Examination
"Electromagnetics and Numerical Calculation of Fields"

March 8, 2004

1) Write down the material equations, that combine E with D, B with H and J with E. Start with the most general expression, then simplify for linear and isotropic materials. Give the units at all fields: E, D, B, H, J. *(10 points)*

<u>Answer:</u>

General case	Linear isotropic material
$D = \varepsilon_0 E + P$	$D = \varepsilon_r \varepsilon_0 E$
$B = \mu_0 (H + M)$	$B = \mu_r \mu_0 H$
$J = \underline{\kappa} E + J^{(i)}$	$J = \kappa E + J^{(i)}$
$\underline{\kappa}$: matrix	

Units:

E: $\frac{V}{m}$

D: $\frac{As}{m^2}$

B: Tesla $= \frac{Vs}{m^2}$

H: $\frac{A}{m}$

J: $\frac{A}{m^2}$

2) How can you find the electric potential Φ of an arbitrary charge distribution $\rho(x,y,z)$?
What type of equation has to be solved?
Give an explicit solution that can be translated to a numerical algorithm.
(3 points)

Answer:

Poisson Equation: $\Delta\phi = -\frac{\rho}{\varepsilon}$

The explicit solution is given by the Coulomb potential:

The Coulomb integral $\phi(\vec{r}) = \frac{1}{4\pi\varepsilon}\iiint \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}dv'$ can be translated into a numerical algorithm.

3) What is the definition of the Green's Function of a Dirichlet Problem?
How can we find the electric potential if we know the Green's Function?
Green's 1st law is

$$\int(\phi\Delta\psi - \psi\Delta\phi)dv = \oint\left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right)d\vec{f}$$

(4 points)

Answer:

The Green's Function of a Dirichlet Problem $G_D(\vec{r},\vec{r}_0)$ solves the following boundary problem:

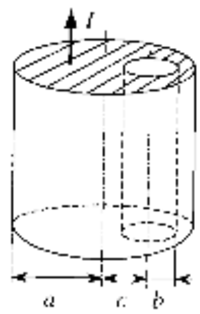
$\Delta G_D(\vec{r},\vec{r}_0) = -\delta(\vec{r}-\vec{r}_0)$ and $G_D = 0$ on the boundary.

$$\Rightarrow \quad \Phi(\vec{r}_0) = \int G_D(\vec{r},\vec{r}_0)\frac{\rho}{\varepsilon}dv' - \oint \phi\frac{\partial G_D}{\partial n}d\vec{f}$$

4) An infinitely long straight conductor is bounded externally by a circular cylinder of radius a and internally by a circular cylinder of radius *b*. The distance between centers is c, with *a>b+c*. The internal cylinder is hollow. The conductor carries a steady current I uniformly distributed over the cross-section. Show that the field within the cylindrical hole is

$$H = \frac{c \cdot I}{2\pi\left(a^2 - b^2\right)}$$

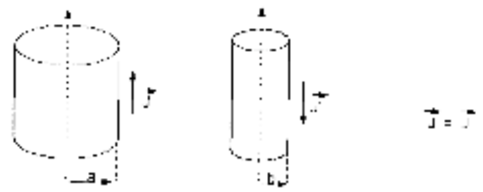
and is directed transverse to the diameter joining the two centers.



(12 points)

Answer:

The problem domain can be decomposed into two conducting cylindrical domains with opposite current densities of equal magnitudes $\vec{J}$ and $\vec{J}'$ as shown below.



