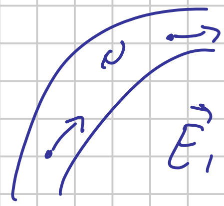
Felder und Wellen Übung 1 28.10.2013

Wiederholung zu HM

Skalarfelder: ordnen jedem Punkt im Raum einen skalaren Wert zu. Bsp: Temperatur, el. Potential

$f: \mathbb{R}^n \to \mathbb{R}$, in F&W meist $\mathbb{R}^3 \to \mathbb{R}$

Vektorfelder: ordnen jedem Punkt einen Vektor zu

Bsp: Strömungen  $\vec{E}, \vec{D}, \vec{B}, \vec{H}$

$f: \mathbb{R}^n \to \mathbb{R}^m$, in F&W $\mathbb{R}^3 \to \mathbb{R}^3$

Partielle Ableitung: geg: Skalarfeld $\Phi(x,y,z)$

PA: $\frac{\partial}{\partial x}\Phi(x,y,z)$, $\frac{\partial}{\partial y}\Phi(x,y,z)$, $\frac{\partial}{\partial z}\Phi(x,y,z)$

Differentialoperatoren:

Gradient: f sei ein Skalarfeld $\mathbb{R}^3 \to \mathbb{R}$

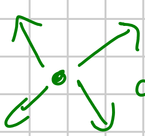
dann $\operatorname{grad}(f) := \begin{pmatrix} \partial f/\partial x \\ \partial f/\partial y \\ \partial f/\partial z \end{pmatrix}$

Vektorfeld
Richtung: steilster Anstieg
Stärke: Änderungsrate

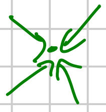
Divergenz: $\vec{f}$ sei ein Vektorfeld, $\vec{f}: \mathbb{R}^3 \to \mathbb{R}^3$ $\left(\vec{f} = \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix}\right)$

dann $\operatorname{div} \vec{f} := \underbrace{\frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z}}$

Skalarfeld: gibt an, wie viel Strömung im jeweiligen Punkt entsteht/endet

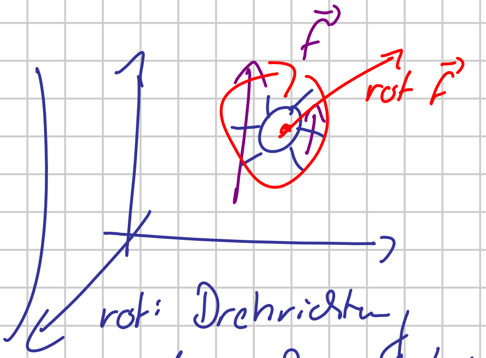
 $\operatorname{div}\vec{f} > 0$

Quelle

 Senke $\operatorname{div}\vec{f} < 0$

Rotation: $\vec{f}: \mathbb{R}^3 \to \mathbb{R}^3$

$$\text{rot}\,\vec{f} := \begin{pmatrix} \partial f_z/\partial y - \partial f_y/\partial z \\ \partial f_x/\partial z - \partial f_z/\partial x \\ \partial f_y/\partial x - \partial f_x/\partial y \end{pmatrix}$$

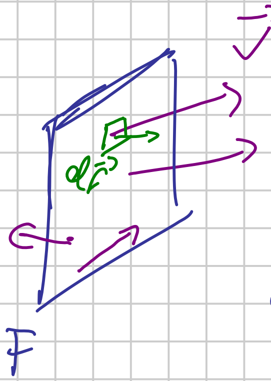


rot: Drehrichtung und -geschwindigkeit

Flächenintegrale über Vektorfelder

$$\underbrace{\iint_F \vec{v}}_{\text{Vektorfeld}}\ \underbrace{d\vec{f}}_{\text{infinitesimales Flächenelement}}$$

Richtung: Normalenvektor der Fläche am jeweiligen Punkt



„Wie viel $\vec{v}$ fließt durch F"

Skalar, nur Anteil $\perp$ F

$\oint\!\!\oint \vec{v}\,d\vec{f}$: Integral über geschlossene Fläche, z.B. Oberfläche einer Kugel K, $F = \underline{\partial K}$

„Rand von K"

Volumenintegrale über Skalarfelder

$$\iiint_V \underbrace{A}_{\text{Skalarfeld}}\ \underbrace{dv}_{\text{infinitesimales Volumenelement}}$$

„Wie viel A ist in V?"

A1) $\vec{v} = \frac{k}{3}\left(x\,\underline{\vec{e_x}} + y\,\underline{\vec{e_y}} + z\,\underline{\vec{e_z}}\right) = \frac{k}{3}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

↑ FdW-Schreibweise

a) div $\vec{v}$ = ?

