# Einleitung:

Wichtige Größen:

$\vec{E}$: el. Feld $\quad \left(E := \frac{\vec{F}}{q}\right)$ $\quad$ Ursache: Ladungen

$\vec{D}$: el. Verschiebungsdichte $\quad \vec{D} = \varepsilon_0 \vec{E} + \vec{P}$ (allg.)

in isotropen Materialen: $\vec{D} = \varepsilon_0 \cdot \varepsilon_r \vec{E}$ (oft, nicht immer; $\varepsilon_r$ Dielektrizitätszahl des Materials)

$\varepsilon_r = 1$ für Vakuum

$\varepsilon_r \approx 80$ für Wasser

$\varepsilon_r > 2000$ für spezielle Materialen, z.B. $BaTiO_3$

$\rho$: Raumladungsdichte $\quad Q = \iiint\limits_V \rho \, dv$ („Ladung pro Volumen")

$\sigma$: Flächenladungsdichte $\quad Q = \iint\limits_F \sigma \, dF$ („Ladung pro Fläche")

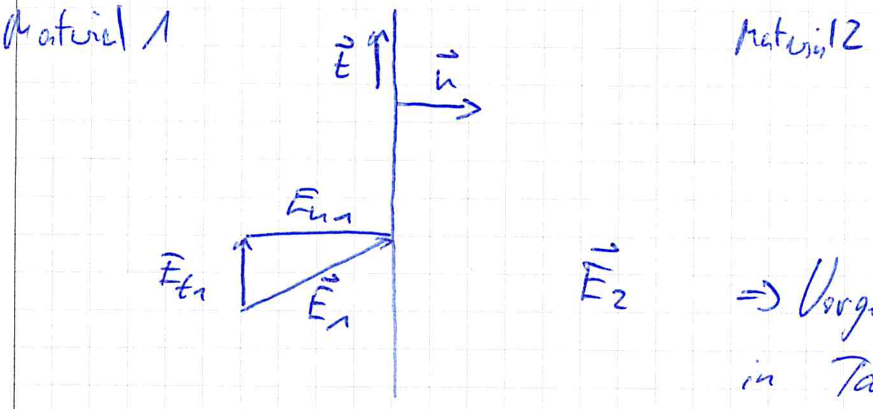
$\phi$: el. Potential: $-\int\limits_{\vec{p}_1}^{\vec{p}_2} \vec{E} \, d\vec{l} = \phi(\vec{p}_2) - \phi(\vec{p}_1)$ (Linien-Integral)

oft $\phi(\infty) = 0$ gesetzt (beliebig, aber nützlich)

Maxwell $\quad \boxed{\operatorname{div} \vec{D} = \rho}$ $\quad$ Satz von Gauß $\Longrightarrow$ $\boxed{\oiint \vec{D} \, d\vec{F} = \iiint \rho \, dv}$

per Def.: $\quad \boxed{\vec{E} = -\operatorname{grad} \phi}$

Grenzübergänge:




$$\boxed{\begin{array}{ll} \text{FS allg.:} & E_{t1} = E_{t2} \\ & D_{n2} - D_{n1} = \sigma \end{array}}$$

$\Rightarrow$ Vorgehen: Felder zerlegen in Tangential- und Normalenanteile

② $D_{n2} = D_{n1} + \sigma$

$\Leftrightarrow \varepsilon_0 \varepsilon_{r2} E_{n2} = \varepsilon_0 \varepsilon_{r1} E_{n1} + \sigma$

$\Leftrightarrow E_{n2} = \dfrac{\sigma / \varepsilon_0}{\varepsilon_{r2}} + \dfrac{\varepsilon_{r1}}{\varepsilon_{r2}} E_{n1}$

① $E_{t1} = E_{t2}$

$\Leftrightarrow \dfrac{1}{\varepsilon_{r1}} D_{t1} = \dfrac{1}{\varepsilon_{r2}} D_{t2}$

