Exercises Physics VI (Nuclei and Particles)

Summer Semester 2009

Exercise sheet Nr. 10 Work out until 16.07.2008

Exercise 1: $SU(2)$ **Symmetry** (Points: 5)

The two dimensional spinor representation of a spin-1/2 system can be analogically applied also to isodublet from u and d quarks. In such a representation, they are:

$$u\text{-Quark} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{(Isospin up)}$$

$$d\text{-Quark} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{(Isospin down)}$$

The transformation of a u quark to a d quark and vice versa is done using a $SU(2)$ transformation. As the isospin is conserved in the strong interaction, such transformations do not have any effect on this interaction. This defines all possible $SU(2)$ transformations in the isospin space. The transformations U have to be unitary ($U^+U = UU^+ = 1$ and $\det U = 1$) and are given by:

$$U = \exp\left(\frac{1}{2}i\theta\hat{\mathbf{n}}\cdot\tau\right) = \mathbf{1}\cos\frac{\theta}{2} + i\hat{\mathbf{n}}\cdot\tau\sin\frac{\theta}{2}$$

Here $\mathbf{1}$ is the unit 2×2 matrix , θ the rotation angle, $\hat{\mathbf{n}}$ is a three component vector of unit vectors, which give the rotation axes and τ is a vector of three 2×2 matrices:

$$\tau = \begin{pmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{pmatrix} \quad \tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Through such a $SU(2)$ transformation any isospinor $\begin{pmatrix} u \\ d \end{pmatrix}$ with the isospin-up component u and the isospin-down component d is transformed into a isospinor $\begin{pmatrix} u' \\ d' \end{pmatrix}$, which is rotated in isospin space. Mathematically it is given by:

$$\begin{pmatrix} u' \\ d' \end{pmatrix} = U\cdot\begin{pmatrix} u \\ d \end{pmatrix}$$

a) Estimate the matrices U_1, U_2 and U_3 for the rotation around the three axes in isospin space ($n_1 = (1,0,0)$, $n_2 = (0,1,0)$, $n_3 = (0,0,1)$) and apply them to the isospinor $\begin{pmatrix} u \\ d \end{pmatrix}$.

b) Show that the transformation matrices U can be used also for antiquarks once the corresponding isospinor is defined as $\begin{pmatrix} \bar{d} \\ -\bar{u} \end{pmatrix}$[1]:

$$\begin{pmatrix} \bar{d}' \\ -\bar{u}' \end{pmatrix} = U \cdot \begin{pmatrix} \bar{d} \\ -\bar{u} \end{pmatrix}$$

Apply charge conjugation ($u \rightarrow \bar{u}$, $d \rightarrow \bar{d}$ and complex conjugation) to the rotated spinors $\begin{pmatrix} u \\ d \end{pmatrix}$ from the previous part. Relate those spinors to the spinors that you obtain from the spinors $\begin{pmatrix} \bar{d} \\ -\bar{u} \end{pmatrix}$ rotated by the U_1, U_2 and U_3 respectively.

c) Show that the ω meson ($\frac{1}{\sqrt{2}}(u\bar{u}+d\bar{d})$) is an isospin singlet as it is invariant under rotations around all three rotation axes. What changes in the argumentation if we talk about an η meson instead of an ω meson? What does the isospin transformation imply for a ϕ meson?

d) Rotate the π^+ meson ($u\bar{d}$) in isospin space by 90° and 180°, once around the first axis and a second time around the second axis ($U_1(90°)\pi^+$, $U_1(180°)\pi^+$, $U_2(90°)\pi^+$, $U_2(180°)\pi^+$). What states do you obtain?

e) The creation and annihilation operators, which raise or lower the I_3 component of the state, are given by

$$I_\pm = \frac{1}{2}(\tau_1 \pm i\tau_2)$$

Write both operators in matrix form and determine how they affect u, d, $\bar{u}$ and $\bar{d}$ quarks. With the help of creation and annihilation operators, show that the ω meson is an isospin singlet and the pions form an isospin triplet. (Note that the operators act on multiquark state as $I_\pm q\bar{q} = (I_\pm q)\bar{q} + q(I_\pm \bar{q})$). How do the isospin singlet and triplet look like for a system composed of a quark and an antiquark ($2 \times \bar{2}$-group) and for a system of two quarks (2×2-group)?

[1] The charge conjugation alter the additive quantum numbers to values with the opposite sign, e.g. $I_3(\bar{u}) = -1/2$ (Isospin down) and $I_3(\bar{d}) = +1/2$ (Isospin up)

<u>Exercise 2</u>: Deutron wave function (Points: 3)

The potential of the proton and neutron in a deuteron can be approximated by a centrally symmetric box potential of depth $-V_0$ and radius $r_0 \approx 1.4$ fm, described as

$$V(r) = \begin{cases} -V_0 \text{ for } r < r_0 \\ 0 \text{ for } r > r_0 \end{cases}$$

Consider the radial Schrödinger equation for the ground state ($l = 0$) of deutrons:

$$\frac{d^2u}{dr^2} + \frac{2m}{\hbar^2}(E - V)u = 0 \qquad \psi(\vec{r}) = \frac{u(r)}{r} Y_0^0$$

What mass m should be used in the solution? Solve the equation for $r < r_0$ and $r > r_0$ under boundary conditions $u(r = 0) = 0$ and $u(r \to \infty) = 0$. In the solution, use the continuity condition of the wave function. Estimate the depth of the potential using the approximation that the binding energy $B = 2.25$ MeV is much smaller that V_0. Is this approximation valid?

How large is the probability that the nucleon is within a radius $r < r_0$?