2.4 Rotationen massiver Objekte

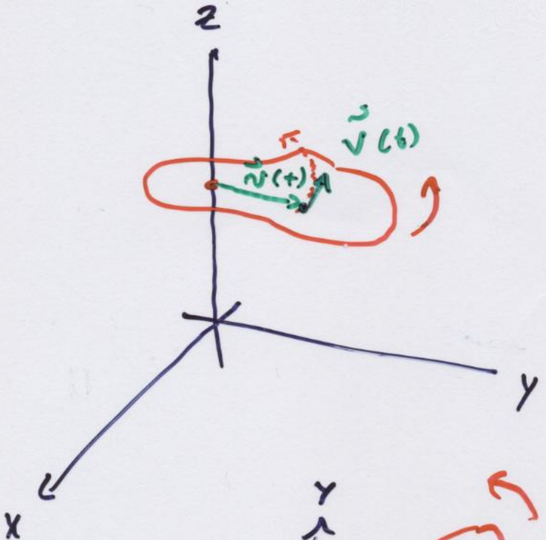
Beschreibung analog wie bei linearer Bewegung:

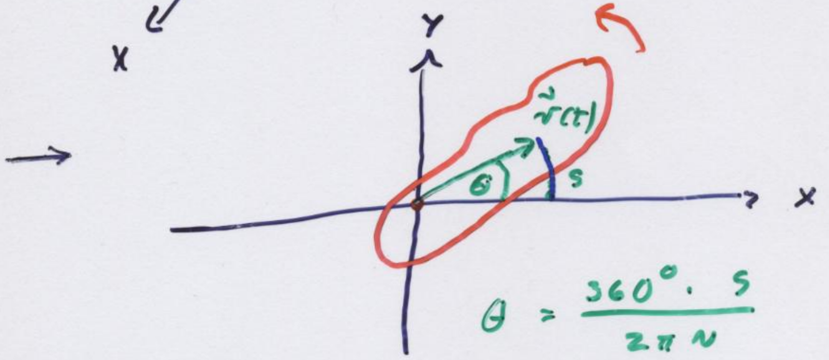
Kinematik	Rotationskinematik
3d Bewegungen	Drehbewegungen
Dynamik	Rotationsdynamik
Beschl. Bewegung	Drehung
Kräften	Drehmoment
Masse	Trägheitsmoment
Energie	Rot. energie
Impuls	Drehimpuls

+ Kombinierte Bewegung

zB Rollen

2.4.1 Rotationskinematik





$$\theta = \frac{360° \cdot s}{2\pi r}$$

$$= \frac{s}{r} \quad \text{in Radian}$$

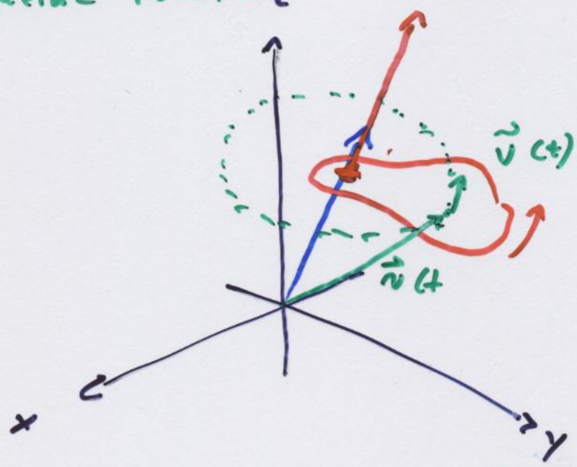
$$[1 \text{ rad} \mathrel{\hat{=}} 57{,}3°]$$

$$\omega = \frac{d\theta}{dt} = \frac{1}{r}\frac{ds}{dt} = \frac{v}{r}$$

Gleichförmige Rotation: $\theta(t) = \omega_0 t + \theta_0$

Gleichförmig beschl. Rotation: $\theta(t) = \frac{\alpha}{2} t^2 + \omega_0 t + \theta_0$