A superposition of these two domains is equivalent to the original problem. The magnitude of the current density is computed as

$$J = \frac{I}{\pi\left(a^2 - b^2\right)}$$

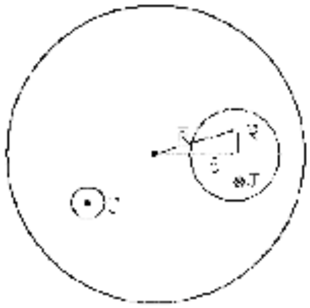
According to Maxwell's equation $\mathrm{rot}\bar{H} = \bar{J}$, magnetic field inside a conducting cylinder is given by

$$H = \frac{1}{2} JR$$

The direction of $\bar{H}$ is taken into account by a vector product:

$$\bar{H} = \frac{1}{2}\bar{J} \times \bar{R}$$

Now, consider the superposition of two domains and compute the magnetic field $\vec{H}(\vec{R}) = \vec{H}_a(\vec{R}) + \vec{H}_b(\vec{R}')$



$$\vec{H}_a(\vec{R}) = \frac{1}{2}\vec{J} \times \vec{R}$$

$$\vec{H}_b(\vec{R}') = \frac{1}{2}\vec{J}' \times \vec{R}'$$

$$\vec{J}' = -\vec{J} \quad \vec{R}' = \vec{R} - \vec{c}$$

$$\Rightarrow \vec{H}(\vec{R}) = \frac{1}{2}\vec{J} \times \vec{c} - \frac{\vec{I} \times \vec{c}}{2\pi\left(a^2 - b^2\right)}$$

The resulting expression gives both magnitude and direction of the unknown magnetic field $\vec{H}$.

5) How can you find the total magnetic field energy of a system of N current carrying wires (electric currents $I_1, I_2, \dots I_N$)? *(3 points)*

Answer:

$$W_m = \frac{1}{2}\sum\sum L_{ik} I_i I_k = \frac{1}{2}(I_1 \cdots I_N)\begin{pmatrix} L_{11} & & L_{1N} \\ & \ddots & \\ L_{N1} & & L_{NN} \end{pmatrix}\begin{pmatrix} I_1 \\ \vdots \\ I_N \end{pmatrix}$$

L_{ii} = self-inductance
L_{ij} = mutual inductance

6) a) How can you find the mutual inductance M between two loops? *(2 points)*

Answer:

$$\Phi_{m12} = I_1 \cdot L_{12} = I_1 \cdot M \rightarrow M = \frac{\Phi_{m12}}{I_1}$$

or

$$\Phi_{m21} = I_2 \cdot L_{21} = I_2 \cdot M \rightarrow M = \frac{\Phi_{m21}}{I_2}$$

$$L_{12} = L_{21} = M$$

Φ_{m12}: flux created in loop 1 and measured in loop 2
Φ_{m21}: flux created in loop 2 and measured in loop 1
I_1: current in loop 1
I_2: current in loop 2

b) M will be a function of vector r_{12} between the centers of the two loops $M=M(r_{12})$. How can you calculate the force between the two loops if $M(r_{12})$ is given? *(2 points)*

Answer:

Magnetic field energy of the system:

$$W_m = \frac{1}{2}L_{11}\ I_1^2 + M(r_{12})I_1I_2 + \frac{1}{2}L_{22}\ I_2^2$$

$$F = -\text{grad}W_m = -I_1I_2\text{grad}M(r_{12})$$

7) Given is a harmonic plane wave, where E is polarized in xy-plane and the direction of propagation is $\vec{e}_x$.

a) Write down the non-zero components of E und H. *(2 points)*

b) Give the relation between E and H in case of the free space, compute its numerical value and give its dimension. *(3 points)*

c) Derive the Poynting vector. Hint: Take real parts of E and H. *(1 point)*

d) Using the result of b) and c), compute the mean energy flow $\overline{S}$ in VA/m² as a function of E_0, where E_0 is the amplitude of the electric field. Show that

$$\overline{S} = 1{,}327 \cdot 10^{-3} E_0^2$$

$$\left(\mu_0 = 4\pi \cdot 10^{-7}\frac{Vs}{Am} \qquad \varepsilon_0 = 8{,}854 \cdot 10^{-12}\frac{As}{Vm}\right)$$ *(3 points)*

