Examination
"Electromagnetics and Numerical Calculation of Fields"

March 6, 2007

1. (a) Two points are given in spherical coordinates as $P_1(r_1,\theta_1,\phi_1)$ and $P_2(r_2,\theta_2,\phi_2)$. Write the vector A connecting P_1(tail) to P_2(head) in Cartesian coordinates. (5P)
   (b) A scalar field is given in Cartesian coordinates as $f(x,y,z)=3x-7zy^2$. Calculate the gradient of the scalar field in cylindrical coordinates. (5P)

   Answer
   (a) The vector A can be expressed in Cartesian coordinates by
   $A=(x_2-x_1)\vec{e}_x+(y_2-y_1)\vec{e}_y+(z_2-z_1)\vec{e}_z$ (1P)
   With the transformation from Cartesian to spherical coordinates:
   $x=r\sin\theta\cos\phi,\ y=r\sin\theta\sin\phi,\ z=r\cos\phi,$ (1P)
   we get
   $$\begin{aligned}A=&(r_2\sin\theta_2\cos\phi_2-r_1\sin\theta_1\cos\phi_1)\vec{e}_x+(r_2\sin\theta_2\sin\phi_2-r_1\sin\theta_1\sin\phi_1)\vec{e}_y\\&+(r_2\cos\phi_2-r_1\cos\phi_1)\vec{e}_z\end{aligned}\qquad\text{(3P)}$$

   (b) The transformation between cylindrical and Cartesian coordinates are:
   $x=R\cos\phi,\ y=R\sin\phi,\ z=z$ (1P)
   Substituting these for x, y, z in the field given in cylindrical coordinates:
   $f(R,\phi,z)=3R\cos\phi-7zR^2\sin^2\phi$ (1P)
   The gradient can now be calculated in cylindrical coordinates:
   $$\begin{aligned}\nabla f&=\left(\vec{e}_R\frac{\partial}{\partial R}+\vec{e}_\phi\frac{1}{R}\frac{\partial}{\partial\phi}+\vec{e}_z\frac{\partial}{\partial z}\right)(3R\cos\phi-7zR^2\sin^2\phi)\\&=\vec{e}_R(3\cos\phi-14zR\sin^2\phi)+\vec{e}_\phi(-3\sin\phi-14zR\sin\phi\cos\phi)-\vec{e}_z7R^2\sin^2\phi\end{aligned}\qquad\text{(3P)}$$

2. (a) Write down Maxwell's equations in differential and integral form. (4P)
   (b) Derive the wave equation for E and H directly from Maxwell's equations (lossless case, linear and isotropic material, $\rho=0$). (8P)

   Answer
   (a)
   $$\begin{aligned}&div\vec{D}=\rho\Leftrightarrow\oint\vec{D}d\vec{a}=\int\rho dv\\&rot\vec{H}=\vec{J}+\dot{\vec{D}}\Leftrightarrow\oint\vec{H}d\vec{l}=\int\left(\vec{J}+\dot{\vec{D}}\right)d\vec{a}\\&rot\vec{E}=-\dot{\vec{B}}\Leftrightarrow\oint\vec{E}d\vec{l}=-\frac{d}{dt}\int\vec{B}d\vec{a}\\&div\vec{B}=0\Leftrightarrow\oint\vec{B}d\vec{a}=0\end{aligned}\qquad\text{(4P)}$$

(b)

$$rot\vec{E} = -\mu\frac{\partial}{\partial t}\vec{H} \qquad rot\vec{H} = \varepsilon\frac{\partial}{\partial t}\vec{E}$$

$$rot\; rot\vec{E} = -\mu\; rot\frac{\partial}{\partial t}\vec{H} \qquad rot\; rot\vec{H} = \varepsilon\; rot\frac{\partial}{\partial t}\vec{E}$$

$$grad\; div\vec{E} - \Delta\vec{E} = -\mu\frac{\partial}{\partial t}rot\vec{H} \quad (4P) \qquad grad\; div\vec{H} - \Delta\vec{H} = \varepsilon\frac{\partial}{\partial t}rot\vec{E} \quad (4P)$$

$$\Delta\vec{E} - \mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2} = 0 \qquad \Delta\vec{H} - \varepsilon\mu\frac{\partial^2\vec{H}}{\partial t^2} = 0$$

