Electromagnetics and Calculation of Fields 2006
Questions and Answers

Question 1)

Given is the following symmetrical spherical arrangement: a metal sphere with radius r = a has the potential Φ = U and will be concentrically surrounded by a grounded (Φ = 0) hollow sphere with radius *r* = *b*. Between these two spheres exists a material with a dielectric constant ε and and the conductivity $\kappa = \kappa_0/r^2$.

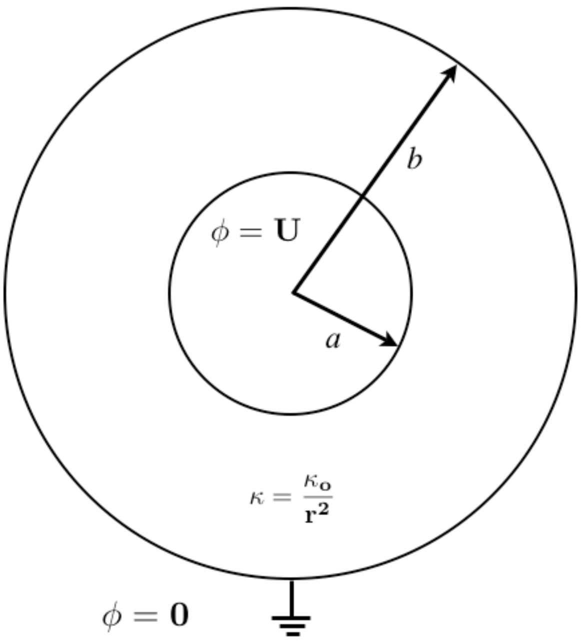
a) Determine $\vec{j}(r)$ with the equation

$$\frac{\kappa}{\epsilon}\varrho + \frac{\epsilon}{\kappa}\vec{j} \cdot grad\left(\frac{\kappa}{\epsilon}\right) = 0$$

as a function of $\varrho(r)$. (4 Points)

b) Determine $\vec{E}(r)$ as a function of $\varrho(r)$. (3 Points)

c) Determine $\vec{E}(r)$ and $\varrho(r)$ explicitely using div $\vec{D} = \varrho$. (7 Points)



Hint: The div-operator in spherical coordinates is:

$$\begin{aligned} div\vec{A} &= \frac{1}{r^2}\frac{d}{dr}(r^2 A_r) \\ &\quad + \frac{1}{r sin\vartheta}\frac{d}{d\vartheta}(A_\vartheta sin\vartheta) \\ &\quad + \frac{1}{r sin\vartheta}\frac{dA_\varphi}{d\varphi} \end{aligned}$$

Solution 1)

a) While the arrangement is spherical-symetrical it is independent of ϑ and φ.

$$\vec{E} = E_r(r)\vec{e}_r$$
$$\vec{j} = j_r(r)\vec{e}_r$$
$$\kappa = \kappa(r)$$

The gradient of $\frac{\kappa}{\epsilon}$ has only a r -component. $\vec{j}$ will be computed as a function of the charge density ϱ.

$$\frac{\kappa}{\epsilon}\varrho + \frac{\epsilon}{\kappa}\vec{j} \cdot grad\left(\frac{\kappa}{\epsilon}\right) = 0$$
$$\frac{\kappa_0\varrho}{\epsilon r^2} + \frac{\epsilon r^2}{\kappa_0} j_r(r)\underbrace{\vec{e}_r\vec{e}_r}\frac{\delta}{\delta r}\left(\frac{\kappa_0}{\epsilon r^2}\right) = 0$$
$$\frac{\kappa_0\varrho}{\epsilon r^2} + \frac{\epsilon r^2}{\kappa_0} j_r(r)\left(-\frac{2\kappa_0}{\epsilon r^3}\right) = 0$$
$$\frac{\kappa_0\varrho}{\epsilon r^2} - \frac{2}{r} j_r(r) = 0$$
$$\frac{\kappa_0\varrho}{\epsilon r} - 2j_r(r) = 0$$
$$2j_r(r) = \frac{\kappa_0\varrho}{\epsilon r}$$
$$\vec{j} = \frac{\kappa_0\varrho}{2\epsilon r}\vec{e}_r$$