$\Leftrightarrow D_{t2} = \dfrac{\varepsilon_{r2}}{\varepsilon_{r1}} D_{t1}$

①, ② $\rightarrow$ liefert $\underbrace{E_{n2}, E_{t1}}_{\vec{E}_2}$ ; $\underbrace{D_{n2}, D_{t1}}_{\vec{D}_2}$

bei geg. $\vec{E}_1$ & $\vec{D}_1$

4. Aufgabe:

Geg.: $\vec{E}_1 = E_0 \cdot \vec{e}_x$ , $\sigma = 0$ , $\varepsilon_{r_1} = 1$ , $\varepsilon_{r_2} = 2$

Ges.: $\vec{E}_2$ , $\vec{D}_1$ , $\vec{D}_2$

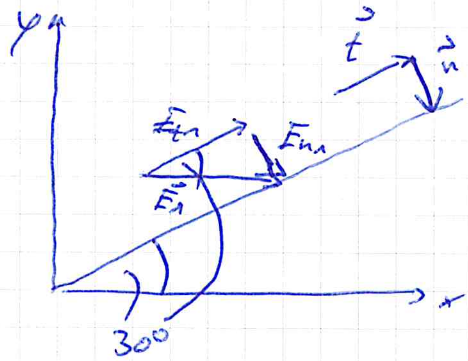
FS allg.: $E_{t1} = E_{t2}$

$D_{n2} - D_{n1} = \sigma$

$$\vec{D}_1 = \varepsilon_0 \varepsilon_{r1} \vec{E}_1 = \underline{\underline{\varepsilon_0 E_0 \cdot \vec{e}_x}}$$

$$E_n = \frac{\sigma / \varepsilon_0}{\varepsilon_{r2}} + \frac{\varepsilon_{r1}}{\varepsilon_{r2}} E_{n1} \Big|_{\sigma = 0}$$

- $E_{n2} = \frac{\varepsilon_{r1}}{\varepsilon_{r2}} E_{n1}$
- $E_{t2} = E_{t1}$
- $E_{t1} = E_0 \cdot \cos\varphi$
- $E_{n1} = E_0 \cdot \sin\varphi$




$\sin\varphi = \frac{E_{n1}}{E_1}$

$\Leftrightarrow E_{n1} = E_1 \cdot \sin\varphi$

$\cos\varphi = \frac{E_{t1}}{E_1}$

$\Leftrightarrow E_{t2} = E_1 \cdot \cos\varphi$

$\vec{E}_2 = E_{n2} \cdot \vec{n} + E_{t2} \cdot \vec{t}$

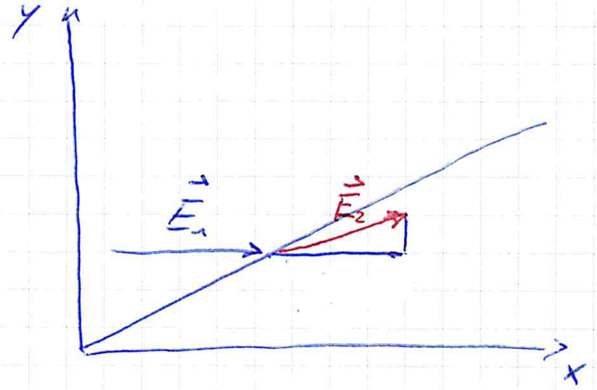
$\vec{t} = \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix}$ , $\vec{n} = \begin{pmatrix} \sin\varphi \\ -\cos\varphi \\ 0 \end{pmatrix}$

$\perp$ , da $\vec{t} \cdot \vec{n} = 0$

$$\vec{E}_2 = \frac{\varepsilon_{r1}}{\varepsilon_{r2}} E_0 \sin\varphi \begin{pmatrix} \sin\varphi \\ -\cos\varphi \\ 0 \end{pmatrix} + E_0 \cos\varphi \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix}$$

$$= \frac{1}{2} E_0 \frac{1}{2} \begin{pmatrix} 1/2 \\ -\sqrt{3}/2 \\ 0 \end{pmatrix} + E_0 \frac{\sqrt{3}}{2} \begin{pmatrix} \sqrt{3}/2 \\ 1/2 \\ 0 \end{pmatrix} =$$

