Eigenschaften von sin, cos

$\forall x \in \mathbb{R}$ gilt (i) $\sin(-x) = -\sin x$, $\cos(-x) = \cos x$

(ii) $\sin(x+\pi) = -\sin x$, $\cos(x+\pi) = -\cos x$

(iii) $\sin(x+2\pi) = \sin x$, $\cos(x+2\pi) = \cos x$

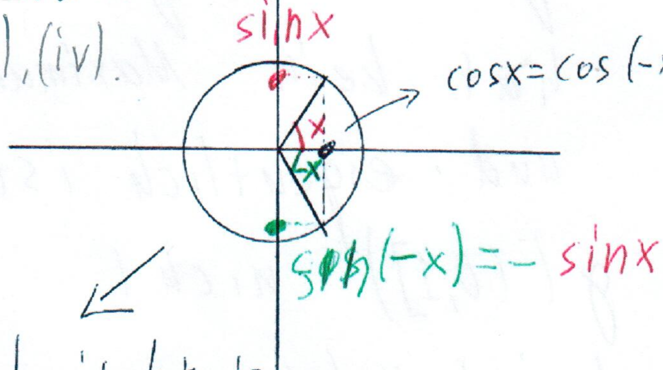
(d.h. sin, cos sind 2π-periodisch).

(iv) $\sin(\pi - x) = \sin x$, $\cos(\pi - x) = -\cos x$.

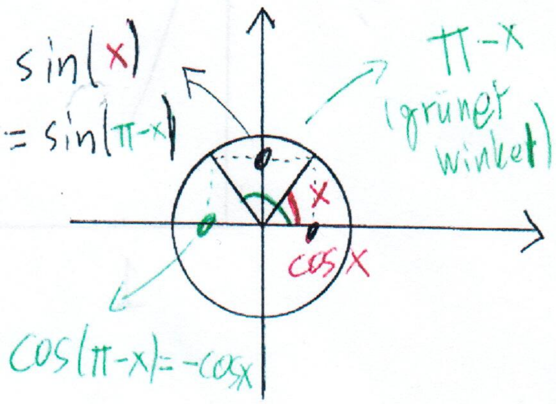
(v) $\sin\left(x+\frac{\pi}{2}\right) = \cos x$, $\cos\left(x+\frac{\pi}{2}\right) = -\sin x$.

(vi) $\forall k \in \mathbb{Z}$ gilt $\cos(k\pi) = (-1)^k$, $\sin(k\pi) = 0$.

Illustration von (i), (iv).



Einheitskreis mit Zentrum in (0,0).



$$\cos(x+y) = \cos x \cos y - \sin x \sin y,$$

Nullstellen von sin, cos: $\forall x \in \mathbb{R}$ gilt

$\cos x = 0 \iff \exists k \in \mathbb{Z}$ mit $x = (2k+1)\frac{\pi}{2}$.

$\sin x = 0 \iff \exists k \in \mathbb{Z}$ mit $x = k\pi$.

Bsp 9.7 Die Zahlen $\cos\left(\frac{317\pi}{6}\right)$, $\sin\left(\frac{317\pi}{6}\right)$ sind gleich.

(1) $\frac{\sqrt{3}}{2}, \frac{1}{2}$

(2) $\frac{1}{2}, \frac{\sqrt{3}}{2}$

(3) $-\frac{1}{2}, -\frac{\sqrt{3}}{2}$

(4) $-\frac{\sqrt{3}}{2}, \frac{1}{2}$ ✓

(5) $-\frac{1}{2}, \frac{\sqrt{3}}{2}$

(6) $\frac{1}{2}, -\frac{\sqrt{3}}{2}$

(7) $-\frac{\sqrt{3}}{2}, -\frac{1}{2}$

(8) $\frac{\sqrt{3}}{2}, -\frac{1}{2}$

(9) keine der obigen Antworten

(10) keine Ahnung.