b) Furthermore it is nessecary: $\kappa \cdot E_r(r) = j_r(r)$

In this equation the result from a) will be inserted:

$$\frac{\kappa_0}{r^2}E_r(r) = \frac{\kappa_0\varrho}{2\epsilon r}$$
$$E_r(r) = \frac{\varrho r}{2\epsilon}$$
$$\varrho = \frac{2\epsilon E_r(r)}{r}$$

c) $\vec{E}$ and ϱ are still unknown. To define the $\vec{E}$ field the following Maxwell equation is needed:

$$div(\epsilon\vec{E}) = \varrho$$

In spherical coordinates

$$\frac{1}{r^2}\frac{\delta}{\delta r}r^2E_r(r) = \frac{\varrho}{\epsilon}$$

$$\frac{1}{r^2}\frac{\delta}{\delta r}r^2E_r(r) = \frac{2E_r(r)}{r}$$

$$\frac{\delta}{\delta r}r^2E_r(r) = 2rE_r(r)$$

$$2rE_r(r) + r^2\frac{\delta E_r}{\delta r} = 2rE_r(r)$$

$$r^2\frac{\delta E_r}{\delta r} = 0 \qquad r \neq 0$$

$$\frac{\delta E_r}{\delta r} = 0$$

$$\Rightarrow E_r = C \qquad \text{with } C = \text{const.}$$

Estimating C

$$\int_a^b C dr = U$$

$$C(b-a) = U$$

$$\Rightarrow C = \frac{U}{(b-a)}$$

$$\Rightarrow \vec{E} = \frac{U}{(b-a)}\vec{e}_r$$

$$\Rightarrow \varrho = \frac{2\epsilon \cdot U}{r(b-a)}$$

Question 2)

A plane cruise with a speed of v = 500 km/h right on top of the magnetic North Pole. The magnetic field is $B = 5 \cdot 10^{-5}$ T.

a) Which electrical field strength could be messured on board? (2 Points)

b) Which voltage exist between the tip of the wings at a wingspan of d = 50 m? (1 Point)

c) How large is the voltage if the plane is crossing the equator flying in the direction of the magnetic North Pole? (1 Point)

Solution 2)

$$\vec{F} = q(\vec{v} \times \vec{B} + \vec{E}) = 0, \qquad \vec{v} \perp \vec{B}$$

$$F = q \cdot v \cdot B + qE = 0$$

a)

$$\Rightarrow E = -v \cdot B = -500 \cdot 10^3 m \cdot \frac{1}{3600s} \cdot 5 \cdot 10^{-5} \frac{Vs}{m^2}$$

$$= -6,9 \cdot 10^{-3} \frac{V}{m}$$

b)