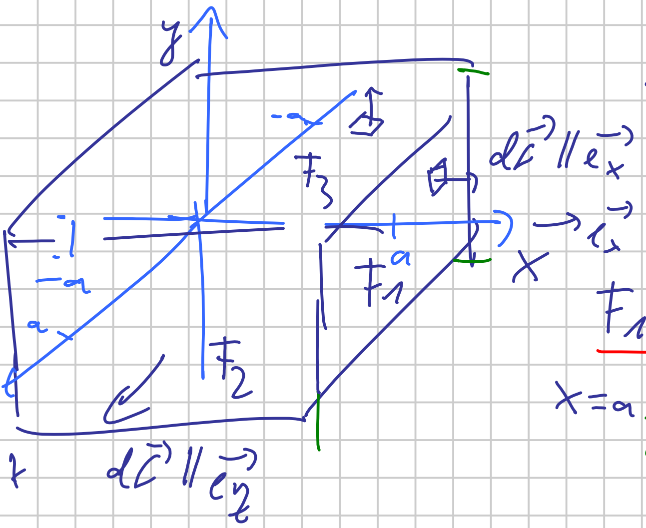
$$= \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

v_x: $k/3 \cdot x$, v_y: $k/3 \cdot y$, v_z: $k/3 \cdot z$

$$= 4/3 + 4/3 + 4/3 = \underline{4} \quad (\text{const.})$$

5) $$\oiint_{\partial W} \vec{v}\, d\vec{f} = ? \qquad = \iint_{F_1} \vec{v}\, d\vec{f} + \iint_{F_2} \vec{v}\, d\vec{f} + \iint_{F_3} \dots$$

$x = a \qquad z = a \qquad y = a \qquad x = -a,\ y = -a,\ z = -a$



$$\underline{F_1}: \int_{z=-a}^{a} \left[\int_{y=-a}^{a} \frac{4}{3}\left(a \cdot \vec{e}_x + y\vec{e}_y + z\vec{e}_z\right)\vec{e}_x \cdot dy \right] dz$$

$x = a$

inneres Integral

$$= \int_{z=-a}^{a} \int_{y=-a}^{a} \frac{4}{3}\left(a\, \underbrace{\vec{e}_x \cdot \vec{e}_x}_{=1} + y\, \underbrace{\vec{e}_y \cdot \vec{e}_x}_{=0} + z\, \underbrace{\vec{e}_z\, \vec{e}_x}_{=0}\right) dy \cdot dz$$

$$= \int_{z=-a}^{a} \left[\int_{y=-a}^{a} \frac{4}{3}\, a \cdot dy \right] dz = \int_{z=-a}^{a} \frac{2 \cdot 4}{3} a^2\, dz = \frac{2 \cdot 4}{3} a^2 \cdot (2a) = \frac{4}{3} 4 a^3$$

$$\left[\frac{4}{3} \cdot a \cdot y\right]_{-a}^{a} = \frac{4}{3} a \cdot \underbrace{(a - (-a))}_{2a} = \boxed{\frac{2}{3} a^2 \cdot 4}$$

F_2: $z = a \quad d\vec{f} : \underline{\vec{e}_z}\, dx\, dy$

$$\to \iint_{F_2} \vec{v}\, d\vec{f} = \int_{y=-a}^{a} \int_{x=-a}^{a} \frac{4}{3}\left(\underbrace{x\vec{e}_x \cdot \vec{e}_z}_{=0} + \underbrace{y\vec{e}_y \cdot \vec{e}_z}_{=0} + a\, \underbrace{\vec{e}_z \cdot \vec{e}_z}_{=1}\right) dx\, dy$$

$$= \int_{y=-a}^{a} \underbrace{\int_{x=-a}^{a} \frac{h}{3}(a)\,dx}_{\frac{h}{3}a\cdot 2a}\,dy = \int_{y=-a}^{a} \frac{2h}{3}a^2\,dy = \frac{2h}{3}a^2\cdot 2a$$

$$= \frac{4}{3}h\cdot a^3$$

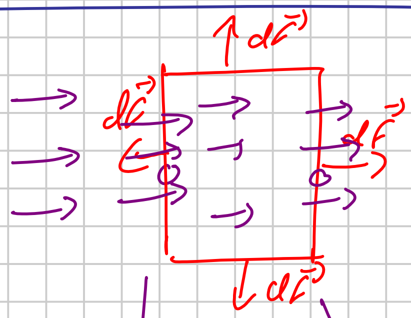
Ebenso: $F_3 .. F_6 = \frac{4}{3} h a^3$

$$\oiint_{\partial W} \vec{V}\,d\vec{f} = \iint_{F_1} + \dots + \iint_{F_6} = 6\cdot\frac{4}{3} h a^3 = 8 h a^3$$

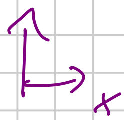
c) ges: $\iiint_{W} \underbrace{\operatorname{div}\vec{V}}_{=h\ (a)}\,dV = \int_{z=-a}^{a}\int_{y=-a}^{a}\int_{x=-a}^{a} h\,dx\,dy\,dz = h\cdot 2a\cdot 2a\cdot 2a$

$= 8 h a^3$

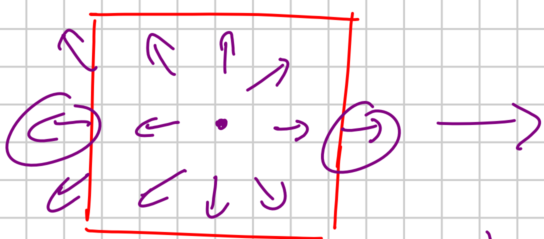
$\hat{=}$ Ergebnis von b)