3. Write down the material equations that combine E with D, B with H and J with E for the most general case and the case of the linear relation and isotropic medium. (6P)

Answer

For dielectrics:

$\vec{D} = \varepsilon_0\vec{E} + \vec{P} \Rightarrow \vec{D} = \varepsilon_0\varepsilon_r \cdot \vec{E}$, where $\varepsilon_0 = \frac{10^7}{c^2 \cdot 4\pi}\frac{As}{Vm}$ (2P)

For magnetic materials:

$\vec{B} = \mu_0\left(\vec{H} + \vec{M}\right) \Rightarrow \vec{B} = \mu_0\mu_r \cdot \vec{H}$, where $\mu_0 = 4\pi \cdot 10^{-7}\frac{Vs}{Am}$ (2P)

For conducting materials:

$\vec{J} = \kappa \cdot \vec{E} + J^{(i)} \Rightarrow \vec{J} = \kappa \cdot \vec{E}$, where $J^{(i)}$ is impressed current density. (2P)

4. What type of functions solve the Laplace equation in Cartesian coordinates, in cylindrical coordinates and in spherical coordinates, respectively? (9P)

Answer

In Cartesian coordinates:

$\sin(\alpha x)/\cos(\alpha x),\ \sin(\beta y)/\cos(\beta y),\ \sinh(\gamma z)/\cosh(\gamma z)$, where $\gamma^2 = \alpha^2 + \beta^2$. (3P)

In cylindrical coordinates:

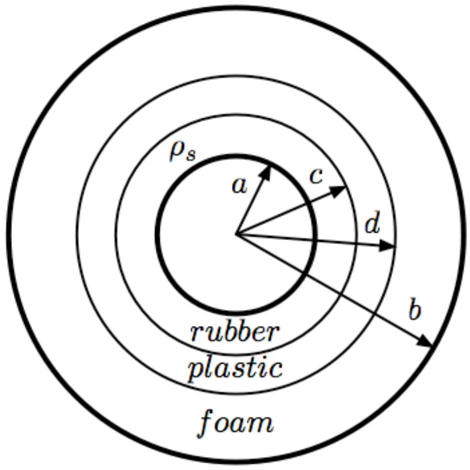
$\sinh(\gamma z)/\cosh(\gamma z),\ \sin(m\phi)/\cos(m\phi),\ J_m(\gamma R)/N_m(\gamma R)$, where J_m and N_m are Bessel Function of 1st and 2nd kind. (3P)

In spherical coordinates:

$r^l/r^{-(l+1)},\ P_l^m(\cos\vartheta)/Q_l^m(\cos\vartheta),\ \sin(m\phi)/\cos(m\phi)$, where P_l^m and Q_l^m are associated Legendre Polynomials. (3P)

5. A long coaxial cable is made with an internal conductor of radius $a = 2mm$ and an external conductor of radius $b = 6mm$. The design calls for three layers of insulation between the two conductors. The inner layer is $1mm$ thick and is made of rubber ($\varepsilon_r = 4.0$), the next layer is a plastic ($\varepsilon_r = 9$) $1mm$ thick, and the third layer is a foam ($\varepsilon_r = 1.5$) $2mm$ thick. (see the figure below) Calculate the capacitance per unit length of the cable. (14P)

(Hint: You can assume a charge per unit surface area of the inner conductor of ρ_s $[C/m^2]$. Next calculate the electric flux density between the conductors at a point $a < R < b$.)