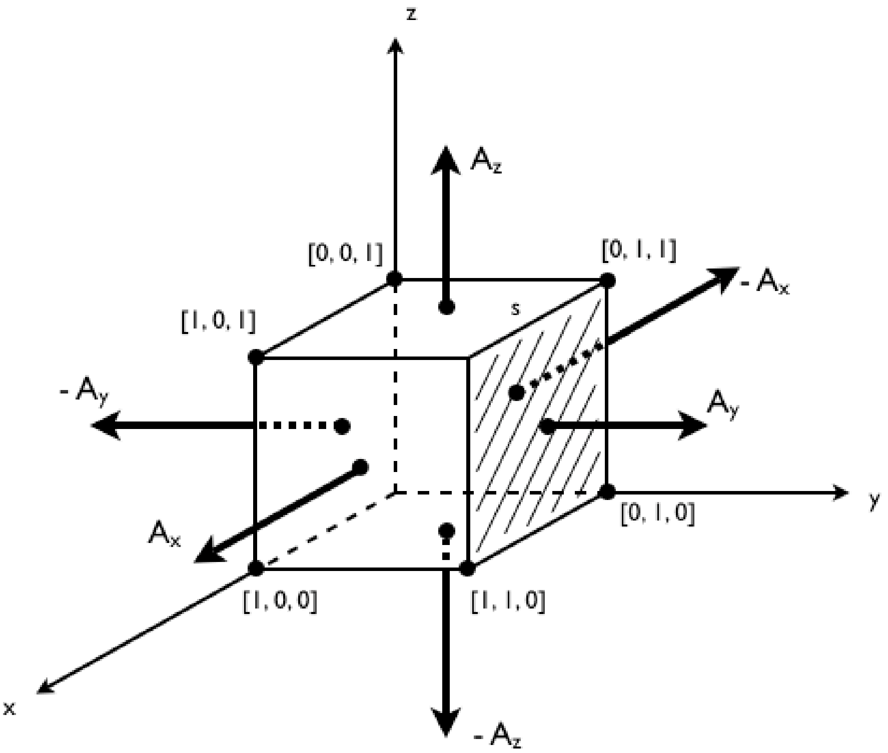
$$U = E \cdot d = 0,347V$$

c)

$$U = 0 \qquad \text{since} \quad \vec{v} \parallel \vec{B}$$

Question 3)

Consider a general vector field given by **A** = $2xy\vec{e}_x + 3\vec{e}_y + z^2y\,\vec{e}_z$. Suppose the surface is a cube with sides of unit area, as show in the figure. Verify the divergence theorem (Gauss' Law) for this vector field and surface. (10 Points)



Solution 3)

The divergenz of **A** is $\nabla \cdot \mathbf{A}$ = 2y + 2zy = 2y(z+1), and a differential volume element is *dx dy dz*. Integrating over the volume *v*, we obtain

$$\int_v \nabla \cdot A \, dv = \int_{z=0}^{1} \int_{y=0}^{1} \int_{x=0}^{1} 2y(z+1) dx dy dz = \frac{3}{2}$$

The integral of the normal component of **A** over the surface of this volume is obtained by integrating those components of **A** that are normal to the surfaces of this cube. From the figure we obtain

$$\oint_s A \cdot ds = \int_{x=0}^{1} \int_{y=0}^{1} A_z \Big|_{z=1} dx dy + \int_{x=0}^{1} \int_{y=0}^{1} (-A_z) \Big|_{z=0} dx dy$$

$$+ \int_{z=0}^{1} \int_{x=0}^{1} A_y \Big|_{y=1} dx dz + \int_{z=0}^{1} \int_{x=0}^{1} (-A_y) \Big|_{y=0} dx dz$$

$$+ \int_{y=0}^{1} \int_{z=0}^{1} A_x \Big|_{x=1} dy dz + \int_{y=0}^{1} \int_{z=0}^{1} (-A_x) \Big|_{x=0} dy dz$$

In each integrant we substitute $A_x = 2xy$, $A_y = 3$ and $A_z = z^2y$ and evaluate them with the contraints, as shown, prior to evaluating the integrals. Hence, we obtain

$$\begin{aligned} \oint_s A \cdot ds &= \int_{x=0}^{1} \int_{y=0}^{1} (y-0) \, dx dy + \int_{z=0}^{1} \int_{x=0}^{1} (3-3) \, dx dz + \int_{z=0}^{1} \int_{y=0}^{1} (2y-0) \, dz dy \\ &= \frac{3}{2} \end{aligned}$$

Question 4:

a. What is the definition of the Poynting Vector (with the unit)? (2 Points)
b. Describe the Poynting Vector in relation to *E, D, B, H, J* using the Gaussian theorem and explain the different parts. (3 Points)