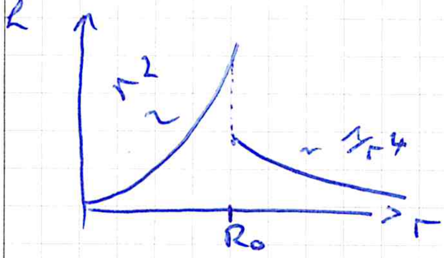
$$\Leftrightarrow E_0\left[\begin{pmatrix} 1/8 \\ -\sqrt{3}/8 \\ 0 \end{pmatrix} + \begin{pmatrix} 3/4 \\ \sqrt{3}/4 \\ 0 \end{pmatrix}\right] = \vec{E}_2 = E_0 \begin{pmatrix} 7/8 \\ \sqrt{3}/8 \\ 0 \end{pmatrix} \approx E_0 \begin{pmatrix} 0{,}87 \\ 0{,}22 \\ 0 \end{pmatrix}$$




Aufgabe 5:

<u>Geg.:</u>

$$\rho(r) = \begin{cases} \frac{\rho_1}{R_0^2} r^2 & ; 0 \le r < R_0 \\ \rho_2 R_0^4 \frac{1}{r^4} & ; R_0 \le r < \infty \end{cases}$$




<u>Ges.:</u> $\vec{E}(r, \vartheta, \varphi) = \vec{E}(r)$

$\vec{E}(r) = E(r) \cdot \vec{e}_r$ wegen Kugelsymmetrie

$$\vec{E}(r) = \frac{1}{\varepsilon_0} \vec{D}(r) : \quad \underbrace{\oint_{\partial K(r)} \vec{D}\, d\vec{f}}_{\boxed{*}} = \underbrace{\iiint_{K(r)} \rho \, dv}_{\boxed{**}}$$

$$\boxed{*} \quad \oint \vec{D}(r)\, d\vec{f} = \int_0^{2\pi}\int_0^{\pi} D(r) \underbrace{\vec{e}_r \cdot \vec{e}_r}_{=1} r^2 \sin\vartheta \, d\vartheta \, d\varphi$$

noch nicht bekannt

$$= D(r)\, r^2 \int_0^{2\pi}\underbrace{\int_0^{\pi} \sin\vartheta \, d\vartheta}_{[-\cos]_0^{\pi} = 2} d\varphi = D(r) \cdot r^2 \cdot 4\pi$$

$$= 4\pi \varepsilon_0 E(r)\, r^2 \Rightarrow E(r) = \frac{1}{4\pi\varepsilon_0 r^2} \cdot \underbrace{\boxed{**}}_{= Q \text{ in } K}$$

$$\boxed{**} = \iiint_{K(r)} \rho(r)\, dV = \int_0^{2\pi}\int_0^{\pi}\int_{r'=0}^{r} \rho(r')\, r'^2 \sin\vartheta \, dr'\, d\vartheta\, d\varphi$$

$$= 2 \cdot 2\pi \int_0^r \rho(r')\, r'^2\, dr'$$

1. Fall : $r \leq R_0$

$$= 4\pi \int_0^r \frac{\rho_1}{R_0^2} r'^2 \cdot r'^2 \, dr' = 4\pi \frac{\rho_1}{R_0^2} \left[ \frac{1}{5} r'^5 \right]_0^r$$

$$= \frac{4\pi}{5} \frac{\rho_1}{R_0^2} \cdot r^5 \qquad \boxed{**}$$

$$\boxed{*} = \boxed{**} \Rightarrow \cancel{4\pi}\, \varepsilon_0 E(r) \cancel{r^2} = \frac{\cancel{4}}{5} \cancel{\pi} \frac{\rho_1}{R_0^2} r^{\cancel{5}\,3}$$

