Examination
“Electromagnetics and Numerical Calculation of Fields”

March 3, 2008

1. A vector is given in cylindrical coordinates as: $\vec{A} = 2\vec{e}_R + 5\vec{e}_\varphi - \vec{e}_z$. Describe this vector in Cartesian coordinates. (6P)

Answer

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_R \\ A_\varphi \\ A_z \end{bmatrix} \qquad \text{(1P)}$$

$$= \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2\cos\varphi - 5\sin\varphi \\ 2\sin\varphi + 5\cos\varphi \\ -1 \end{bmatrix} \qquad \text{(1P)}$$

$$= \begin{bmatrix} \frac{2x}{\sqrt{x^2+y^2}} - \frac{5y}{\sqrt{x^2+y^2}} \\ \frac{2y}{\sqrt{x^2+y^2}} + \frac{5x}{\sqrt{x^2+y^2}} \\ -1 \end{bmatrix}$$

where $\cos\varphi = \frac{x}{R},\ \sin\varphi = \frac{y}{R},\ R = \sqrt{x^2+y^2}$ (3P)

Thus, vector $\vec{A}$ is

$$\vec{A} = \left(\frac{2x}{\sqrt{x^2+y^2}} - \frac{5y}{\sqrt{x^2+y^2}} \right)\vec{e}_x + \left(\frac{2y}{\sqrt{x^2+y^2}} + \frac{5x}{\sqrt{x^2+y^2}} \right)\vec{e}_y - \vec{e}_z \qquad \text{(1P)}$$

2. (a) Explain Gauss’ Law and Stokes’ Law with formulas and text. (4P)
(b) Write down Maxwell’s equations in differential and integral form. (4P)
(c) Derive the wave equation for E and H directly from Maxwell’s equations (lossless case, linear and isotropic material, $\rho = 0$). (8P)

Answer

(a)

Gauss’ Law: $\int div\vec{A}dv = \oint \vec{A}d\vec{a}$ (1P)

The volume integral of the divergence of a vector field $\vec{A}$ equals the flux of this vector field through the surface enclosing this volume. (1P)

Stokes' Law: $\int rot\vec{A}d\vec{a} = \oint \vec{A}d\vec{l}$ (1P)

The area integral of the normal component of $rot\vec{A}$ (flux of $rot\vec{A}$) equals the line integral of $\vec{A}$ along the contour encircling the area. (1P)

(b)

$$div\vec{D} = \rho \Leftrightarrow \oint \vec{D}d\vec{a} = \int \rho dv$$

$$rot\vec{H} = \vec{J} + \dot{\vec{D}} \Leftrightarrow \oint \vec{H}d\vec{l} = \int \left(\vec{J} + \dot{\vec{D}}\right)d\vec{a}$$

$$rot\vec{E} = -\dot{\vec{B}} \Leftrightarrow \oint \vec{E}d\vec{l} = -\frac{d}{dt}\int \vec{B}d\vec{a}$$

$$div\vec{B} = 0 \Leftrightarrow \oint \vec{B}d\vec{a} = 0$$

(4P)

(c)

$$rot\vec{E} = -\mu\frac{\partial}{\partial t}\vec{H}$$

$$rot\, rot\vec{E} = -\mu\, rot\frac{\partial}{\partial t}\vec{H}$$

$$grad\, div\vec{E} - \Delta\vec{E} = -\mu\frac{\partial}{\partial t}rot\vec{H}$$

$$\Delta\vec{E} - \mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2} = 0$$

(4P)

$$rot\vec{H} = \varepsilon\frac{\partial}{\partial t}\vec{E}$$

$$rot\, rot\vec{H} = \varepsilon\, rot\frac{\partial}{\partial t}\vec{E}$$

$$grad\, div\vec{H} - \Delta\vec{H} = \varepsilon\frac{\partial}{\partial t}rot\vec{E}$$

$$\Delta\vec{H} - \varepsilon\mu\frac{\partial^2\vec{H}}{\partial t^2} = 0$$

(4P)

3. (a) What type of equation has to be solved for finding the electric potential Φ of an arbitrary charge distribution $\rho(x,y,z)$? (2P)
   (b) What are Dirichlet boundary conditions and Neumann boundary conditions? (2P)
   (c) What type of differential equation has to be solved for Φ, when a variable voltage is given on the surface of a metallic ball and no charge inside? Which coordinate system should be chosen? What type of functions do we get in this coordinate system? (4P)

