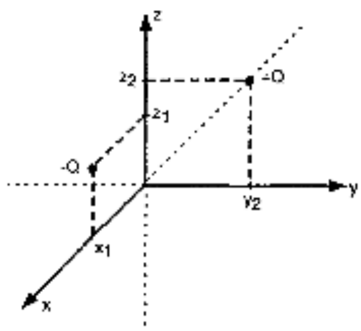
Examination
"Electromagnetics and Numerical Calculation of Fields"

2. March 2004

1) Write down the names and units of E, D, P, J, B, H, M (in terms of m, kg, s, A and V).
(7 Points)

2) Given is the electric flux density:
$D = (2(-x+y)e_x + (x+3y)e_y)$ (As/m^2)
Find the corresponding charge density ρ. *(3 Points)*

3) Consider the electrostatic point charge distribution of the figure



$-Q$ at $(x_1, 0, z_1)$
$+Q$ at $(0, y_2, z_2)$

The x-y-plane is perfectly electrically conducting and grounded.
In the upper half space $\varepsilon_r = 1$.

a) Find the electrostatic potential $\phi(x, y, z)$.
b) Determine the electric surface charge density $\eta(x, y)$ in the x, y-plane at $z = 0$.
(10 Points)

4) Given is a cylindrical metal tube with no charges inside.
a) What type of differential equation has to be solved for the electric potential Φ inside the tube?
b) What type of functions solve the equation (choose the best coordinate system)?
(8 Points)

5) Given is the electric current density *(8 Points)*

$$\vec{J}(\vec{r}) = \frac{I_0}{\pi R_0^2}\vec{e}_z \quad \text{for} \quad 0 \le R \le R_0$$
$$\vec{J}(\vec{r}) = 0 \qquad \text{else}$$

a) Determine the magnetic field strength $\vec{H}$ for $0 \le R \le \infty$
b) Sketch $\vec{H}$ for $0 \le R \le \infty$

6) a) Write down the sinusoidal plane wave solution of the wave equation for E and H (two types of solutions, no losses).
b) What is the difference between the two types of solutions?
c) Derive the frequency and the wavelength from the solution.
d) What is the phase velocity (meaning and value)?
e) Derive the Poynting Vector S from these solutions. Explain the direction of S.
(12 Points)

7) a) Write down the general wave equation for conducting, dielectric and paramagnetic materials.
b) What is the prerequisite so that we can reduce the equation to the case of good conductors?
c) How does the equation look like? What type of equation is it?
d) When can we reduce the equation to the lossless case?
(8 Points)

8) Give a sketch of the "computing molecule" (FDM) for the
a) wave equation
b) diffusion equation
Caption the axes.
Mark the values that are used to calculate the next value.
How does it all start?
What do you do at the boundaries? *(8 Points)*

9 a) Give a sketch of Yee's lattice (FDTD).
b) What is the ingenious idea of Yee's lattice?
c) How does the FDTD method proceed, step by step?
(8 Points)

10) What type of boundary conditions are implicitly used in FEM if nothing is specified about the boundaries? *(4 Points)*

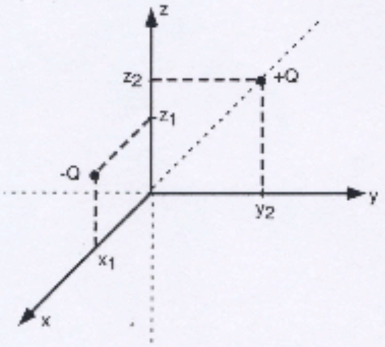
1) Write down the names and units of E, D, P, J, B, H, M (in terms of m, kg, s, A and V). 7p

Answer:
E: Electric Field Strength in V/m
D: Electric Flux Density in As/m^2
P: Electric Polarization in As/m^2
J: (free) Electric Current Density in A/m^2
B: Magnetic Flux Density or Magnetic Induction in Tesla = Vs/m^2
H: Magnetic Field Strength in A/m
M: Magnetization in A/m

2) Given is the electric flux density: 3p
$D = (2(-x+y)e_x + (x+3y)e_y)\ (As/m^2)$
Find the corresponding charge density ρ.

Answer:
$$div\vec{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = -2 + 3 = 1$$
$$\rho = 1As/m^3$$

3) Consider the electrostatic point charge distribution of the figure 10p



$-Q$ at$(x_1, 0, z_1)$
$+Q$ at$(0, y_2, z_2)$

The x-y-plane is perfectly electrically conducting and grounded. In the upper half space $\epsilon_r = 1$.

a) Find the electrostatic potential φ(x, y, z).

b) determine the electric surface charge density η(x, y) the x, y-plane at z = 0.