Solution 4:

a)

$$\boxed{\vec{S} = \vec{E} \times \vec{H}}$$ Poynting Vector

$$\text{unit } \vec{S} = \frac{V}{m} \cdot \frac{A}{m} = \frac{\text{Watt}}{m^2} = \frac{\text{Joule}}{m^2 \cdot s}$$

b)

$$\operatorname{div}\left(\vec{E} \times \vec{H}\right) = -\vec{E} \cdot \vec{J} - \vec{E}\frac{\partial \vec{D}}{\partial t} - \vec{H}\frac{\partial \vec{B}}{\partial t}$$

$$\operatorname{div}\vec{S} = -\frac{\partial}{\partial t}\left(w_j + w_e + w_m\right)$$ using Gaussian theorem

$$\oint \vec{S} d\vec{a} = -\frac{\partial}{\partial t}\int \left(w_j + w_e + w_m\right) dv$$

with $w_e = \frac{1}{2}\vec{E} \cdot \vec{D}$ electrostatic energy density

$w_m = \frac{1}{2}\vec{H} \cdot \vec{B}$ magnetic energy density

$w_j = \int_0^t \vec{J}\vec{E} dt$ energy density dissipated due to ohmic losses

The integral power flux out of a volume is equal to the sum of

- ohmic loss
- change of electroststic field energy
- change of magnetic field energy

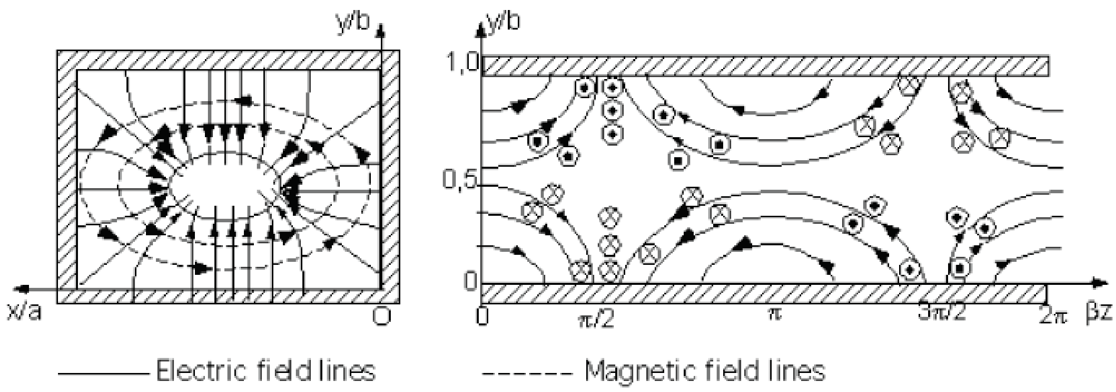
Question 5)

a. Sketch the field lines of the electric and magnetic field of the TM11 mode inside a rectangular waveguide. Please mark the different lines clearly (especially if you choose one graph)! (2 Points)

b. What are the general characteristics for TM and for TE waves in a waveguide? (2 Points)

Solution 5)

a)



b) For TE waves the electric field is completely transverse to the direction off propagation, $E_z = 0$.
For TM waves the magnetic field is completely transverse to the direction of propagation, $H_z = 0$.

Question 6:

Using Euler's differential equations the Laplace equation can be translated to a variational problem.

a. Which type of integral is minimized for the example of electric potential theory? (2 Points)
b. What is the physical meaning of the integral? (1 Point)
c. What type of boundary conditions are often implicitly incorporated? (1 Point)

Solution 6)

a)

$$\left.\begin{array}{l}\text{Find the function}\\ \text{to minimize}\\ I=\int(\mathrm{grad}\,u)^2\,dv\\ \text{with } u(\bar{r})=b(\bar{r})\\ \text{on the boundary}\end{array}\right\} \leftrightarrow \left\{\begin{array}{l}\Delta u(\bar{r})=0\\ u(\bar{r})=b(\bar{r})\\ \dfrac{\partial u}{\partial n}=0\\ \text{on the boundary}\end{array}\right.$$

b) The integral that is minimized is proportional to the field energy.

c) Neumann's boundary conditions with $\frac{du}{dn}=0$ are incorporated if simply the field energy is minimized.