$a = 2mm$
$b = 6mm$
$c = 3mm$
$d = 4mm$

Answer

The electric flux density between the conductors at a point $a < R < b$ is given as

$$\int \vec{D} \cdot d\vec{a} = q \Rightarrow D 2\pi RL = \rho_s 2\pi aL \qquad \text{(1P)}$$

The electric flux density is in the R direction and equals

$$D = \vec{e}_R \frac{\rho_s a}{R}\left[\frac{C}{m^2}\right] \qquad \text{(1P)}$$

The electric field in the three layers is:

- In rubber ($2 < R < 3mm$), $\varepsilon_r = 4$:

$$\vec{E}_{rubber} = \frac{\vec{D}}{4\varepsilon_0} = \vec{e}_R \frac{\rho_s a}{4\varepsilon_0 R}\left[\frac{V}{m}\right] \qquad \text{(1P)}$$

- In plastic ($3 < R < 4mm$), $\varepsilon_r = 9$:

$$\vec{E}_{plastic} = \frac{\vec{D}}{9\varepsilon_0} = \vec{e}_R \frac{\rho_s a}{9\varepsilon_0 R}\left[\frac{V}{m}\right] \qquad \text{(1P)}$$

- In foam ($4 < R < 6mm$), $\varepsilon_r = 1.5$:

$$\vec{E}_{foam} = \frac{\vec{D}}{1.5\varepsilon_0} = \vec{e}_R \frac{\rho_s a}{1.5\varepsilon_0 R}\left[\frac{V}{m}\right] \qquad \text{(1P)}$$

The potential difference between the plates (integrating from outer to inner shell against the field) is

$$V_{ab} = -\int_b^a \vec{E} \cdot dl = -\int_b^{b-0.002} E_{foam} dR - \int_{b-0.002}^{b-0.003} E_{plastic} dR - \int_{b-0.003}^{a} E_{rubber} dR \qquad \text{(3P)}$$

$$= -\frac{\rho_s a}{\varepsilon_0}\left(\frac{1}{1.5}\ln\frac{b-0.002}{b} + \frac{1}{9}\ln\frac{b-0.003}{b-0.002} + \frac{1}{4}\ln\frac{a}{b-0.003}\right) \qquad \text{(1P)}$$

With $a = 0.002m$ and $b = 0.006m$, we get

$$V_{ab} = -\frac{\rho_s \times 0.002}{\varepsilon_0}\left(\frac{1}{1.5}\ln\frac{0.004}{0.006} + \frac{1}{9}\ln\frac{0.003}{0.004} + \frac{1}{4}\ln\frac{0.002}{0.003}\right) \qquad \text{(1P)}$$

$$= 8.073 \times 10^{-4} \frac{\rho_s}{\varepsilon_0}[V]$$

The total charge per unit length of the inner conductor ($L = 1m$) is $2\pi a \rho_s$. Dividing the charge per unit length by the potential difference gives

$$C = \frac{Q}{V_{ab}} = \frac{2\pi a \rho_s}{8.073 \times 10^{-4} \rho_s / \varepsilon_0} \qquad \text{(2P)}$$

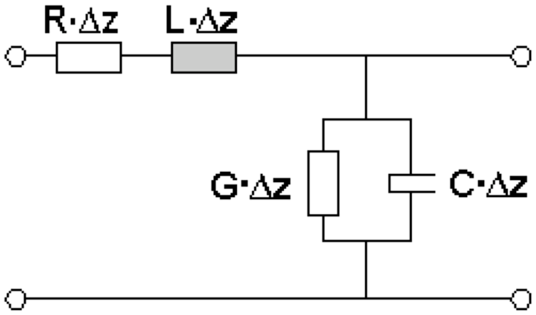
$$= \frac{2\pi a \varepsilon_0}{8.073 \times 10^{-4}} = \frac{2 \times \pi \times 0.002 \times 8.854 \times 10^{-12}}{8.073 \times 10^{-4}} \qquad \text{(1P)}$$

$$= 1.378 \times 10^{-10} \left[\frac{F}{m}\right] = 137.8 [pF/m] \qquad \text{(1P)}$$

6. What is the definition of the magnetic flux Φ_m? How can you find the magnetic flux Φ_m directly from the magnetic vector potential $\vec{A}$? (3P)

Answer
The definition of magnetic flux
$\Phi_m = \int \vec{B} d\vec{a}$ (1P)
Substituting $\vec{B} = rot\vec{A}$
$\Phi_m = \int rot\vec{A} da$ (1P)
Applying Stokes' theorem
$\Phi_m = \oint \vec{A} d\vec{l}$ (1P)

7. Give the equivalent circuit of a two-conductor lossy transmission line for a differential length Δz. (3P)



(3P)