Answer:
a) We use the method of mirror charges:

$$Q_1 = -Q: \ \vec{r}_1 = x_1\vec{e}_x + z_1\vec{e}_z$$
$$Q_2 = +Q: \ \vec{r}_2 = y_2\vec{e}_y + z_2\vec{e}_z$$
(charges)

$$Q_3 = +Q: \ \vec{r}_3 = x_1\vec{e}_x - z_1\vec{e}_z$$
$$Q_4 = -Q: \ \vec{r}_4 = y_2\vec{e}_y - z_2\vec{e}_z$$
(images)

$$\phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{n=1}^{4}\frac{Q_n}{|\vec{r}-\vec{r}_n|} \qquad z \geq 0$$
$$\phi(\vec{r}) = 0 \qquad z < 0$$

b)

We find the electric surface charge density as the normal component of the electric flux density at the xy plane:

$$\eta(r)\big|_{z=0} = \vec{n}\vec{D}(\vec{r},t)\big|_{z=0}$$
($\vec{n}$ = surface normal unit vectors)

$$\vec{D}(\vec{r}) = \varepsilon_0\vec{E}(\vec{r}) = \frac{1}{4\pi}\sum_{n=1}^{4}Q_n\frac{\vec{r}-\vec{r}_n}{|\vec{r}-\vec{r}_n|^3} \qquad z \geq 0$$
$$\vec{D}(\vec{r}) = \varepsilon_0\vec{E}(\vec{r}) = 0 \qquad z < 0$$
$$\eta(r)\big|_{z=0} = \vec{e}_z\frac{1}{4\pi}\sum_{n=1}^{4}Q_n\frac{\vec{r}-\vec{r}_n}{|\vec{r}-\vec{r}_n|^3}\Bigg|_{z=0}$$
$$\eta(x,y) = \vec{e}_z\cdot\vec{D}(x,y) = \frac{Q}{2\pi}\left[\frac{z_1}{\left[(x-x_1)^2+y^2+z_1^2\right]^{3/2}} - \frac{z_2}{\left[x^2+(y-y_2)^2+z_2^2\right]^{3/2}}\right]$$

4) Given is a cylindrical metal tube with no charges inside. 8P

a) What type of differential equation has to be solved for the electric potential Φ inside the tube?
b) What type of functions solve the equation (choose the best coordinatee system)?

Answer:

a) A Laplace equation has to be solved:
$\Delta\phi = 0$

b) The best choice for the coordinate system are cylindrical coordinates. The type of solutions are:
$\sinh(\gamma z)/\cosh(\gamma z)$, $\sin(m\phi)/\cos(m\phi)$, $J_m(\gamma R)/N_m(\gamma R)$

J_m and N_m: Bessel/Neumann functions

5) Given is the electric current density 8 P

$$\vec{J}(\vec{r}) = \frac{I_0}{\pi R_0^2}\vec{e}_z \quad \text{for} \quad 0 \le R \le R_0$$

$$\vec{J}(\vec{r}) = 0 \qquad \text{else}$$

a) Determine the magnetic field strength $\vec{H}$ for $0 \le R \le \infty$

b) Sketch $\vec{H}$ for $0 \le R \le \infty$

Answer:

a)

$$\int_{\partial s} \vec{H} d\vec{R} = \iint_{s} \vec{J} d\vec{s}$$

Cylindrical symmetry:

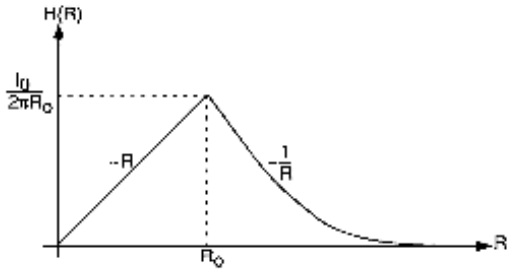
$$\vec{H} = H(R)\vec{e}_\varphi$$

$$\int_{\partial s} \vec{H} d\vec{R} = \int_0^{2\pi} H(R)\vec{e}_\varphi \cdot R\vec{e}_\varphi \, d\varphi = H(R) 2\pi R$$

$$\iint_{s} \vec{J} d\vec{s} = \begin{cases} I_0\left(\frac{R}{R_0}\right)^2 & \text{for } 0 \le R \le R_0 \\ I_0 & \text{for } > R_0 \end{cases}$$

$$\vec{H}(\vec{R}) = \begin{cases} \frac{I_0}{2\pi R_0^2} R\vec{e}_\varphi & \text{for } 0 \le R \le R_0 \\ \frac{I_0 \vec{e}_\varphi}{2\pi R} & \text{for } > R_0 \end{cases}$$

b)