$$E(r) = \frac{\rho_1}{R_0^2 \cdot 5 \varepsilon_0} r^3 \, , \quad r \leq R_0$$

2. Fall : $r > R_0$

$$4\pi \int_0^r \rho(r')\, r'^2 \, dr' = 4\pi \int_0^{R_0} \frac{\rho_1}{R_0^2} r'^2 \cdot r'^2 \, dr + \quad (\text{bis } R_0)$$

$$+ 4\pi \int_{R_0}^r \rho_2 R_0^4 \cdot \frac{1}{r'^{\cancel{4}\,2}} \cdot \cancel{r'^2} \, dr'$$

$$= \frac{4\pi}{5} \rho_1 R_0^3 + 4\pi \rho_2 R_0^4 \int_{R_0}^r \frac{1}{r'^2} \, dr' = \frac{4}{5}\pi \rho_1 R_0^3 +$$

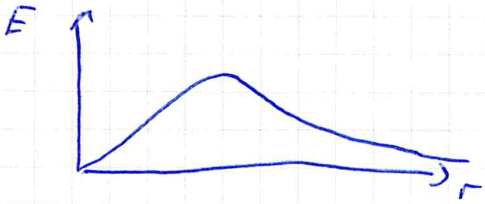
$$+ 4\pi \rho_1 R_0^4 \left[ \frac{1}{R_0} - \frac{1}{r} \right]$$

$\boxed{*} = \boxed{**} \Rightarrow 4\pi\,\varepsilon_0\, E(r)\, r^2 = 4\pi\, \varrho_0^3 \left[ \frac{\varrho_1}{5} + \varrho_2 R_0 \left( \frac{1}{R_0} - \frac{1}{r} \right) \right]$

$$E(r) = \frac{\varrho_0^3}{\varepsilon_0 r^2} \left[ \frac{\varrho_1}{5} + \varrho_2 R_0 \left( \frac{1}{R_0} - \frac{1}{r} \right) \right], \quad r > R_0$$

Aufgabe 6:

geg.: $\vec{E} = \frac{E_0}{r_0} r \cdot e^{-\frac{r}{r_0}} \cdot \vec{e}_r$, $\varepsilon = \varepsilon_0$, $\phi(\infty) = 0$




$\phi(r, \vartheta, \varphi) = \phi(r)$ wegen Kugelsymmetrie

FS: $- \int_{r_1}^{r_2} \vec{E}\, d\vec{s} = \phi(\vec{r}_2) - \phi(\vec{r}_1)$

Integration entlang $\vec{e}_r$: $- \int_{r_1}^{r_2} \frac{E_0}{r_0} r' e^{-\frac{r}{r_0}} \underbrace{\vec{e}_r \cdot \vec{e}_r}_{=1} dr$

$$= - \frac{E_0}{r_0} \int_{r_1}^{r_2} r' e^{-\frac{r'}{r_0}} dr' = \phi(r_2) - \phi(r_1)$$

$$\overset{r_2 \to \infty}{=} 0 - \phi(r_1) \overset{r_1 = r}{=} - \phi(r)$$

$\Rightarrow + \phi(r) = + \frac{E_0}{r_0} \int_r^\infty r' e^{-\frac{r'}{r_0}} dr' = \frac{E_0}{r_0} \left[ \frac{-\frac{1}{r_0} r' - 1}{\frac{1}{r_0^2}} e^{-\frac{r'}{r_0}} \right]_r^\infty$

Math FS: $\int x e^{ax} dx$

$$= \frac{ax - 1}{a^2} \cdot e^{ax} \Big| \; a = -\frac{1}{r_0}$$

$$= \frac{E_0}{r_0}\left[(-r_0 r' - r_0^2)\, e^{-\frac{r'}{r_0}}\right]_r^\infty = \frac{E_0}{r_0}\left[0-(-r_0 r - r_0^2)\, e^{-\frac{r}{r_0}}\right]$$

$$= E_0\,(r + r_0)\cdot e^{-r/r_0}$$

b) Geg.: $\vec{E}$

Ges.: $\varrho(r)$

$$\operatorname{div} \vec{D} = \varrho \iff \operatorname{div} \varepsilon_0 \vec{E} = \varrho$$