8. (a) What is a TEM wave? What are the characteristics of TM wave as well as of TE waves in a waveguide? What does the cut-off frequency for a specific mode in a waveguide mean? (4 P)
(b) A rectangular waveguide has internal dimensions $a = 19.05mm$ and $b = 9.53mm$. Find the lowest possible TM mode at which the wave may be excited and calculate its cut-off frequency. (7P)
(c) Give a sketch of the electric and magnetic field of the TE_{10} mode inside a rectangular waveguide. (3P)

Answer

(a) A TEM wave is a Transverse Electric and Magnetic wave, $E_z = 0$ and $H_z = 0$, where z is the direction of propagation; For TM waves the magnetic field is completely transverse to the direction of propagation, $E_z \neq 0$ and $H_z = 0$; for TE waves the electric field is completely transverse to the direction of propagation, $E_z = 0$ and $H_z \neq 0$. The cut-off frequency is the lowest frequency that allows for undamped propagation of waves for a specific mode. (4P)

(b) The cut-off frequency for the TM modes is:

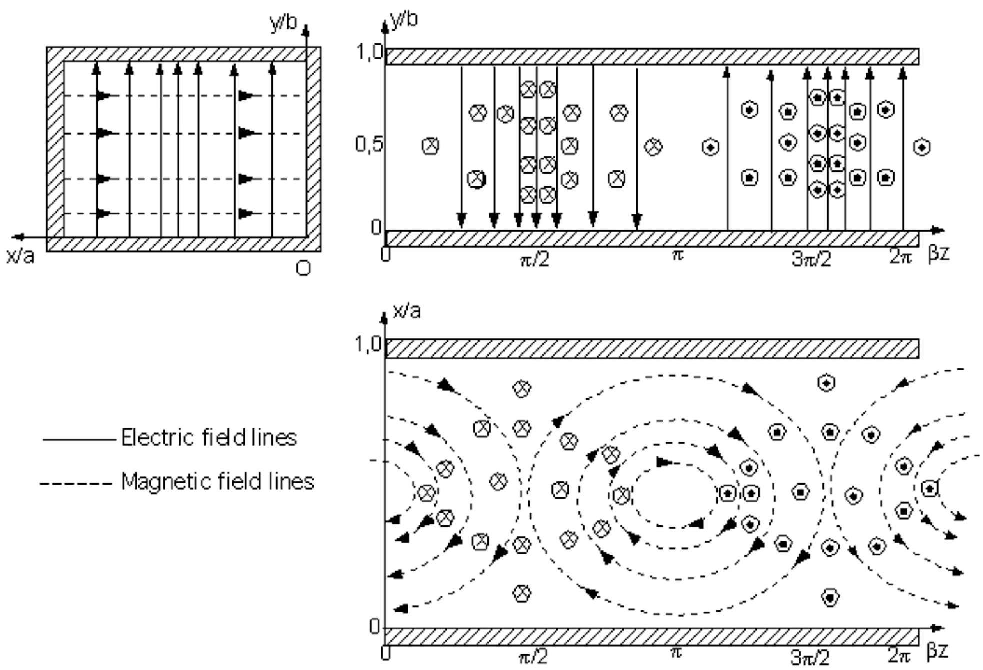
$$f_{cmn} = \frac{1}{2\pi\sqrt{\mu_0\varepsilon_0}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} = \frac{1}{2\sqrt{\mu_0\varepsilon_0}}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2} \qquad \text{(1P)}$$

$$= 1.5 \times 10^8 \sqrt{\left(\frac{m}{0.01905}\right)^2 + \left(\frac{n}{0.00953}\right)^2}\,[Hz] \qquad \text{(1P)}$$

Since, in TM modes, m or n cannot be zero, only $f_{c11} = 17.60GHz$ corresponds to a possible TM mode. Thus, the only possible mode is the TM_{11} mode. (3P)
Its cut-off frequency is:

$$f_{c11} = 1.5 \times 10^8 \sqrt{\left(\frac{1}{0.01905}\right)^2 + \left(\frac{1}{0.00953}\right)^2} = 17.60[GHz] \qquad \text{(2P)}$$

(c)



(3P)

9. The electric field of an electromagnetic wave is given as:

$E_x = 10\sin\left(\frac{2\pi 10^8}{s} \cdot t - \frac{2\pi}{m} \cdot z\right)\left[\frac{V}{m}\right]$, where the units of t and z are s and m, respectively.

