{D}(r)}\cdot\underbrace{\underbrace{(\vec{e}_r\cdot\vec{e}_r)}_{=1}r^2\sin\vartheta\,d\vartheta\,d\varphi}_{d\vec{f}} = r^2 D(r)\underbrace{\int_0^{2\pi}}_{=2\pi}\underbrace{\int_0^{\pi}\sin\vartheta\,d\vartheta}_{=[-\cos\vartheta]_0^{\pi}=1+1=2}d\varphi$$

$$= r^2 D(r)\cdot 4\pi = 4\pi\varepsilon_0 E(r) r^2$$

rechte Seite: $\iiint\limits_{K(r)} \rho \, dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{r} \rho(r')\, r'^2 \underbrace{\sin\vartheta \, dr' \, d\vartheta}_{= 2} \underbrace{d\varphi}_{= 2\pi} = 4\pi \int_0^r \rho(r')\, r'^2 \, dr'$

(xx) $$4\pi \int_0^r \rho(r')\, r'^2 \, dr' \underset{\text{für } r \le R_0}{=} 4\pi \int_0^r \rho_1/R_0^2 \cdot r'^2 \, r'^2 \, dr' = 4\pi \frac{\rho_1}{R_0^2} \int_0^r r'^4 \, dr'$$

$$\left[\tfrac{1}{5} r'^5\right]_0^r$$

$$= \frac{4\pi \rho_1 r^5}{5 R_0^2}$$

(*) = (xx) $$\cancel{4\pi}\varepsilon_0 E(r) \cancel{r^2} = \frac{\cancel{4\pi}}{5R_0^2}\rho_1 r^{\cancel{5}\,3} \quad \Big| : \varepsilon_0 \Rightarrow \boxed{E(r) = \frac{\rho_1}{5\varepsilon_0 R_0^2} r^3}$$

$r \le R_0$

$$4\pi \int_0^r \rho(r')\, r'^2 \, dr' \underset{r > R_0}{=} \underbrace{4\pi \int_0^{R_0} \rho(r')\, r'^2 \, dr'}_{\frac{4\pi}{5\cancel{R_0^2}}\rho_1 R_0^{\cancel{5}\,3}} + 4\pi \int_{R_0}^{r} \rho(r')\, r'^2 \, dr' = \frac{4\pi}{5}\rho_1 R_0^3 + 4\pi \int_{R_0}^{r} \rho_2 R_0^4 \cdot 1/r'^{\cancel{4}\,2} \cdot \cancel{r'^2} \, dr'$$

$$= \frac{4\pi}{5}\rho_1 R_0^3 + 4\pi \rho_2 R_0^4 \int_{R_0}^{r} \underbrace{\frac{1}{r'^2}}_{= r'^{-2}} dr' = \frac{4\pi}{5}\rho_1 R_0^3 + 4\pi \rho_2 R_0^4 \left(-\frac{1}{r} + \frac{1}{R_0}\right)$$

$$\left[-r'^{-1}\right]_{R_0}^{r} = \left[-1/r'\right]_{R_0}^{r}$$

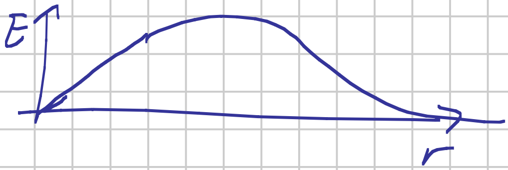
$$= 4\pi R_0^3 \left(\frac{\rho_1}{5} + \rho_2 R_0 \left(\frac{1}{R_0} - \frac{1}{r}\right)\right)$$

(*) = (xx) $$\cancel{4\pi}\varepsilon_0 E(r) r^2 = \cancel{4\pi} R_0^3 \left(\frac{\rho_1}{5} + \rho_2 R_0 \left(\frac{1}{R_0} - \frac{1}{r}\right)\right) \quad \Big| : \varepsilon_0 \cdot r^2$$

$$\boxed{E(r) = \frac{R_0^3}{\varepsilon_0 r^2}\left(\frac{\rho_1}{5} + \rho_2 \left(1 - \frac{R_0}{r}\right)\right)}$$

A6 Kugelkoordinaten, geg $\vec E$, ges Φ, ρ, $\varepsilon = \varepsilon_0$

$$\vec E = \frac{E_0}{r_0}\, r e^{-r/r_0}\,\vec e_r$$



a) ges Φ, $\Phi(\infty) = 0$

$\Phi(r,\vartheta,\varphi) = \Phi(r)$ $\qquad -\int_{\vec p_1}^{\vec p_2} \vec E\, d\vec l = \Phi(\vec p_2) - \Phi(\vec p_1)$

Integration entlang $\vec e_r$: $\quad -\int_{r_1}^{r_2} \vec E\, d\vec l = \int_r^\infty \underbrace{\frac{E_0}{r_0} r' e^{-r'/r_0}\,\vec e_r}_{\vec E} \cdot \underbrace{\vec e_r\, dr}_{d\vec l}$ $\qquad \vec e_r \cdot \vec e_r = 1$

$$= -\frac{E_0}{r_0}\int_r^\infty r' e^{-r'/r_0}\,dr = -\frac{E_0}{r_0}\left[\frac{-1/r_0\cdot r' - 1}{1/r_0^2}\, e^{-r'/r_0}\right]_r^\infty$$

math. FS: $\int x e^{ax}\,dx = \frac{ax-1}{a^2} e^{ax}$

hier: $a = -1/r_0$

$$= -\frac{E_0}{r_0}\left[(-r_0 r' - r_0^2)\, e^{-r'/r_0}\right]_r^\infty = -\frac{E_0}{\cancel{r_0}}\left(0 - (-\cancel{r_0} r - r_0^2)\, e^{-r/r_0}\right)$$

$\lim_{r'\to\infty} r' e^{-r'/r_0} \to 0$

$$= -E_0(r + r_0)\, e^{-r/r_0} = \underbrace{\Phi(\infty)}_{=0} - \Phi(r)$$

$$\boxed{\Phi(r) = E_0\,(r + r_0)\, e^{-r/r_0}}$$

b) ges: $\rho(r)$ $\qquad \mathrm{div}\,\vec D = \rho = \mathrm{div}\,\varepsilon_0 \vec E = \varepsilon_0\, \mathrm{div}\left(\frac{E_0}{r_0} r e^{-r/r_0}\,\vec e_r\right)$

$$= \varepsilon_0 \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{E_0}{r_0} r e^{-r/r_0}\right)$$

$$= \varepsilon_0 \frac{E_0}{r_0}\frac{1}{\cancel{r^2}}\, \cancel{r^2} e^{-r/r_0}\left(3 - r/r_0\right) = \varepsilon_0 \frac{E_0}{r_0} e^{-r/r_0}\left(3 - \frac{r}{r_0}\right)$$

$\frac{\partial}{\partial r}\left(r^3 e^{-r/r_0}\right)$

$u = r^3$, $v = e^{-r/r_0}$

$(u\cdot v)' = u'v + uv'$

$u' = 3r^2$, $v' = -\frac{1}{r_0} e^{-r/r_0}$

$$= 3r^2 e^{-r/r_0} + r^3\left(-\frac{1}{r_0}\right) e^{-r/r_0} = r^2 e^{-r/r_0}\left(3 - r/r_0\right)$$