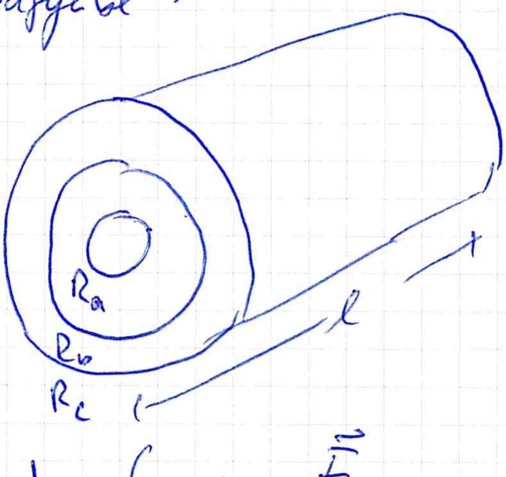
$$\Rightarrow \varrho = \operatorname{div}\left(\varepsilon_0 \frac{E_0}{r_0}\, r\, e^{-r/r_0}\, \vec{e}_r\right)$$

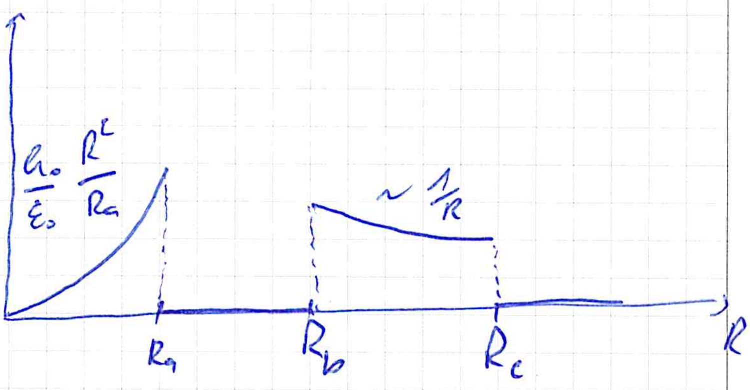
$$= \varepsilon_0 \left(\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{E_0}{r_0}\, r\, e^{-r/r_0}\right)$$

Produktregel $(uv)' = u'v + uv'$

$$= \varepsilon_0 \frac{E_0}{r_0}\, \frac{1}{\cancel{r^2}} \cdot \cancel{r^2}\, e^{-r/r_0}\left(3 - \frac{r}{r_0}\right)$$

Aufgabe 7:



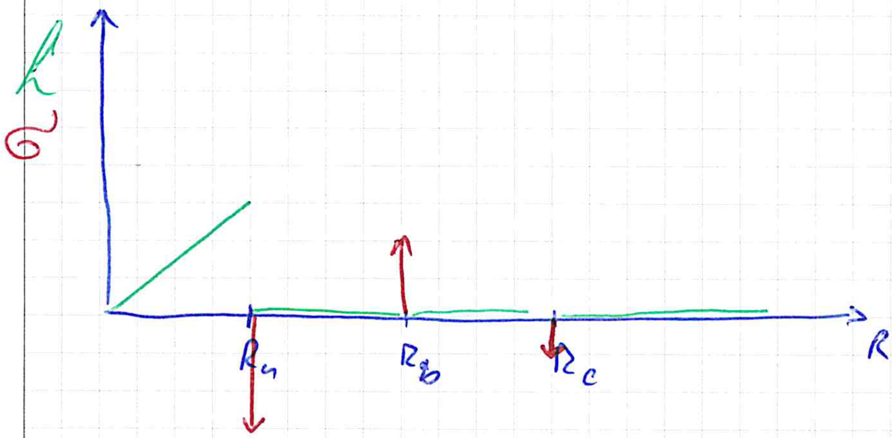




a) Geg.: $\vec{E}$

Ges.: $\varrho$, $\sigma$

Raumladungsdichte:

FS: $\operatorname{div} \vec{D} = \varrho$




Weiteres Vorgehen, siehe Musterlösung.