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MATHEMATISCHES INSTITUT I
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WS 2002/2003
07.02.2003

Lösungsvorschläge zum 15. Übungsblatt

Höhere Mathematik I für die Fachrichtungen
Elektroingenieurwesen, Physik und Geodäsie

Aufgabe 1 **a)** $V_{x-Achse} = \pi \int_0^a (x^2)^2\,dx = \frac{\pi x^5}{5}\Big|_0^a = \frac{\pi a^5}{5}$

a) $V_{y-Achse} = \pi \int_0^{a^2} (\sqrt{x})^2\,dx = \frac{\pi x^2}{2}\Big|_0^{a^2} = \frac{\pi a^4}{2}$

Aufgabe 2 $V = \pi \int_0^{\frac{\pi}{2}} \sin^2 x\,dx - \pi \int_0^{\frac{\pi}{2}} (\frac{2x}{\pi})^2\,dx = \pi(\frac{1}{2}(x - \sin x\cos x) - \frac{4x^3}{3\pi^2})\Big|_0^{\frac{\pi}{2}} = \frac{\pi^2}{12}$

$M = 2\pi \int_0^{\frac{\pi}{2}} \sin x\sqrt{1+\cos^2 x}\,dx + 2\pi \int_0^{\frac{\pi}{2}} \frac{2x}{\pi}\sqrt{1+(\frac{2}{\pi})^2}\,dx$

$= 2\pi \int_0^1 \sqrt{1+t^2}\,dt + 4\sqrt{1+(\frac{2}{\pi})^2}\frac{x^2}{2}\Big|_0^{\frac{\pi}{2}} = \pi(x\sqrt{1+x^2} + \ln(x+\sqrt{1+x^2}))\Big|_0^1 + \frac{\pi}{2}\sqrt{\pi^2+4} =$
$\pi(\sqrt{2} + \ln(1+\sqrt{2}) + \frac{\sqrt{\pi^2+4}}{2})$

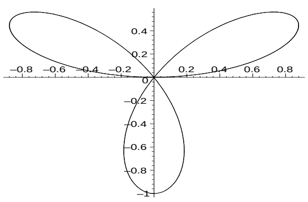
Aufgabe 3 **a)** $\dot{c}(t) = e^{-2t}\begin{pmatrix} -2\cos t - \sin t \\ -2\sin t + \cos t \end{pmatrix} \implies \|\dot{c}(t)\|_2 = \sqrt{5}e^{-2t}$

$\implies L(\mathbf{c}) = \int_0^\infty \|\dot{c}(t)\|_2\,dt = \lim_{a\to\infty} \int_0^a \sqrt{5}e^{-2t}\,dt = \lim_{a\to\infty} -\frac{\sqrt{5}e^{-2t}}{2}\Big|_0^a$
$= \lim_{a\to\infty}(-\frac{\sqrt{5}e^{-2a}}{2} + \frac{\sqrt{5}}{2}) = \frac{\sqrt{5}}{2}$

b) $\dot{c}(t) = \begin{pmatrix} \cos t - t\sin t \\ \sin t + t\cos t \\ 1 \end{pmatrix} \implies \|\dot{c}(t)\|_2 = \sqrt{2+t^2}$

$\implies L(\mathbf{c}) = \int_0^a \|\dot{c}(t)\|_2\,dt = \int_0 = \int_0^a \sqrt{2+t^2} = \sqrt{2}\int_0^a \sqrt{1+(\frac{t}{\sqrt{2}})^2}\,dt$
Substitution: $x = \frac{t}{\sqrt{2}} \implies dt = \sqrt{2}\,dx$
$\implies \sqrt{2}\int_0^a \sqrt{1+(\frac{t}{\sqrt{2}})^2}\,dt = 2\int_0^{\frac{a}{\sqrt{2}}} \sqrt{1+x^2}\,dx = (x\sqrt{1+x^2} + \ln(x+\sqrt{1+x^2}))\Big|_0^{\frac{a}{\sqrt{2}}}$
$= \frac{a}{\sqrt{2}}\sqrt{1+\frac{a^2}{2}} + \ln(\frac{a}{\sqrt{2}} + \sqrt{1+\frac{a^2}{2}})$