(a) Find the wavelength, the phase velocity and the relative dielectric constant of the medium ($\mu_r = 1$). (6P)

(b) Find the corresponding magnetic field H (value and direction). (6P)

Answer

(a)

From $E_x = 10\sin\left(\frac{2\pi 10^8}{s} \cdot t - \frac{2\pi}{m} \cdot z\right)\left[\frac{V}{m}\right]$ we get

$$\omega = \frac{2\pi 10^8}{s} = 2\pi f \Rightarrow f = 10^8\left[s^{-1}\right] \qquad \text{(1P)}$$

$$\frac{\kappa}{m} = \frac{2\pi}{m} = \frac{2\pi}{\lambda} \Rightarrow \lambda = 1[m] \qquad \text{(1P)}$$

$$c = \lambda \cdot f = 10^8\left[\frac{m}{s}\right] = \frac{1}{\sqrt{\mu_0\varepsilon_0\varepsilon_r}} \qquad \text{(1P)}$$

$$c_{vac} = 3 \cdot 10^8\left[\frac{m}{s}\right] = \frac{1}{\sqrt{\mu_0\varepsilon_0}} \qquad \text{(1P)}$$

$$\Rightarrow \frac{c_{vac}}{c} = \sqrt{\frac{\mu_0\varepsilon_0\varepsilon_r}{\mu_0\varepsilon_0}} = \sqrt{\varepsilon_r} = 3 \qquad \text{(1P)}$$

$$\Rightarrow \varepsilon_r = \left(\frac{c_{vac}}{c}\right)^2 = 9 \qquad \text{(1P)}$$

(b)

$$H_y = \frac{E_y}{\Gamma} = E_y \cdot \sqrt{\frac{\varepsilon_0\varepsilon_r}{\mu_0}} \qquad \text{(2P)}$$

$$= E_y \cdot \sqrt{\frac{10^7/\left(c^2 \cdot 4\pi\right) \cdot 9}{4\pi \cdot 10^{-7}}}$$

$$= E_y \cdot \sqrt{\frac{10^{14} \cdot 9}{16\pi^2 \cdot c^2}}$$

$$= E_y \cdot \frac{3 \cdot 10^7}{4\pi c}$$

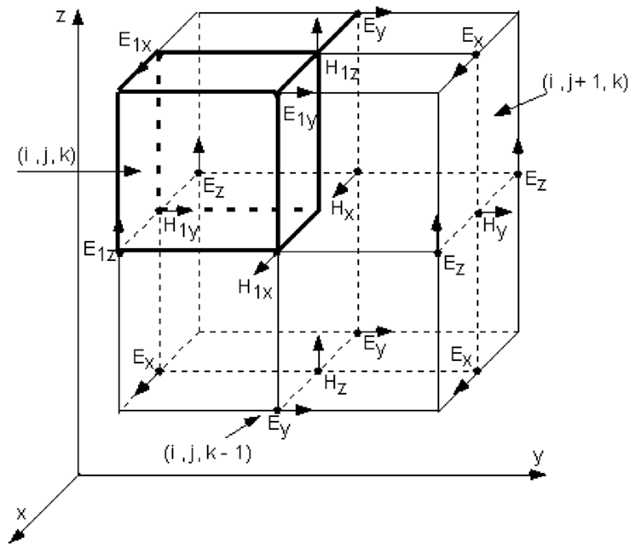
$$= 10 \cdot \frac{3 \cdot 10^7}{4\pi \cdot 3 \cdot 10^8} \qquad \text{(3P)}$$

$$= \frac{1}{4\pi} = 0.08\left[\frac{A}{m}\right] \qquad \text{(1P)}$$

10. What is the ingenious idea of Yee's lattice? Give a sketch of Yee's lattice that can be applied in the finite integration technique. Write down the Maxwell's Equation called "Law of Induction", in the discrete formulation in this case. (8P)

Answer

The interlaced lattices for E and H allow for an easy translation of Maxwell's equations into an algorithm. E_x, E_y and E_z are not defined at the same position but at the edges a cube.