Question 7)

Look at the Finite Element Method FEM:

a. Describe how the scalar function (like *Φ(x)*) can be approximated using the linear node shape functions. Give a sketch of the linear node shape functions. (4 Points)

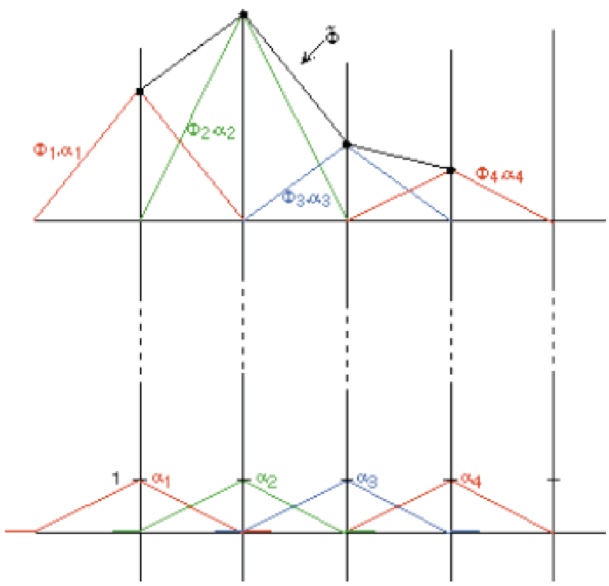
b. What is the main difference as compared to a Fourier expansion? (1 Point)

c. What is a Residuum, e.g. in case of a Poisson equation? (3 Points)

d. What was the idea of Galerkin? (2 Points)

Solution 7)

a)



$$\tilde{\Phi} = \sum \alpha_k(x,y,z) \cdot \Phi_k$$ k = 1......p p: number of nodes

α_K: node shape functions
Φ_K: node voltages

b) Node shape functions are local basis functions. Sine and cosine are global basis functions.

c) $$\Delta\Phi + \frac{\rho}{\varepsilon} = 0$$

$$\Delta\tilde{\Phi} + \frac{\rho}{\varepsilon} = R \qquad \tilde{\Phi} = \sum \alpha_k(x,y,z)\cdot\phi_k \qquad R = \text{Residuum}$$

d) The best approximation means: $\int w\, R\, dv = 0$ with w: weighting function

Galerkin's choice of w: $w_\ell(x,y,z) = \alpha_\ell(x,y,z)$ $\quad \ell = 1......p$

The weighting functions are equal to the node shape functions.

Question 8)

Describe the basic idea and equations of the Method of Moments MoM. Assume an arbitray linear integral operator L and a set of orthonormal basis functions α_n.
(3 Points)

Solution 8)

$L \cdot u = g$	L: linear operator
$u^* = \sum c_n \cdot \alpha_n$	
$R = \sum c_n \cdot L \cdot \alpha_n - g$	Residuum
$\int R \cdot w \cdot dv = 0$	weighted residuals

Question 9:

Look at the Transmission Line Matrix Method TLM.

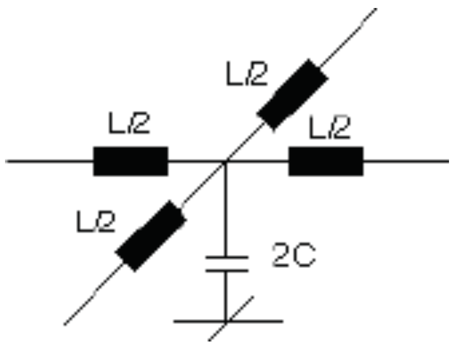
a. Write down the lumped element circuit of a single element in 2D (lossless case) (1 Point)

b. Find the scattered signals from the incident signals for one node inside a homogeneous region. (3 Points)