6)

a) Write down the sinusoidal plane wave solution of the wave equation for E and H (two types of solutions, no losses). 12 P

b) What is the diffference between the two types of solutions?

c) Derive the frequency and the wavelength from the solution.
d) What is the phase velocity (meaning and value)?
e) Derive the Poynting Vector S from these solutions. Explain the direction of S.
Answer:

$E_y = E_0 \cdot e^{j(\omega t - kx)}$ $\qquad$ $E_y = E_0 \cdot e^{j(\omega t + kx)}$

$H_z = H_0 \cdot e^{j(\omega t - kx)}$ $\qquad$ $H_z = H_0 \cdot e^{j(\omega t + kx)}$

$S_x = E_0 H_0 e^{j(2\omega t - 2kx)}$ $\qquad$ $S_x = -E_0 H_0 e^{j(2\omega t - 2kx)}$

$\omega = 2\pi f \quad k = \frac{2\pi}{\lambda} \quad c_{phase} = \frac{\omega}{k}$

b) The solution for the -kx case moves to the right, the solution for the +kx case moves to the left.
c) The phase velocity is the velocity of the wave-maximum of a continuous sinusoidal wave.
d) The Poynting vector (direction of energy flow) shows into the positive x-direction for the -kx case and into the negative x-direction for the +kx case.

7) a) Write down the general wave equation for conducting, dielectric and paramagnetic materials.
b) What is the prerequisite so that we can reduce the equation to the case of good conductors?
c) How does the equation look like? What type of equation is it?
d) When can we reduce the equation to the lossless case? 9 P

Answer:
a)

$$\Delta\vec{H} = \kappa\mu\frac{\partial\vec{H}}{\partial t} + \varepsilon\mu\frac{\partial^2\vec{H}}{\partial t^2}$$

$$\Delta\vec{E} = \kappa\mu\frac{\partial\vec{E}}{\partial t} + \varepsilon\mu\frac{\partial^2\vec{E}}{\partial t^2}$$

b)

$\frac{\kappa}{\omega\varepsilon} >> 1 \rightarrow$ good conductors

c)

$\Delta\vec{H} = \kappa\mu\frac{\partial\vec{H}}{\partial t}$ diffusion equation

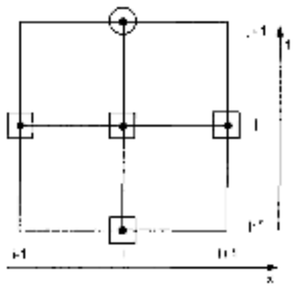
d)

$\frac{\kappa}{\omega\varepsilon} << 1 \rightarrow$ lossless media $\quad \Delta\vec{H} = \varepsilon\mu\frac{\partial^2\vec{H}}{\partial t^2}$ lossless wave equation

8) Give a sketch of the "computing molecule" (FDM) for the
a) wave equation
b) diffusion equation

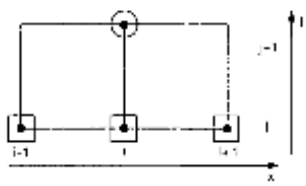
Caption the axes. 8 P
Mark the values that are used to calculate the next value.
How does it all start?
What do you do at the boundaries?

Answer:
a)



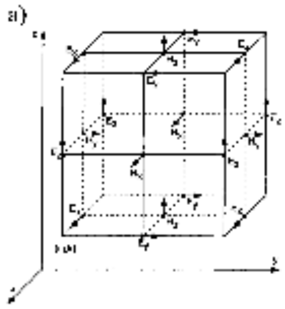
The initial values in the first two lines of time must be given.
At the borders the type of boundary condition must be given: e.g. absorbing or reflecting. This way the "missing values" for the computing molecule can be calculated.

b)



The initial values in the first line must be given.
At the borders the type of boundary condition must be given: e.g. absorbing or reflecting. This way the "missing values" for the computing molecule can be calculated.

9) a) Give a sketch of Yee's lattice (FDTD).
b) What is the ingenious idea of Yee's lattice?
c) How does the FDTD method proceed, step by step?
Answer:
a)



b) The "interlaced lattices" for E and H allow for an easy translation of Maxwell's equations into an algorithm. The interlaced timesteps allow for a successive calculation of H and E from the past.
c) With "interlaced timesteps" we can calculate H(t+1/2) from and H(t-1/2) and E(t) and we can calculate E(t+1) from E(t) and H(t+1/2). Then we replace t+1 by t and start all over again.

10) What type of boundary conditions is implicitly used in FEM if nothing is specified about the boundaries?

Answer:
Neumann boundary conditions, the normal derivative is zero,

$$\frac{\partial \Phi}{\partial n} = 0$$