Lösung

Aufgabe 1 a) $z^4 = -i \Leftrightarrow z^4 = e^{i\frac{3}{2}\pi} \Rightarrow z = e^{i(\frac{3}{8}+\frac{2\pi k}{4})}$

$z_1 = e^{i(\frac{3}{8}\pi + 2\pi n)}$, $z_2 = e^{i(\frac{7}{8}\pi + 2\pi n)}$, $z_3 = e^{i(\frac{11}{8}\pi + 2\pi n)}$,

$z_4 = e^{i(\frac{15}{8}\pi + 2\pi n)}$.

b) IA. $n = 1$ $\quad 1 \cdot 3 = \frac{1\cdot 2\cdot 9}{6}$ ✓

IS. Annahme: für n gilt

$$\sum_{k=1}^{n} k(2k+1) = \frac{n(n+1)(4n+5)}{6}$$

Wir werden beweisen, dass

$$\sum_{k=1}^{n+1} k(2k+1) = \frac{(n+1)((n+1)+1)(4(n+1)+5)}{6}.$$

Es gilt

$$\sum_{k=1}^{n+1} k(2k+1) = \sum_{k=1}^{n} k(2k+1) + (n+1)(2n+2+1) =$$

$$= \frac{n(n+1)(4n+5)}{6} + (n+1)(2n+3) =$$

$$= \frac{(n+1)[n(4n+5) + 6(2n+3)]}{6} = \frac{n+1}{6}\cdot[4n^2 +$$

$$+ 5n + 12n + 18] = \frac{(n+1)}{6}[(n+2)(4n+9)] =$$

$$= \frac{(n+1)(n+2)(4(n+1)+5)}{6}$$ ✓

c) i) $\lim\limits_{n\to\infty}\left(\sqrt{n^4+n^{3/2}\cos n}-n^2\right) =$

$= \lim\limits_{n\to\infty} \dfrac{n^4+n^{3/2}\cos n - n^4}{\sqrt{n^4+n^{3/2}\cos n}+n^2} =$

$= \lim\limits_{n\to\infty} \dfrac{\frac{n^{3/2}}{n^2}\cos n}{2} = 0$

ii) $\lim\limits_{x\to 0} \dfrac{x-\tan x}{e^{x^2}-e^{-x^2}} \underset{=}{\overset{l'Hospital}{}}$ Fall $\frac{0}{0}$

$= \lim\limits_{x\to 0} \dfrac{1-\frac{1}{\cos^2 x}}{2x\left[e^{x^2}+e^{-x^2}\right]} = \lim\limits_{x\to 0} \dfrac{-\sin^2 x}{\cos^2 x \cdot 4x} =$

$= \lim\limits_{x\to 0} -\frac{1}{4}\sin x = 0$

Aufgabe 2 a) i) $f'(x) = 2e^{2x} - \frac{1}{1+x^2} + \cancel{2}\, 1 >$

$> 2e^{2x} > 0$. Die Funktion ist streng ~~monoton~~ monoton

$\Rightarrow$ injektiv.

$f(-\infty) = -\infty$, $f(+\infty) = +\infty \Rightarrow$

$f(\mathbb{R}) = (-\infty, \infty)$.

ii) $f^{-1}(1) = 0$, weil $f(0) = e^{2\cdot 0} - \arctan g(0) +$

$+ 0 = 1 - 0 + 0 = 1$

$$(f^{-1})'(1) = \frac{1}{f'(0)} = \frac{1}{2\cdot e^0 - \frac{1}{1+0} + 1} = \frac{1}{2}$$

b) Es gilt

$$|\ln(1+\sin x) - \ln(1+\sin y)| = |[\ln(1+\sin t)]'| \cdot$$

$$\cdot |x-y|, \quad \text{wobei } t \in [0, \tfrac{\pi}{2}].$$

$$|[\ln(1+\sin t)]'| = \left|\frac{1}{1+\sin t}\cdot \cos t\right| \leq 1$$

für $t \in [0, \frac{\pi}{2}]$.

<u>Aufgabe 3</u>

a) i) $$\int_0^{\sqrt{\pi/2}} x \sin(x^2)\,dx \overset{u=x^2}{=} \frac{1}{2}\int_0^{\pi/2} \sin u\,du =$$

$$= -\frac{1}{2}\cos u\Big|_0^{\pi/2} = \frac{1}{2}$$

ii) $$\int_0^{\pi} e^x \sin^2 x\,dx = e^x \sin^2 x\Big|_0^{\pi} - \int_0^{\pi} 2 e^x \sin x \cdot$$

$$\cdot \cos x\,dx = 0 - \int_0^{\pi} \sin 2x \cdot e^x\,dx =$$

$$= -\int_0^{\pi} e^x \sin 2x\,dx$$

Wir rechnen nun

$$\int_0^{\pi} e^x \sin 2x\,dx = e^x \sin 2x \Big|_0^{\pi} - \int_0^{\pi} e^x \cdot 2\cos 2x\,dx =$$

$$= -2e^x \cos 2x \Big|_0^{\pi} + 4\int_0^{\pi} e^x(-\sin 2x)\,dx$$

$$\Rightarrow 5\int_0^{\pi} e^x \sin 2x\,dx = -2\left[e^{\pi}\cdot 1 - e^0\cdot 1\right] =$$

$$= 2\left[1 - e^{\pi}\right] \Longrightarrow$$

$$\Rightarrow \int_0^{\pi} e^x \sin^2 x\,dx = \frac{2}{5}\left[e^{\pi} - 1\right].$$

b) Auf dem Interval $[1,4]$ gilt

$|2x - \frac{1}{x}| = 2x - \frac{1}{x}$.