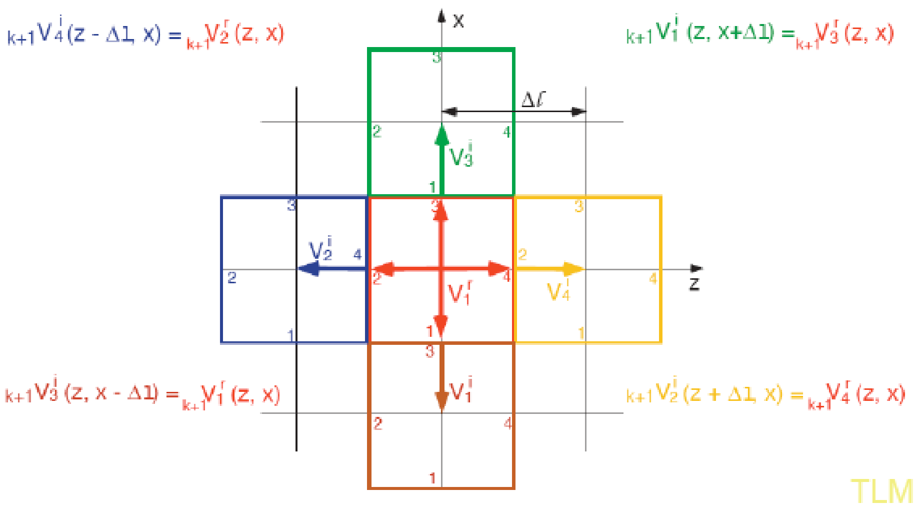
c. How does the algorithm proceed? (2 Points)

Solution 9:

a)



b)



$$_{k+1}V_1^r = \frac{1}{2}\left(-_kV_1^i + _kV_2^i + _kV_3^i + _kV_4^i\right)$$

$$_{k+1}V_2^r = \frac{1}{2}(\ldots) \qquad _{k+1}V_3^r = \frac{1}{2}(\ldots) \qquad _{k+1}V_4^r = \frac{1}{2}(\ldots)$$

c)

$$
{}_{k+1}\begin{bmatrix}V_1\\V_2\\V_3\\V_4\end{bmatrix}^r = \frac{1}{2}\begin{bmatrix}-1&1&1&1\\1&-1&1&1\\1&1&-1&1\\1&1&1&-1\end{bmatrix}_k \begin{bmatrix}V_1\\V_2\\V_3\\V_4\end{bmatrix}^i
$$

$$
{}_{k+1}V_1^i(z, x+\Delta l) = {}_{k+1}V_3^r(z, x)
$$

$$
\vdots
$$

Question 10)

a. Describe the 3 golden rules for good accuracy in time dependent numerical field calculation. Explain each rule with some words. (6 Points)
b. Describe two methods to evaluate the accuracy of the solution. (4 Points)

Solution 10)

a)

1. Mesh must fit to the geometry of the problem
   - Small objects > fine mesh
   - Large objects > coarse mesh

2. Mesh must fit to the interpolation function, e.g. linear shape functions must be an accaptable approximation inside the elements
   - fine mesh is required near to sources e.g. ρ, J
   - fine mesh is required near to boundaries with large changes in ε, μ, κ

3. Time step must fit to the mesh
   - $\Delta t \leq \Delta x / c_{mat}$ $\quad c_{mat} = c_{vac} / \sqrt{\varepsilon\mu}$ $\quad$ c: velocity of light
     (if smallest Δx = 1mm and material = vacuum, then Δt = 3 ps)

b)

1. Numerical accuracy
   e.g. if $A\vec{\Phi} = \vec{b}$ then test, if $|A\vec{\Phi} - \vec{b}| \leq 10^{-12}$

2. „Physical" accuracy
   e.g. test the equation of conservation of charge

$$\oint \vec{J} d\vec{f} + \frac{\delta}{\delta t} \oint \rho dv = 0$$

3. Large relative change e.g. of el. potential between neighbouring nodes is dangerous.