Aufgabe 4 **a)**



b)

$$c(\varphi) = \begin{pmatrix} x(\varphi) \\ y(\varphi) \end{pmatrix} = r(\varphi)\begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix} = \sin(3\varphi)\begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}$$

$$\implies \dot{\mathbf{c}}(\varphi) = \begin{pmatrix} 3\cos(3\varphi)\cos\varphi - \sin(3\varphi)\sin\varphi \\ 3\cos(3\varphi)\sin\varphi + \sin(3\varphi)\cos\varphi \end{pmatrix}$$

c)Polarkoordinaten: $x(\varphi)\dot{y}(\varphi) - \dot{x}(\varphi)y(\varphi)$
$= r(\varphi)\cos\varphi(\dot{r}(\varphi)\sin\varphi + r(\varphi)\cos\varphi) - r(\varphi)\sin\varphi(\dot{r}(\varphi)\cos\varphi - r(\varphi)\sin\varphi) = r^2(\varphi).$

Mit $\varphi \in [0, \pi]$ werden alle drei Blätter erzeugt. Ein Blatt ergibt sich für $\varphi \in [0, \frac{\pi}{3}]$. Die Fläche eines Blattes ergibt sich somit durch

$$F(\mathbf{c}) = \frac{1}{2} \int_0^{\frac{\pi}{3}} r^2(\varphi)\, d\varphi = \frac{1}{2} \int_0^{\frac{\pi}{3}} \sin^2(3\varphi)\, d\varphi$$

Substitution: $t = 3\varphi \Longrightarrow dt = \frac{1}{3}\, d\varphi$

$$\Longrightarrow F(\mathbf{c}) = \frac{1}{6} \int_0^{\pi} \sin^2 t\, dt = \frac{1}{12} \int_0^{\pi} (1 - \cos(2t))\, dt = \frac{1}{12}(t - \frac{1}{2}\sin(2t))\Big|_0^{\pi} = \frac{\pi}{12}$$

Aufgabe T1 *i)* $V_{Rot} = \pi \int_a^b (f(x))^2\, dx$

$V = \pi \int_{-1}^1 (3 - x^2)^2\, dx - \pi \int_{-1}^1 (x^2 + 1)^2\, dx = 16\pi \int_0^1 (1 - x^2)\, dx = 16\pi(-\frac{1}{3}x^3 + x)\Big|_0^1 = \frac{32}{3}\pi$

ii) $M_{Rot} = 2\pi \int_a^b y(x)\sqrt{1 + (y'(x))^2}\, dx$

$M = 2\pi \int_{-1}^1 (3 - x^2)\sqrt{1 + 4x^2}\, dx + 2\pi \int_{-1}^1 (x^2 + 1)\sqrt{1 + 4x^2}\, dx = 16\pi \int_0^1 \sqrt{1 + (2x)^2}\, dx$

Substitution: $t = 2x, \Longrightarrow dx = \frac{1}{2}\, dt$

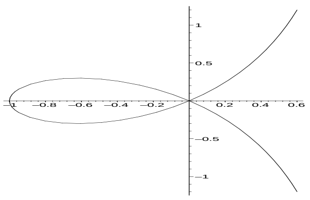
$$\Longrightarrow M = 8\pi \int_0^2 \sqrt{1 + t^2}\, dt = 4\pi\big(t\sqrt{1 + t^2} + \ln(t + \sqrt{1 + t^2})\big)\Big|_0^2 = 4\pi(2\sqrt{5} + \ln(2 + \sqrt{5}))$$

Aufgabe T2 $V = 2\pi \int_0^a \big(R + b\sqrt{1 - \frac{x^2}{a^2}}\big)^2\, dx - 2\pi \int_0^a \big(R - b\sqrt{1 - \frac{x^2}{a^2}}\big)^2\, dx = 8\pi b R \int_0^a \sqrt{1 - \frac{x^2}{a^2}}\, dx$

Substitution: $x = a\sin t, \Longrightarrow dx = a\cos t$

$$\Longrightarrow V = 8\pi abR \int_0^{\frac{\pi}{2}} \cos^2 t\, dt = 4\pi abR \int_0^{\frac{\pi}{2}} (1 + \cos 2t)\, dt = 4\pi abR(t + \frac{1}{2}\sin(2t))\Big|_0^{\frac{\pi}{2}} = 2\pi^2 abR$$