Bsp 9.7 $\frac{317\pi}{6} = \frac{317}{12} 2\pi.$

$$\begin{array}{r|l} 317 & 12 \\ -24 & 26 \\ \hline 77 & \\ -72 & \\ \hline 5 & \end{array}$$

Also $317 = 26 \cdot 12 + 5$

$\Rightarrow \frac{317}{12} 2\pi = 26 \cdot 2\pi + \frac{5}{12} \cdot 2\pi.$

$\Rightarrow \cos\left(\frac{317\pi}{6}\right) = \cos\left(26 \cdot 2\pi + \frac{5\pi}{6}\right) = \cos\left(\frac{5\pi}{6}\right).$

Ähnlich $\sin\left(\frac{317\pi}{6}\right) = \sin\left(\frac{5\pi}{6}\right).$

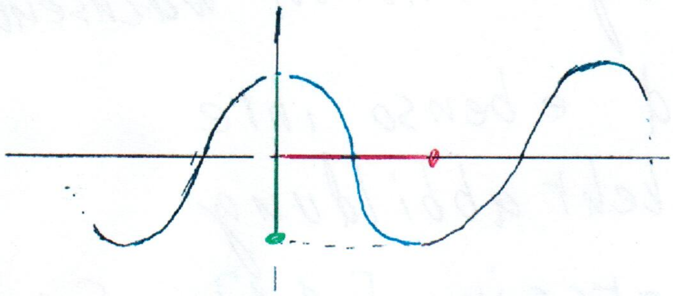
Aber $\cos\left(\frac{5\pi}{6}\right) = \cos\left(\pi - \frac{\pi}{6}\right) \overset{(iv)}{=} -\cos\frac{\pi}{6} = -\frac{\sqrt{3}}{2}.$

und $\sin\left(\frac{5\pi}{6}\right) = \sin\left(\pi - \frac{\pi}{6}\right) \overset{(v)}{=} \sin\frac{\pi}{6} = \frac{1}{2}.$

Deshalb ist (4) die richtige Antwort.

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin x$	$\frac{\sqrt{0}}{2}=0$	$\frac{\sqrt{1}}{2}=\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{4}}{2}=1$
$\cos x$	$\frac{\sqrt{4}}{2}=1$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{1}}{2}=\frac{1}{2}$	$\frac{\sqrt{0}}{2}=0$

Arcuscosinus: $\cos: \mathbb{R} \to [-1,1]$ ist nicht injektiv.

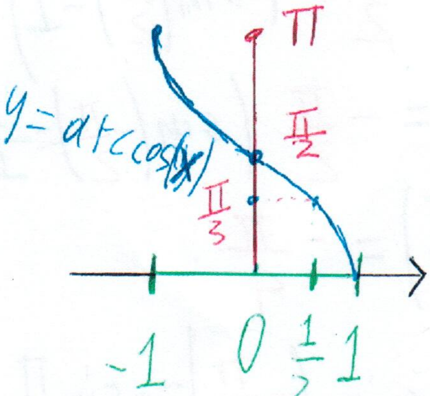


Aber

$\cos: [0,\pi] \to [-1,1]$

ist bijektiv, stetig und streng monoton fallend, und ebenso ihre

$\arccos: [-1,1] \to [0,\pi]$.



Bsp 9.8 $\arccos\left(\frac{1}{2}\right) = ?$

Welcher Winkel in $[0,\pi]$ hat cos ?

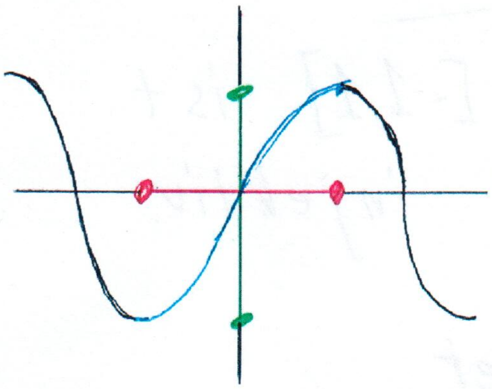
Antwort: $\arccos\left(\frac{1}{2}\right) = \frac{\pi}{3}$ da $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$ und $\frac{\pi}{3} \in [0,\pi]$.