Answer

(a) Poisson equation has to be solved: $\Delta\Phi = -\dfrac{\rho}{\varepsilon}$

or

Coulomb integral: $\Phi(\vec{r}) = \dfrac{1}{4\pi\varepsilon}\iiint \dfrac{\rho(\vec{r}\,')}{\left|\vec{r} - \vec{r}\,'\right|}dv'$ (2P)

(b) Dirichlet boundary conditions: $\Phi\big|_{\text{boundary}} = \tilde{\Phi}$ (1P)

Neumann boundary conditions: $\dfrac{\partial\Phi}{\partial n}\Big|_{\text{boundary}} = \tilde{\Phi}$ (1P)

(c) Laplace equation has to be solved: $\Delta\Phi = 0$ (1P)

Spherical coordinate system could be a good choice. (1P)

In spherical coordinates the solutions are linear combinations of the following functions:

$r^l / r^{-(l+1)}$, $P_l^m(\cos\vartheta)/Q_l^m(\cos\vartheta)$, $\sin(m\phi)/\cos(m\phi)$, where P_l^m and Q_l^m are associated Legendre Polynomials. (2P)

4. A long cylindrical conductor of radius $a = 20mm$, charged with a surface charge density $\rho_a = 0.01 C/m^2$, is located in free space:
   (a) Calculate the energy stored per unit length of the conductor within a radius $b = 100m$ from the conductor. (7P)
   (b) What is the energy per unit length stored inside the conductor? (3P)

   Answer
   (a) Using Gauss' law, the electric field intensity at a distance R from the cylinder is
   $$\oint_a \vec{E}\cdot d\vec{a} = \frac{q}{\varepsilon_0} \Rightarrow E 2\pi R L = \frac{\rho_a 2\pi a L}{\varepsilon_0} \quad \text{(1P)}$$
   Where the integration is over the area of the Gaussian surface a. The electric field intensity is in the R direction and equals
   $$\vec{E} = \vec{e}_R \frac{\rho_a a}{\varepsilon_0 R}\left[\frac{V}{m}\right] \quad \text{(1P)}$$
   The energy density in space is therefore
   $$w_e = \frac{\varepsilon_0 E^2}{2} = \frac{\rho_a{}^2 a^2}{2\varepsilon_0 R^2}\left[\frac{J}{m^3}\right] \quad \text{(1P)}$$
   We define a shell of thickness dr and length L and write the volume of this shell at R as $dv = 2\pi R L dR$. The total energy stored is then
   $$W_e = \int_v w_e dv = \int_{R=a}^{R=b} \frac{\pi \rho_a{}^2 a^2}{\varepsilon_0 R} L dR = \frac{\pi \rho_a{}^2 a^2}{\varepsilon_0} L \ln\frac{b}{a} \quad \text{(3P)}$$
   For $a = 0.02m$, $b = 100m$, $\rho_s = 10^{-2} C/m^2$, we get
   $$\frac{W_e}{L} = \frac{\pi \times 10^{-4} \times (0.02)^2}{8.854 \times 10^{-12}} \ln\frac{100}{0.02} = 1.21 \times 10^5 \left[\frac{J}{m}\right] \quad \text{(1P)}$$

   (b) The electrostatic energy stored in the conductor itself is zero because the electric field intensity inside the conductor is zero. (3P)

5. (a) What is the meaning of the Poynting vector? (3P)
   (b) Calculate the Poynting vector of a plane wave that generates an electric field intensity $\vec{E} = -\vec{e}_y E_0 \cos(\omega t - kz)[V/m]$. What is the direction of propagation of this wave? (Lossless propagation in free space) (8P)
   Hint: Only consider the real part of $\vec{E}$ and $\vec{H}$ in the calculation.

   Answer
   (a) The Poynting vector is the power per area that is transmitted through a given area via electromagnetic fields. (3P)

   (b) First we have to find $\vec{H}(t,z)$. The electric field has only a y component and varies only with z:

$$-\vec{e}_x \frac{\partial E_y}{\partial z} = \vec{e}_x j\omega\mu H_x \quad \text{(2P)}$$

Using the phasor form (taking the real part) of $\vec{E}$

$$\vec{E} = -\vec{e}_y E_0 e^{-jkz}\left[\frac{V}{m}\right] \quad \text{(1P)}$$

get

$$\vec{H} = \vec{e}_x \frac{k}{\omega\mu_0} E_0 e^{-jkz} = \vec{e}_x \frac{1}{\eta_0} E_0 e^{-jkz}\left[\frac{A}{m}\right] \qquad \text{(1P)}$$

where η_0 is the impedance of vacuum. Then transform back to the time domain

$$\vec{H} = \vec{e}_x \frac{1}{\eta_0} E_0 \cos(\omega t - kz)\left[\frac{A}{m}\right]$$