Answer:

a)

$$E_y = E_0 \cdot e^{j(\omega t - kx)}$$

$$H_z = H_0 \cdot e^{j(\omega t - kx)}$$

b)

$$\frac{E_y}{H_z} = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 376{,}7 \ \text{Ohms}$$

c)

$$S_x = E_0H_0\cos^2(\omega t + kx)$$

d)

Mean value of $\cos^2$:

$$\frac{1}{T}\int_0^T \cos^2(\omega t + kx) = \frac{1}{2}; \quad T = \frac{2\pi}{\omega}$$

$$\overline{S} = \overline{S}_x = \frac{1}{2}E_0H_0 = \frac{1}{2 \cdot 376{,}7}E_0^2 = 1{,}327 \cdot 10^{-3}E_0^2\frac{VA}{m^2}$$

8) a) What type of differential equation for $\vec{E}(\vec{r},t)$ and $\vec{H}(\vec{r},t)$ is valid inside a waveguide? *(2 points)*

b) What is the recommended approach to translate this equation into a differential equation for $\underline{\vec{E}}_0(x,y)$?

(cartesian coordinates, propagation in z-direction) *(3 points)*

c) What is the recommended approach to solve this differential equation for $\underline{\vec{E}}_0(x,y)$? *(3 points)*

d) What is the "mode" of a propagating wave in a waveguide? *(2 points)*

<u>Answer:</u>

a) The classical wave equation for time harmonic case is valid:

$$\Delta\vec{E} + \omega^2\mu\varepsilon\vec{E} = 0$$

$$\Delta\vec{H} + \omega^2\mu\varepsilon\vec{H} = 0$$

b)

$$\underline{\vec{E}} = \underline{\vec{E}}_0(x,y)e^{j(\omega t - k_z z)}$$

$$\Delta\underline{\vec{E}} = \left(\nabla_{xy}^2 + \nabla_z^2\right)\underline{\vec{E}} = \nabla_{xy}^2\underline{\vec{E}} - k_z^2\underline{\vec{E}}$$

$$\Rightarrow \nabla_{xy}^2\underline{\vec{E}}_0(x,y) + \left(\omega^2\mu\varepsilon - kz^2\right)\underline{\vec{E}}_0(x,y) = 0$$

c) Separation of variables, e.g.

$$E_z = E_0 \sin(k_x x)\sin(k_y y)$$

d) Due to the boundary conditions only specific values for k_x and k_y are possible. They are numbered. One pair of numbers is one mode.

9) Given is a uniform rectangular 2D-Grid. The grid step in x- and y-direction is h.

a) Translate the Laplace equation into finite difference approximation using central difference formula. *(2 points)*

<u>Answer</u>:

$$\frac{\Phi(i+1,j)-2\Phi(i,j)+\Phi(i-1,j)}{h^2}+\frac{\Phi(i,j+1)-2\Phi(i,j)+\Phi(i,j-1)}{h^2}=0$$

or

$$\Phi(i+1,j)-4\Phi(i,j)+\Phi(i-1,j)+\Phi(i,j+1)+\Phi(i,j-1)=0$$

b) Translate Dirichlet condition $\Phi\big|_{x=0}=V_0$

and Neumann condition $\left.\frac{\partial\Phi}{\partial x}\right|_{y=0}=-E_0$ into their finite difference equivalents. *(2 points)*

<u>Answer</u>:

$$\Phi(0,j)=V_0$$
$$\Phi(i,1)-\Phi(i,0)=-hE_0$$

10) Translate the following Maxwell equation into a linear equation using the Finite Integration Technique:

$$\int\vec{E}d\vec{s}=-\iint\frac{\partial\vec{B}}{\partial t}d\vec{a}$$

(2 points)

<u>Answer</u>:

$$g\cdot\left(E_z(i,j,k)+E_y(i,j,k)-E_z(i,j+1,k)-E_y(i,j,k-1)\right)=-g^2\dot{B}_x(i,j,k)$$

g: side length of the grid