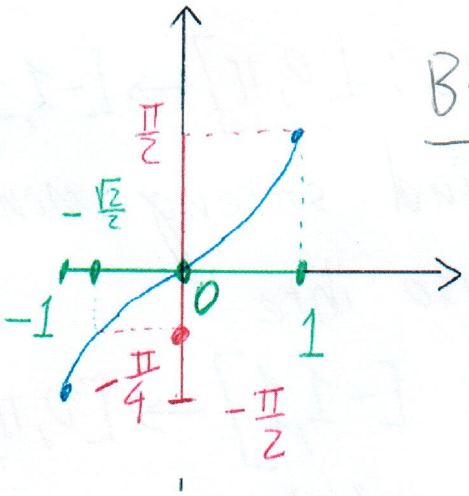
Ähnlich: $\arccos(0) = \frac{\pi}{2}$ $\left(\cos\left(\frac{\pi}{2}\right) = 0\right)$, $\arccos(1) = 0$ $(\cos(0) = 1)$, $\arccos(-1) = \pi$ $(\cos\pi = -1)$.

Achtung: $\cos\left(\frac{7\pi}{3}\right) = \frac{1}{2}$ aber

$\arccos\left(\frac{1}{2}\right) \neq \frac{7\pi}{3}$ da $\frac{7\pi}{3} \notin [0, \pi]$.

Arcussinus: Die Funktion $\sin: \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1, 1]$ ist bijektiv, stetig und streng monoton wachsend und ebenso ihre Umkehrabbildung





Bsp 9.9 $\arcsin: [-1, 1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

$\arcsin(0) = 0 \quad (\sin(0) = 0)$

$\arcsin(1) = \frac{\pi}{2} \quad \left(\sin\left(\frac{\pi}{2}\right) = 1\right)$

$\arcsin(-1) = -\frac{\pi}{2} \quad \left(\sin\left(-\frac{\pi}{2}\right) = -1\right)$.

$\arcsin\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$.

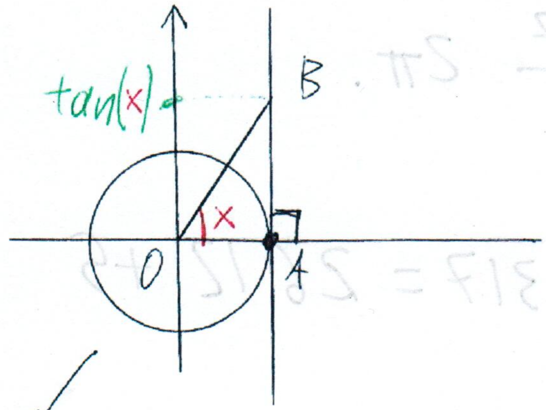
$\left(\sin\left(-\frac{\pi}{4}\right) = -\sin\frac{\pi}{4} = -\frac{\sqrt{2}}{2}\right)$.

Die Funktion $\tan: \mathbb{R} \setminus A \to \mathbb{R}$

$x \mapsto \tan x = \frac{\sin x}{\cos x}$

wobei $A = \left\{\frac{\pi}{2} + k\pi, k \in \mathbb{Z}\right\}$ (Nullstellen von cos)