The Poynting vector is

$$\begin{aligned}\vec{S} = \vec{E} \times \vec{H} &= \left[-\vec{e}_y E_0 \cos(\omega t - kz)\right] \times \left[\vec{e}_x \frac{1}{\eta_0} E_0 \cos(\omega t - kz)\right] \\ &= \vec{e}_z \frac{E_0^{\ 2}}{\eta_0} \cos^2(\omega t - kz)\left[\frac{W}{m^2}\right]\end{aligned} \qquad \text{(3P)}$$

The direction of the Poynting vector, i.e. the direction of flow of power is the z direction. This is also the direction of propagation of this wave. (1P)

6. (a) What is a TEM wave? What are the characteristics of TM wave as well as of TE waves in a waveguide? What does the cut-off frequency for a specific mode in a waveguide mean? (4 P)
(b) A rectangular waveguide has internal dimensions $a = 8.64mm$ and $b = 4.32mm$. Find the lowest possible mode at which the wave may be excited and calculate its cut-off frequency. (8P)
(c) Give a sketch of the electric and magnetic field of the TM_{11} mode inside a rectangular waveguide. (3P)

Answer

(a) A TEM wave is a Transverse Electric and Magnetic wave, $E_z = 0$ and $H_z = 0$, where z is the direction of propagation; For TM waves the magnetic field is completely transverse to the direction of propagation, $E_z \neq 0$ and $H_z = 0$; for TE waves the electric field is completely transverse to the direction of propagation, $E_z = 0$ and $H_z \neq 0$. The cut-off frequency is the lowest frequency that allows for undamped propagation of waves for a specific mode. (4P)

(b) The cut-off frequency is:

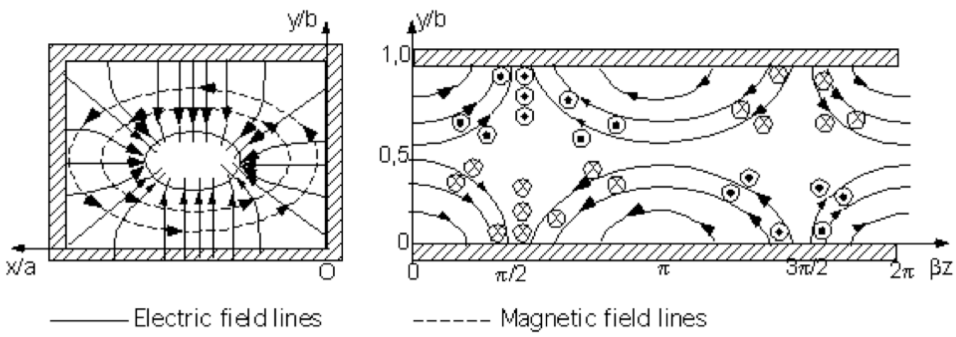
$$f_{cmn} = \frac{1}{2\pi\sqrt{\mu_0\varepsilon_0}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} = \frac{1}{2\sqrt{\mu_0\varepsilon_0}}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2} \qquad \text{(1P)}$$

$$= 1.5 \times 10^8 \sqrt{\left(\frac{m}{0.00864}\right)^2 + \left(\frac{n}{0.00432}\right)^2}\,[Hz] \qquad \text{(1P)}$$

Since m or n cannot be zero in TM modes and $a > b$, the lowest possible model should be TE_{10} mode. (4P)
Its cut-off frequency is:

$$f_{c11} = 1.5 \times 10^8 \sqrt{\left(\frac{1}{0.00864}\right)^2 + \left(\frac{0}{0.00432}\right)^2} = 17.361[GHz] \quad \text{(2P)}$$

(c)



(3P)

7. Write down the wave equation of a scalar function $\Phi(x,t)$ in the general case and translate it into a finite difference approximation. Please also give the sketch of the "computing molecule" for this case. (8P)

Answer

Wave equation: $u^2 \frac{\partial^2 \Phi}{\partial x^2} = \frac{\partial^2 \Phi}{\partial t^2}$ (1P)