$d\vec{f}\cdot\vec{V} < 0$ $\quad$ $d\vec{f}\cdot\vec{V} > 0$



$\vec{V} = h\cdot\vec{e_x}$



$\vec{V} = h\left(x\vec{e_x} + y\vec{e_y} + z\vec{e_z}\right)$

$d\vec{f}\cdot\vec{V} > 0$ $\quad$ $d\vec{f}\cdot\vec{V} > 0$

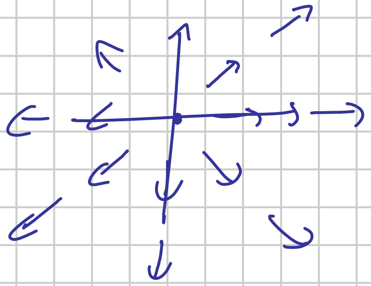
A2 | Kugelkoordinaten: Radius r, 2 Winkel ϑ, φ

$\vartheta \in [0, \pi]$, $\varphi \in [0 .. 2\pi]$

$\vec{e_r} \cdot \vec{e_r} = 1$

$\vec{e_r} \cdot \vec{e_\varphi} = 0$

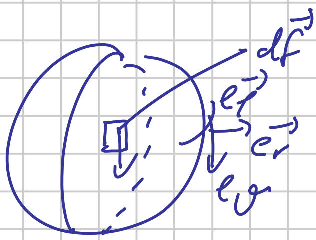
geg: $\vec{V} = \frac{k}{3} \cdot r \cdot \vec{e_r} + 0 \cdot \vec{e_\varphi} + 0 \cdot \vec{e_\vartheta}$



ges: $\operatorname{div} \vec{V} \underset{FS}{=} \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 V_r\right) + \frac{1}{r \sin\vartheta} \frac{\partial}{\partial \vartheta}\left(\underbrace{\cancel{V_\vartheta}}_{=0} \sin\vartheta\right) + \frac{1}{r \sin\vartheta} \frac{\partial \cancel{V_\varphi}}{\partial \varphi}$ $=0$

$= \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{k}{3} \cdot r\right) = \frac{1}{\cancel{r^2}} \frac{k}{\cancel{3}} \cancel{3} \cdot \cancel{r^2} = \underline{\underline{k}}$

b) ges: $\underset{\partial K}{\oiint} \underline{\vec{V}} \, d\vec{f}$



$d\vec{f} \underset{FS}{=} \underline{\vec{e_r}\, r^2 \sin\vartheta \, d\vartheta \, d\varphi}$, $\vartheta = 0 .. \pi$, $\varphi = 0 .. 2\pi$

$\underset{\underline{r=a}}{\oiint} \vec{V} d\vec{f} = \int_{\varphi=0}^{2\pi} \int_{\vartheta=0}^{\pi} \frac{k}{3} a \underbrace{\vec{e_r} \cdot \vec{e_r}}_{=1} a^2 \sin\vartheta \, \underline{d\vartheta} \, d\varphi = \frac{k}{3} \cdot a^3 \int_{\varphi=0}^{2\pi} \int_{\underline{\vartheta=0}}^{\pi} \underline{\sin\vartheta \, d\vartheta} \cdot d\varphi$

$= \frac{k}{3} a^3 \int_{\varphi=0}^{2\pi} \left[-\cos\vartheta\right]_0^\pi d\varphi = \frac{k}{3} a^3 \int_{\varphi=0}^{2\pi} 2 \, d\varphi = \frac{k}{3} a^3 \cdot 2 \cdot 2\pi$

$-(-1) + (1) = 2$

$= \underline{\underline{\frac{4\pi}{3} k \cdot a^3}}$

c) $\underset{K}{\iiint} \underbrace{\operatorname{div} \vec{V}}_{=k} \cdot \underline{dv} \underset{FS}{=} \int_{\varphi=0}^{2\pi} \int_{\vartheta=0}^{\pi} \int_{r=0}^{a} k \cdot \underline{r^2 \sin\vartheta \, dr \, d\vartheta \, d\varphi}$

$$= \int_{\varphi=0}^{2\pi} \int_{\vartheta=0}^{\pi} \underbrace{\left[k \cdot \sin\vartheta \cdot \frac{1}{3} r^3\right]_0^a}_{k \sin\vartheta \cdot \frac{1}{3} a^3} \, d\vartheta \, d\varphi$$

$$= k \cdot \frac{1}{3} a^3 \int_{\varphi=0}^{2\pi} \underbrace{\int_{\vartheta=0}^{\pi} \sin\vartheta \, d\vartheta}_{[-\cos\vartheta]_0^\pi = 2} \, d\varphi = k \cdot \frac{2}{3} a^3 \underbrace{\int_{\varphi=0}^{2\pi} d\varphi}_{=2\pi}$$

$$= \frac{4\pi}{3} k \cdot a^3$$

$\hat{=}$ Ergebnis aus b)