heißt Tangens



$x \in \left(0, \frac{\pi}{2}\right) \Rightarrow$

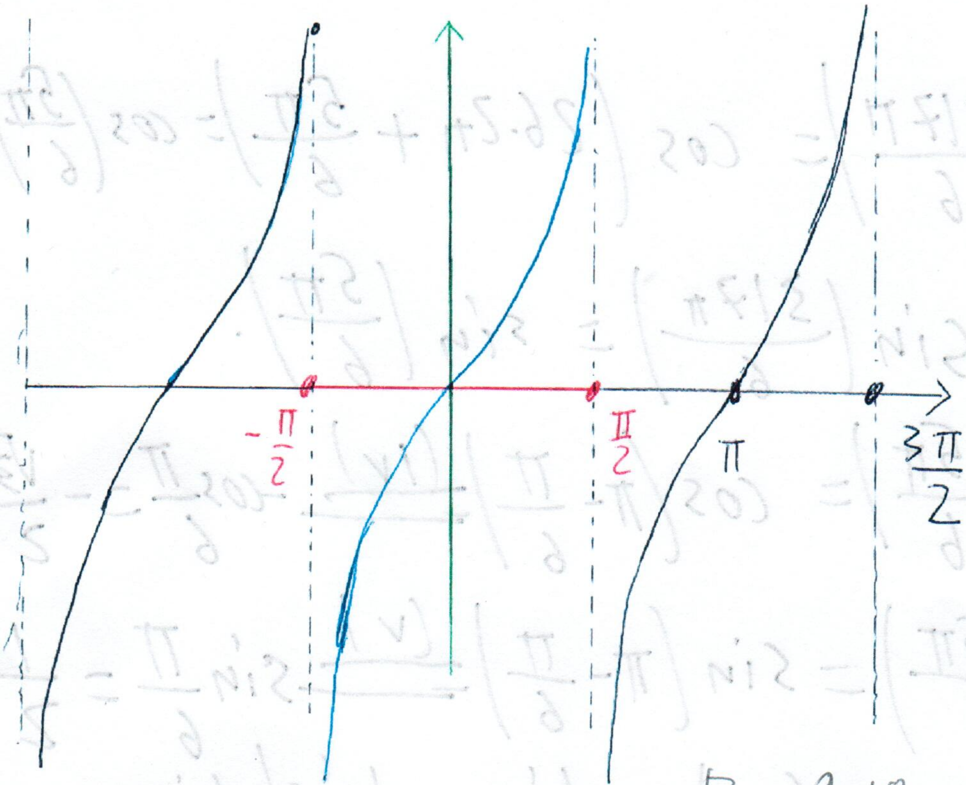
$\tan x = \frac{|AB|}{|0A|} \overset{(*)}{=} |AB|$

Einheitskreis

$|0A| = 1$ (*)

Es gilt auch $\lim\limits_{x \to \frac{\pi}{2}-} \tan x = \infty$.

$\lim\limits_{x \to -\frac{\pi}{2}+} \tan x = -\infty$.

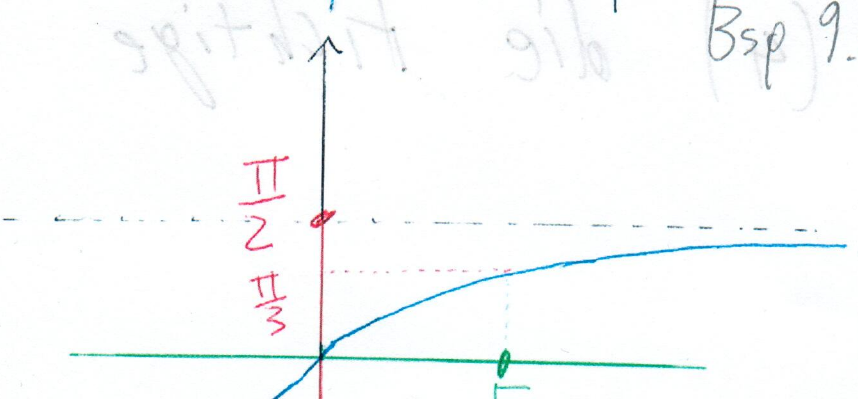


$\tan : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$ ist streng monoton wachsend stetig und bijektiv, und ebenso ihre Umkehrabbildung

Bsp 9.10 $\arctan : \mathbb{R} \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\arctan(0) = 0 \quad (\tan 0 = 0)$

$\arctan(\sqrt{3}) = \frac{\pi}{3} \quad \left(\tan \frac{\pi}{3} = \sqrt{3}\right)$



$\lim\limits_{y \to \infty} \arctan(y) = \frac{\pi}{2}$.

$\left(\lim\limits_{x \to \frac{\pi}{2}-} \tan(x) = \infty\right)$.

$\lim\limits_{y \to -\infty} \arctan(y) = -\frac{\pi}{2} \quad \left(\lim\limits_{x \to -\frac{\pi}{2}+} \tan(x) = -\infty\right)$

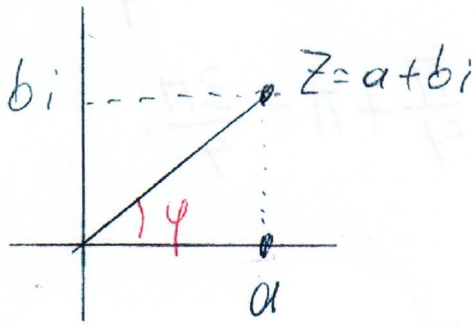
Anwendung von arctan in der Polardarstellung