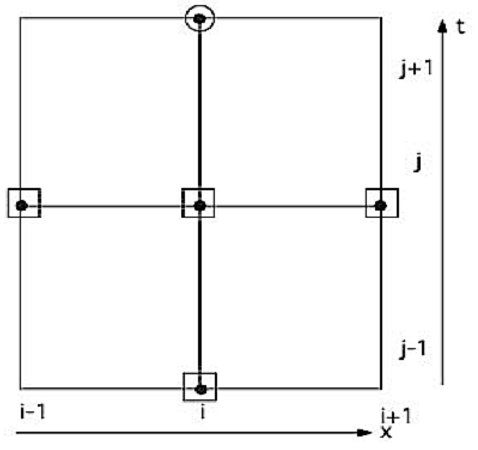
Translate in a finite difference equation:

$$u^2 \frac{\Phi(i+1,j) - 2\Phi(i,j) + \Phi(i-1,j)}{(\Delta x)^2} = \frac{\Phi(i,j+1) - 2\Phi(i,j) + \Phi(i,j-1)}{(\Delta t)^2} \quad \text{(2P)}$$

$$\Phi(i,j+1) = 2(1-r)\cdot\Phi(i,j) + r\left[\Phi(i+1,j) + \Phi(i-1,j) - \Phi(i,j-1)\right]$$

where $r = \left(\frac{u\Delta t}{\Delta x}\right)^2$ (1P)

The computing molecule used here:

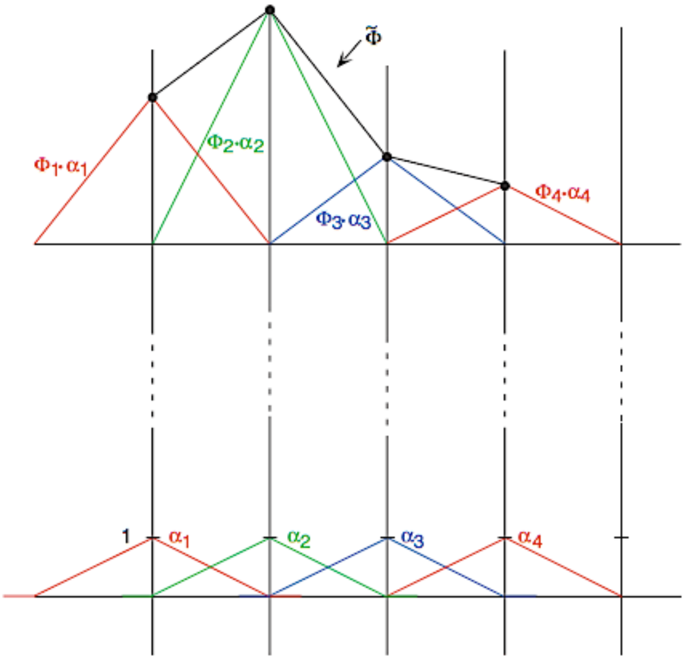


(4P)

8. (a) Describe how a scalar function (like $\Phi(x)$) can be approximated using linear node shape functions, with the aid of giving a sketch of linear node shape functions in 1D case. (4P)
(b) Describe the basic idea and equations of the finite element method cooperating with the method of weighted residual in the case of solving the Poisson's equation for piecewise homogenous materials. If the Galerkin's choice is applied here, how should the weighting function look like? Derive the equation until the second derivative of Φ is removed. (7P)

Answer

(a)



(3P)

$\tilde{\Phi} = \sum_{k=1}^{nodes} \alpha_k(x,y,z)\cdot\Phi_k$, where Φ_k is node potential. (1P)

(b)

Poisson's equation: $\Delta\Phi = -\dfrac{\rho}{\varepsilon}$ (1P)

Ideal solution: $\Delta\Phi + \dfrac{\rho}{\varepsilon} = 0$ (1P)

Approximated solution: $\Delta\tilde{\Phi} + \dfrac{\rho}{\varepsilon} = R$ (1P)

Best approximation: $\int w \cdot R dv = \int \left(\Delta\tilde{\Phi} + \dfrac{\rho}{\varepsilon}\right)\cdot w dv = 0$ (1P)

where w is weighting function.
Using Green's 2nd law:

$$\int \left(grad\tilde{\Phi}\right)\cdot(grad w)dv - \oint w\frac{\partial\tilde{\Phi}}{\partial n}da - \int \frac{\rho}{\varepsilon} w dv = 0 \quad (2P)$$

In the Galerkin's method, the weighting functions are chosen to be the basis functions: $w_l(x,y,z) = \alpha_l(x,y,z)$. (1P)

9. What are the prerequisites for accuracy in numerical field calculation? What types of accuracy have to be checked? (5P)

Answer

Prerequisites:

1) Mesh must fit to the geometry of the problem (1P)
2) Mesh must fit to the interpolation function (1P)
3) Time step must fit to the mesh (1P)

Test of accuracy:

1) Numerical accuracy has to be checked (1P)
2) “Physics accuracy” has to be checked (1P)

1	2	3	4	5	6	7	8	9	Sum
6	16	8	10	11	15	8	11	5	90