Aufgabe T3 a)



b) $F(\mathbf{c}) = \frac{1}{2} \int_a^b (x(t)\dot{y}(t) - \dot{x}(t)y(t))\, dt$

Mit $x(t) = \frac{t^2 - 1}{t^2 + 1}$ und $y(t) = tx(t) = \frac{t(t^2 - 1)}{t^2 + 1}$ gilt

$x\dot{y} - \dot{x}y = x(x + t\dot{x}) - \dot{x}(tx) = x^2 = \big(\frac{t^2 - 1}{t^2 + 1}\big)^2 = \big(1 - \frac{2}{t^2 + 1}\big)^2$

$$F(\mathbf{c}) = \frac{1}{2} \int_{-1}^1 (x(t)\dot{y}(t) - \dot{x}(t)y(t))\, dt = \frac{1}{2} \int_{-1}^1 (1 - \frac{4}{t^2 + 1} + \frac{4}{(t^2 + 1)^2})\, dt$$

$$F(\mathbf{c}) = \big(t - 4\arctan t + 2\arctan t + \frac{2t}{t^2 + 1}\big)\Big|_0^1 = 2 - \frac{\pi}{2}$$

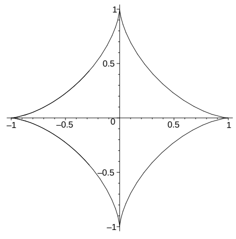
$\big(\int \frac{1}{t^2 + 1}\, dt = \int \frac{t^2 + 1}{(t^2 + 1)^2}\, dt = \int \frac{t}{2} \frac{2t}{(t^2 + 1)^2}\, dt + \int \frac{1}{(t^2 + 1)^2}\, dt$

$\int \frac{2t}{(t^2 + 1)^2}\, dt \overset{u = t^2 + 1}{=} \int \frac{du}{u^2} = -\frac{1}{u} = -\frac{1}{t^2 + 1}$

$\int \frac{t}{2} \frac{2t}{(t^2 + 1)^2} = \frac{t}{2}(-\frac{1}{t^2 + 1}) + \frac{1}{2} \int \frac{1}{t^2 + 1}\, dt$

$\Longrightarrow \int \frac{1}{(t^2 + 1)^2}\, dt = \frac{1}{2} \int \frac{1}{t^2 + 1}\, dt + \frac{t}{2(t^2 + 1)} \big)$

Aufgabe T4 a) Skizze: Asteroide mit $R = 1$



b) $L(\mathbf{c}) = \int_a^b \|\dot{\mathbf{c}}(t)\|_2 \, dt$

$$\mathbf{c}(t) = R \begin{pmatrix} \cos^3 t \\ \sin^3 t \end{pmatrix} \quad \Longrightarrow \quad \dot{\mathbf{c}} = 3R \sin t \cos t \begin{pmatrix} -\cos t \\ \sin t \end{pmatrix}$$

$L(\mathbf{c}) = \int_0^{2\pi} \|\dot{\mathbf{c}}(t)\|_2 \, dt = \int_0^{2\pi} 3R|\sin t \cos t| \, dt = 12R \int_0^{\frac{\pi}{2}} \sin t \cos t \, dt.$

Substitution: $x = \sin t$

$$\Longrightarrow L(\mathbf{c}) = 12R \int_0^1 x \, dx = 12R\big(\frac{1}{2}x^2\big)\Big|_0^1 = 6R$$

c) $F(\mathbf{c})) = \frac{1}{2} \int_0^{2\pi} (x(t)\dot{y}(t) - \dot{x}(t)y(t)) \, dt$

$x(t)\dot{y}(t) - \dot{x}(t)y(t) = 3R^2 \sin^2 t \cos^4 t + 3R^2 \sin^4 t \cos^2 t = 3R^2 \sin^2 t \cos^2 t$

$$\begin{aligned} \Longrightarrow F(\mathbf{c}) &= \frac{3R^2}{2} \int_0^{2\pi} \sin^2 t \cos^2 t \, dt = \frac{3R^2}{8} \int_0^{2\pi} \sin^2(2t) \, dt \overset{Sub:2t=x}{=} \frac{3R^2}{16} \int_0^{4\pi} \sin^2 x \, dx \\ &= \frac{3R^2}{32} \int_0^{4\pi} (1 - \cos 2x) \, dx = \frac{3R^2}{32} \big(x - \frac{1}{2} \sin 2x\big)\Big|_0^{4\pi} = \frac{3R^2\pi}{8} \end{aligned}$$