Sei $z = a + bi \neq 0$, $a, b \in \mathbb{R}$. Dann $\exists\, \varphi \in (-\pi, \pi]$

mit $$z = |z|(\cos\varphi + i\sin\varphi) = |z|e^{i\varphi}$$

$$\arg z := \varphi \quad (\text{Argument von } z)$$

Es gilt $\tan\varphi = \frac{b}{a}$, aber



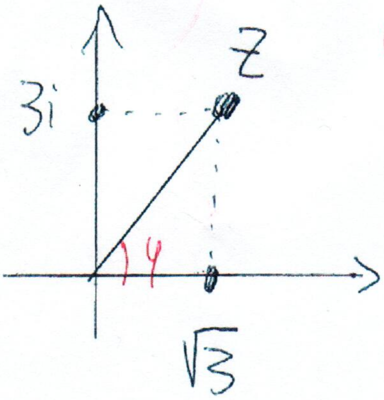
nicht immer

$$\varphi = \arctan\frac{b}{a}.$$

da $\arctan(\mathbb{R}) = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

aber φ nicht immer in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Fall 1: $a > 0$. Dann $\varphi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ also



$$\varphi = \arctan\frac{b}{a}.$$

9.11

Beispiel: $z = \sqrt{3} + 3i$.

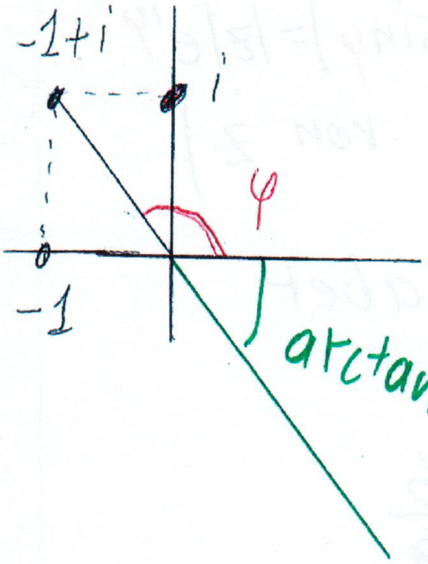
Also $\varphi = \arctan\frac{3}{\sqrt{3}} =$

$$= \arctan\sqrt{3} = \frac{\pi}{3}$$

Fall 2: $a = 0$. Dann $b \neq 0$. (Da $z \neq 0$).

$$\varphi = \begin{cases} \frac{\pi}{2}, & b > 0 \\ -\frac{\pi}{2}, & b < 0. \end{cases}$$

Fall 3: $a<0$, $b\geq 0$. Dann



$$\varphi = \arctan\frac{b}{a} + \pi. \quad (1)$$

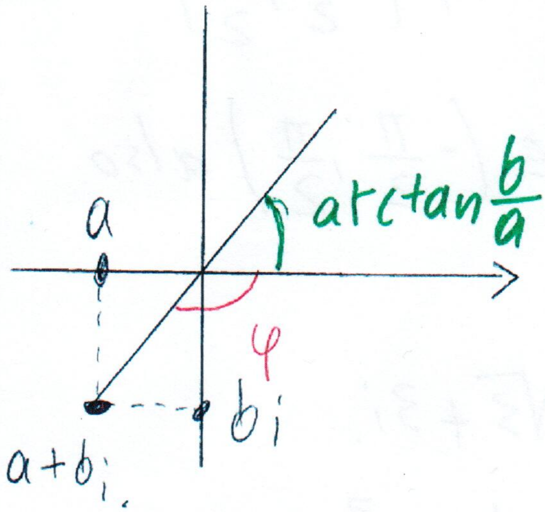
9.12.

Beispiel: $z = -1+i$.

$$\arctan\frac{1}{-1} = \arctan(-1) = -\frac{\pi}{4}.$$

Also $\varphi \overset{(1)}{=} -\frac{\pi}{4} + \pi = \frac{3\pi}{4}.$

Fall 4: $a<0$, $b<0$. Dann



$$\varphi = \arctan\frac{b}{a} - \pi.$$