Allgemeine Rotation



$$\vec\omega(t) = \frac{d\vec\theta}{dt}$$

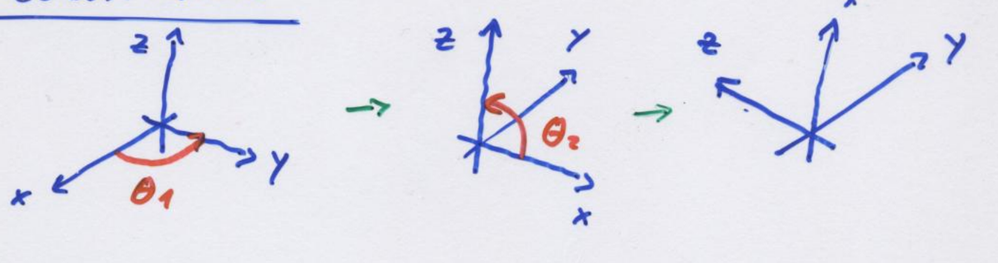
$$\perp \vec v(t)$$

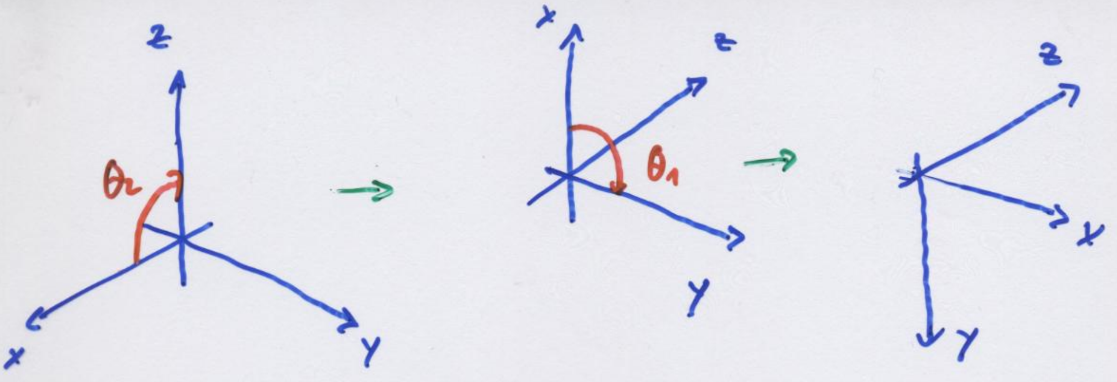
$\vec\omega$ ist ein Vektor

θ kann <u>nicht</u> ein Vektor sein,

$\theta_1 + \theta_2 \neq \theta_2 + \theta_1$ (nicht in 3d)

<u>Illustration</u> :





Also : Kommutativgesetz gilt nicht

[Nur bei kleinen Winkeln:

$$\Delta\theta_1 + \Delta\theta_2 \approx \Delta\theta_2 + \Delta\theta_1 \quad]$$

Bewegungsgleichungen (2d)

$$\vec{r}(t) = r \begin{pmatrix} \sin\theta(t) \\ \cos\theta(t) \end{pmatrix} = r \cdot \vec{e}_r(t)$$

$$\vec{v}(t) = r\,\omega(t) \cdot \vec{e}_\theta(t)$$

konstant bei Rotationen festen Körper

$$\vec{a}(t) = r \cdot \alpha(t)\,\vec{e}_\theta(t) - r\,\omega^2(t)\,\vec{e}_r(t)$$

2.4.2 Rotationsdynamik

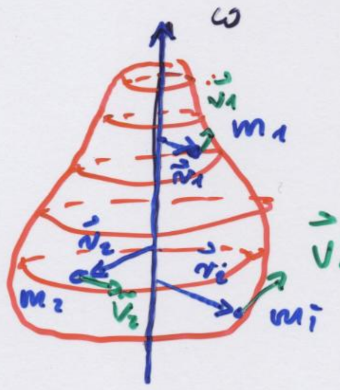
1] Trägheitsmoment

• Lineare Dynamik: Masse

$$M = \sum m_i$$

$$M = \int dm$$

• Rotationsdynamik:



$$E_K = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 + \dots \frac{1}{2} m_i v_i^2 + \dots$$

$$= \frac{1}{2} \underbrace{\sum_1^N m_i r_i^2}_{J} \omega^2$$

Trägheitsmoment

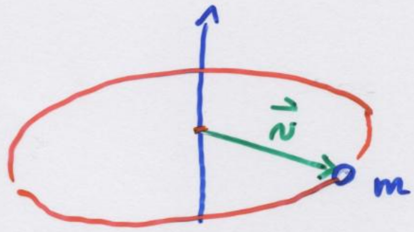
$$E_K = \frac{1}{2} J \omega^2$$

Trägheitsmoment:

Für $N \to \infty$ $\qquad J = \int r^2 \, dm$

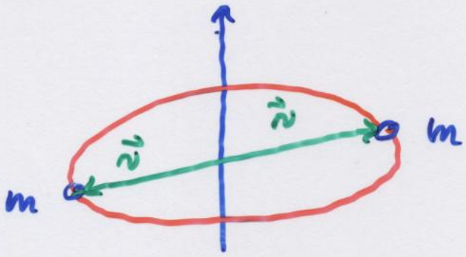
Beispiele

a)