$f'(x) = 2 + \frac{1}{x^2} > 0 \Rightarrow f_{min} = f(1) = 1$

$f_{max} = f(4) = 4\cdot 2 - \frac{1}{4} = 7\frac{3}{4}$.

c) Es gilt $|\sin^2(\frac{\pi}{2}x)| \leq 1$.

$$\int_0^{\infty} \frac{1}{x^2+1}\,dx = \arctan x \Big|_0^{\infty} = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

Nach Majoranten Kriterium ist das Integral $\int_0^{\infty} \frac{\sin^2(\frac{\pi}{2}x)}{1+x^2}\,dx$ konvergent.

Außerdem gilt es

$$\int_0^\infty \frac{\sin^2(\frac{\pi}{2}x)}{1+x^2}\,dx = \int_0^1 \frac{\sin^2(\frac{\pi}{2}x)}{1+x^2}\,dx + \int_1^\infty \frac{\sin^2(\frac{\pi}{2}x)}{1+x^2}\,dx \le$$

$$\le \int_0^1 \frac{1}{2}(1-\cos \pi x)\,dx + \arctan x \Big|_1^\infty =$$

$$= \frac{1}{2} - \frac{1}{2}\int_0^1 \cos \pi x\,dx + \frac{\pi}{4} = \frac{1}{2} - \frac{1}{2\pi}\sin \pi x\Big|_0^1 + \frac{\pi}{4} =$$

$$= \frac{1}{2} + \frac{\pi}{4}.$$

Aufgabe 4

a)

$$\left(\begin{array}{ccccc|c} 1 & 2 & 3 & 4 & 5 & 8 \\ 3 & 2 & 1 & 0 & -1 & -6 \\ 7 & 6 & 5 & 4 & 3 & -4 \end{array}\right) \xrightarrow[Z_3 \to Z_3 - 7Z_1]{Z_2 \to Z_2 - 3Z_1} \left(\begin{array}{ccccc|c} 1 & 2 & 3 & 4 & 5 & 8 \\ 0 & -4 & -8 & -12 & -16 & -30 \\ 0 & -8 & -16 & -24 & -32 & -60 \end{array}\right) \to$$

$$\xrightarrow[Z_2 \to -\frac{1}{4}Z_2]{Z_3 \to Z_3 - 2Z_2} \left(\begin{array}{ccccc|c} 1 & 2 & 3 & 4 & 5 & 8 \\ 0 & 1 & 2 & 3 & 4 & 7{,}5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{Z_1 \to Z_1 - 2Z_2} \left(\begin{array}{ccccc|c} 1 & 0 & -1 & -2 & -3 & -7 \\ 0 & 1 & 2 & 3 & 4 & 7{,}5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\Rightarrow \mathrm{Kern}(A) = \mathrm{lin}\left\{ \begin{pmatrix} -1 \\ 2 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 3 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 4 \\ 0 \\ 0 \\ -1 \end{pmatrix} \right\},$$

Lösung: $\vec{x} = \begin{pmatrix} -7 \\ 7{,}5 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ -1 \\ 0 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ 3 \\ 0 \\ -1 \\ 0 \end{pmatrix} + \nu \begin{pmatrix} -3 \\ 4 \\ 0 \\ 0 \\ -1 \end{pmatrix}.$

$\dim \text{Bild}(A) = 5 - \dim \text{Kern}(A) = 2$.

Die Vektoren $\begin{pmatrix}1\\3\\7\end{pmatrix}$ und $\begin{pmatrix}2\\2\\6\end{pmatrix}$ sind linear unabhängig. $\text{Bild}(A) = \text{lin}\left\{\begin{pmatrix}1\\3\\7\end{pmatrix}, \begin{pmatrix}1\\1\\3\end{pmatrix}\right\}$.

b) $\left(\sum_{k=0}^{n} \frac{3^k}{k!}\right) \xrightarrow{n\to\infty} e^3 \Longrightarrow$

$n = 2p+1 \qquad b_n \xrightarrow{n\to\infty} e^{-3\cdot 2n}$

$n = 2p \qquad b_n \xrightarrow{n\to\infty} e^{3\cdot 2n}$

$$R = \frac{1}{\limsup \sqrt[n]{\frac{b_n}{n}}} = \frac{1}{\limsup \sqrt[n]{b_n}} =$$

$$= \frac{1}{\lim\limits_{n\to\infty} \sqrt[n]{e^{6n}}} = e^{-6}.$$