(4P)

"Law of Induction" in the analytical and discrete formulations:

$$\oint \vec{E}d\vec{l} = -\iint \dot{\vec{B}}d\vec{a} \Rightarrow$$

$$g\left[E_z(i,j,k) + E_y(i,j,k) - E_z(i,j+1,k) - E_y(i,j,k-1)\right] = -g^2\dot{B}_x(i,j,k)$$

$$g\left[E_z(i,j,k) - E_x(i,j,k) - E_z(i-1,j,k) + E_x(i,j,k-1)\right] = -g^2\dot{B}_y(i,j,k)$$

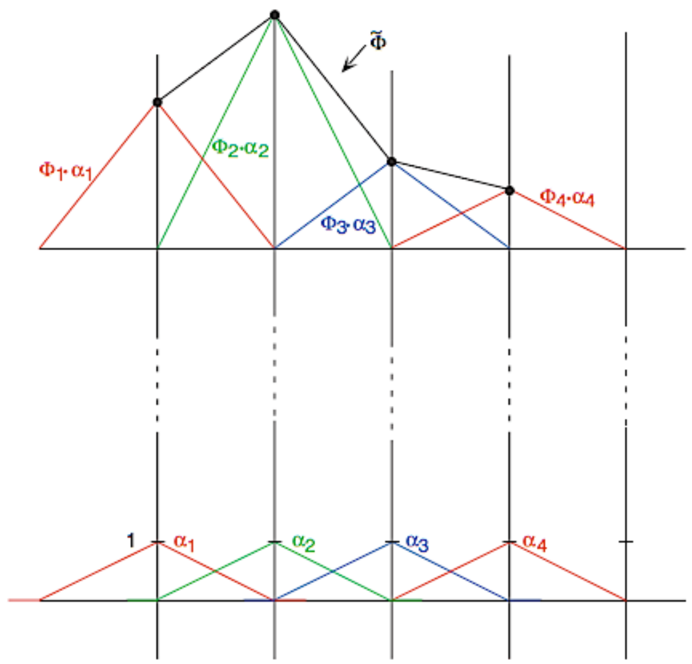
$$g\left[E_x(i,j,k) + E_y(i,j,k) - E_x(i,j+1,k) - E_y(i-1,j,k)\right] = -g^2\dot{B}_z(i,j,k)$$

(4P)

11. (a) Describe how a scalar function (like $\Phi(x)$) can be approximated using linear node shape function, with the aid of giving a sketch of linear node shape functions in 1D case. (3P)
(b) Describe the basic idea and equations of the finite element method cooperating with the method of weighted residual in the case of solving the Poisson's equation for piecewise homogenous materials. If the Galerkin's choice is applied here, how should the weighting function look like? Derive the equation until the second derivative of Φ is removed. (6P)

Answer

(a)



(2P)

$\tilde{\Phi} = \sum_{k=1}^{nodes} \alpha_k(x,y,z) \cdot \Phi_k$, where Φ_k is node potential. (1P)

(b)

Poisson's equation: $\Delta\Phi = -\frac{\rho}{\varepsilon}$ (1P)

Ideal solution: $\Delta\Phi + \frac{\rho}{\varepsilon} = 0$ (1P)

Approximated solution: $\Delta\tilde{\Phi} + \frac{\rho}{\varepsilon} = R$ (1P)

Best approximation: $\int w \cdot R dv = \int \left(\Delta\tilde{\Phi} + \frac{\rho}{\varepsilon}\right) \cdot w dv = 0$ (1P)

where w is weighting function.
Using Green's 2nd law:

$\int \left(grad\tilde{\Phi}\right) \cdot (grad w) dv - \oint w \frac{\partial \tilde{\Phi}}{\partial n} da - \int \frac{\rho}{\varepsilon} w dv = 0$ (1P)

In the Galerkin's method, the weighting functions are chosen to be the basis functions: $w_l(x,y,z) = \alpha_l(x,y,z)$. (1P)

1	2	3	4	5	6	7	8	9	10	11	Sum
10	12	6	9	14	3	3	14	12	8	9	100