$J = m r^2$

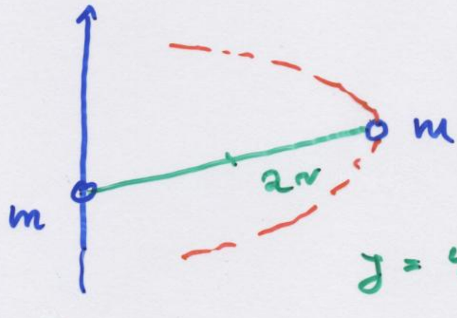
b)



$J = 2 m r^2$

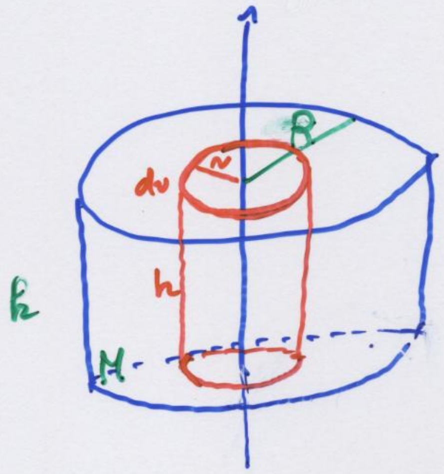
Hantel

c)



$J = 4 m r^2$

d) Zylinder



$$M = \rho \cdot V$$

$$dm = \rho \, dV$$

$$dV = h \cdot 2\pi r \, dr$$

$$J = \int r^2 \, dm$$

$$= \int_0^R r^2 \, \rho \cdot h \cdot 2\pi r \, dr$$

$$= \int_0^R 2\pi \rho h \, r^3 \, dr = \underbrace{\pi h R^2 \rho}_{M} \cdot \frac{R^2}{2}$$

$$J = \frac{1}{2} M R^2$$

e) Hohlzylinder

$$J = \int_{R_1}^{R_2} 2\pi \varrho h r^3 dr = \frac{\pi \varrho h}{2} (R_2^4 - R_1^4)$$

$$M = \int_{R_1}^{R_2} \varrho \, dV = \pi \varrho h (R_2^2 - R_1^2)$$

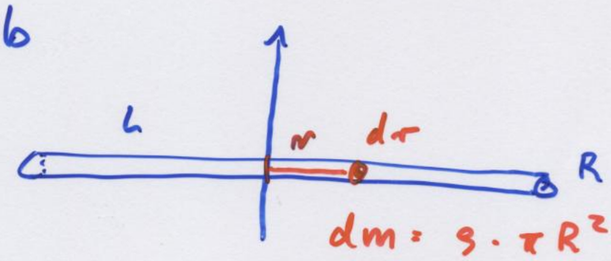
$$\Rightarrow \quad J = \frac{1}{2} M (R_1^2 + R_2^2)$$

Extremfall:

$R_1 \to 0 \qquad J = \frac{1}{2} M R_2^2$ ✓

$R_1 \to R_2 \qquad J = M R_2^2$ ✓

f) Stab



$dm = \varrho \cdot \pi R^2 \cdot dr$

$$J = \int r^2 dm$$

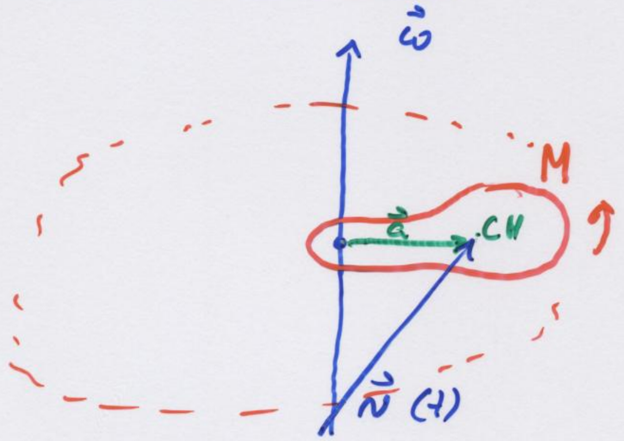
$$= \int_{-L/2}^{+L/2} r^2 \cdot \varrho \pi R^2 dr = M \frac{L^2}{12}$$

Allgemeine rotationssymmetrische Objekte:

$$J = \kappa \cdot M R^2$$

$\hookrightarrow \quad \kappa = 0 \ldots 1$

Satz von Steiner

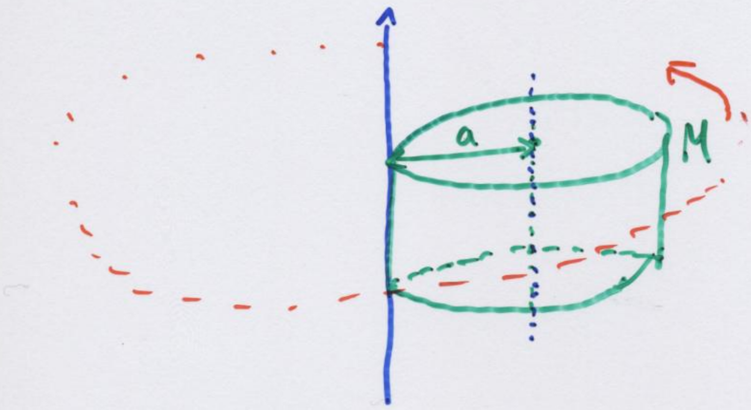


$$E_K = \frac{1}{2} J_{CM}\, \omega^2 + \frac{1}{2} M \omega^2 a^2$$

$$= \frac{1}{2} \left(J_{CM} + M a^2 \right) \omega^2$$

$$\boxed{J = J_{CM} + M a^2}$$

Beispiel : Zylinder



$$\left.\begin{aligned} J_{CM} &= \frac{1}{2} M a^2 \\ J_a &= M a^2 \end{aligned}\right\} J = \frac{3}{2} M a^2$$

$$[= 3 \cdot J_{CM}]$$

2] Drehimpuls

- Lineare Dynamik : Impuls $\vec{p} = \sum m_i \vec{v}_i$

$$= M \cdot \vec{v}_{CM}$$

$= const.$ ohne äußere Kräfte

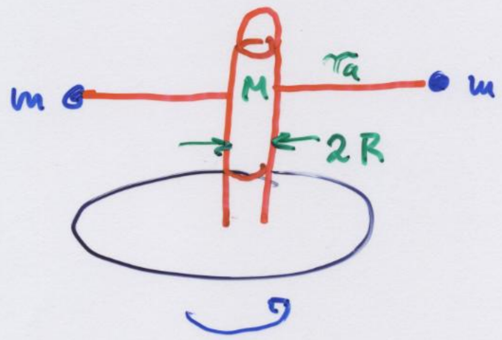
• Rotation:

Drehimpuls:

$$\vec{L} = J \cdot \vec{\omega}$$

= const. ohne äußere Drehmomente

Demonstration:



a) $r_a = 0{,}8\,m$

$M = 50\,kg$

$R = 14\,cm$

$m = 2\,kg$

Trägheitsmoment

$$J_a = \frac{1}{2} M R^2 + 2 m r_a^2$$

$= 0{,}5\ kg\,m^2$

$+ 2{,}6\ kg\,m^2$

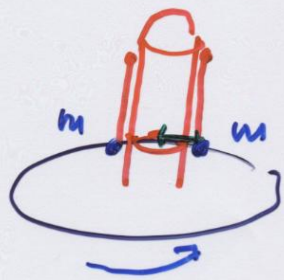
$= 3{,}1\ kg\,m^2$

$T_a = 2\,s$

Drehimpuls $L_a = J_a \cdot \omega_a$

$= 9{,}3\ kg\ m^2/s$

b)



$r_b = 0{,}2\,m$

$$J_b = \frac{1}{2} M R^2 + 2 \cdot m \cdot r_b^2$$

$$= 0{,}7\ kg\,m^2$$

Mit $L_a = J_a\,\omega_a$

$$= J_b\,\omega_b = L_b = \text{const}:$$

$$\rightarrow \omega_b = \omega_a \cdot \frac{J_a}{J_b}$$

$$\rightarrow T_b = T_a \cdot \frac{J_b}{J_a}$$

$$= 2\,s \cdot \frac{0{,}7}{3{,}1}$$

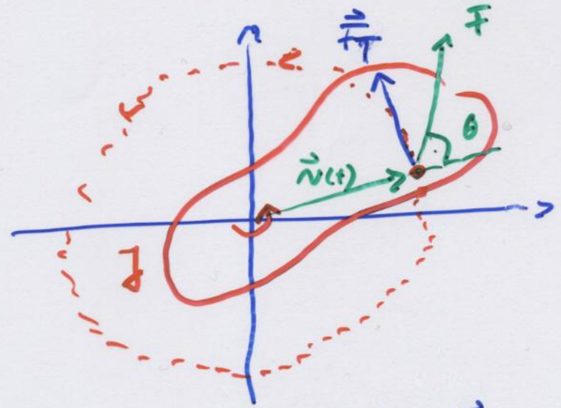
$$\underline{= 0{,}4\ s}$$

3] Drehmoment

Linear : Kraft : $\vec{F} = m \cdot \vec{a}$

Rotation : Drehmoment $\vec{M} \equiv \vec{r} \times \vec{F}$

$$\vec{M} = J \cdot \vec{\alpha}$$



Kraftkomponente $\perp \vec{r}$ führt zu beschleunigter Rotation

$$\vec{F}_T = |\vec{F}| \cdot \sin\theta$$

$$|\vec{M}| = |\vec{r} \times \vec{F}| = r \cdot F_T$$
$$= r \cdot F \cdot \sin\theta$$