Examination
"Electromagnetics and Numerical Calculation of Fields"

March 12, 2009

1. In Cartesian coordinate system, prove that
   (a) $\nabla\times(\nabla\cdot\phi)=0$, (3P)
   (b) $\nabla\cdot(\nabla\times\vec{A})=0$, (3P)
   where ϕ and $\vec{A}$ are scalar and vector fields, respectively. (Do not use the short-cut with the "direction of Nabla".)

   Answer
   (a) $\nabla\times(\nabla\cdot\phi)=0$

$$\nabla\times(\nabla\cdot\phi)=\nabla\times\left(\vec{e}_x\frac{\partial}{\partial x}\phi+\vec{e}_y\frac{\partial}{\partial y}\phi+\vec{e}_z\frac{\partial}{\partial z}\phi\right)$$

$$=\vec{e}_x\left[\frac{\partial}{\partial y}\left(\frac{\partial}{\partial z}\phi\right)-\frac{\partial}{\partial z}\left(\frac{\partial}{\partial y}\phi\right)\right]+\vec{e}_y\left[\frac{\partial}{\partial z}\left(\frac{\partial}{\partial x}\phi\right)-\frac{\partial}{\partial x}\left(\frac{\partial}{\partial z}\phi\right)\right]+$$

$$\vec{e}_z\left[\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\phi\right)-\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\phi\right)\right] \qquad \text{(2P)}$$

$$=0\vec{e}_x+0\vec{e}_y+0\vec{e}_z \qquad \text{(1P)}$$

$$=0$$

   (b) $\nabla\cdot(\nabla\times\vec{A})=0$

$$\nabla\cdot(\nabla\times\vec{A})=\nabla\cdot\left[\vec{e}_x\left(\frac{\partial}{\partial y}A_z-\frac{\partial}{\partial z}A_y\right)+\vec{e}_y\left(\frac{\partial}{\partial z}A_x-\frac{\partial}{\partial x}A_z\right)+\vec{e}_z\left(\frac{\partial}{\partial x}A_y-\frac{\partial}{\partial y}A_x\right)\right]$$

$$=\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}A_z-\frac{\partial}{\partial z}A_y\right)+\frac{\partial}{\partial y}\left(\frac{\partial}{\partial z}A_x-\frac{\partial}{\partial x}A_z\right)+\frac{\partial}{\partial z}\left(\frac{\partial}{\partial x}A_y-\frac{\partial}{\partial y}A_x\right) \qquad \text{(1P)}$$

$$=\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}A_z\right)-\frac{\partial}{\partial x}\left(\frac{\partial}{\partial z}A_y\right)+\frac{\partial}{\partial y}\left(\frac{\partial}{\partial z}A_x\right)-\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}A_z\right)+$$

$$\frac{\partial}{\partial z}\left(\frac{\partial}{\partial x}A_y\right)-\frac{\partial}{\partial z}\left(\frac{\partial}{\partial y}A_x\right) \qquad \text{(1P)}$$

$$=\left[\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}A_z\right)-\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}A_z\right)\right]+\left[\frac{\partial}{\partial y}\left(\frac{\partial}{\partial z}A_x\right)-\frac{\partial}{\partial z}\left(\frac{\partial}{\partial y}A_x\right)\right]+$$

$$\left[\frac{\partial}{\partial z}\left(\frac{\partial}{\partial x}A_y\right)-\frac{\partial}{\partial x}\left(\frac{\partial}{\partial z}A_y\right)\right]$$

$$=0+0+0$$

$$=0 \qquad \text{(1P)}$$

2. (a) Explain Gauss' Law and Stokes' Law with formulas and text. (4P)
(b) Write down Maxwell's equations in differential and integral form. (4P)
(c) Derive the wave equation for E and H directly from Maxwell's equations (lossless case, linear and isotropic material, $\rho = 0$). (4P)

Answer

(a) Gauss' Law: $\int div\vec{A}dv = \oint \vec{A}d\vec{a}$ (1P)

The volume integral of the divergence of a vector field $\vec{A}$ equals the flux of this vector field through the surface enclosing this volume. (1P)

Stokes' Law: $\int rot\vec{A}d\vec{a} = \oint \vec{A}d\vec{l}$ (1P)

The area integral of the normal component of $rot\vec{A}$ (flux of $rot\vec{A}$) equals the line integral of $\vec{A}$ along the contour encircling the area. (1P)

(b)

$$div\vec{D} = \rho \Leftrightarrow \oint \vec{D}d\vec{a} = \int \rho dv$$

$$rot\vec{H} = \vec{J} + \dot{\vec{D}} \Leftrightarrow \oint \vec{H}d\vec{l} = \int \left(\vec{J} + \dot{\vec{D}}\right)d\vec{a}$$

$$rot\vec{E} = -\dot{\vec{B}} \Leftrightarrow \oint \vec{E}d\vec{l} = -\frac{d}{dt}\int \vec{B}d\vec{a}$$

$$div\vec{B} = 0 \Leftrightarrow \oint \vec{B}d\vec{a} = 0$$

(4P)

(c)

$$rot\vec{E} = -\mu\frac{\partial}{\partial t}\vec{H}$$

$$rot\ rot\vec{E} = -\mu\ rot\frac{\partial}{\partial t}\vec{H}$$

$$grad\ div\vec{E} - \Delta\vec{E} = -\mu\frac{\partial}{\partial t}rot\vec{H}$$

$$\Delta\vec{E} - \mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2} = 0$$

(2P)

$$rot\vec{H} = \varepsilon\frac{\partial}{\partial t}\vec{E}$$

$$rot\ rot\vec{H} = \varepsilon\ rot\frac{\partial}{\partial t}\vec{E}$$

$$grad\ div\vec{H} - \Delta\vec{H} = \varepsilon\frac{\partial}{\partial t}rot\vec{E}$$

$$\Delta\vec{H} - \varepsilon\mu\frac{\partial^2\vec{H}}{\partial t^2} = 0$$

(2P)

3. (a) What is the definition of the Green's function of a Dirichlet problem? How can we find the electric potential if the Green's function is given? (4P)
(b) What is the definition of the Green's function of a Neumann problem? How can we find the electric potential if the Green's function is given? (4P)

Answer

(a) Green's function of a Dirichlet problem:
$\Delta G_D(\vec{r},\vec{r}_0) = -\delta(\vec{r} - \vec{r}_0)$ and $G_D = 0$ on the boundary (1P)
From Green's first law it follows:

$$\int (\Phi \Delta G_D - G_D \Delta\Phi) dv = \oint \left(\Phi \frac{\partial G_D}{\partial n} - G_D \frac{\partial \Phi}{\partial n} \right) d\vec{a}$$

$$\int \left[-\Phi \delta(\vec{r} - \vec{r}_0) - G_D \cdot \left(-\frac{\rho}{\varepsilon} \right) \right] dv = \oint \Phi \frac{\partial G_D}{\partial n} d\vec{a}$$

(3P)

$$-\Phi(\vec{r}_0) + \int G_D \cdot \frac{\rho}{\varepsilon} dv = \oint \Phi \frac{\partial G_D}{\partial n} d\vec{a}$$

$$\Phi(\vec{r}_0) = \int G_D(\vec{r},\vec{r}_0) \cdot \frac{\rho}{\varepsilon} dv - \oint \Phi \frac{\partial G_D}{\partial n} d\vec{a}$$

(b) Green's function of a Neumann problem:

$\Delta G_N(\vec{r},\vec{r}_0) = -\delta(\vec{r} - \vec{r}_0)$ and $\frac{\partial G_N}{\partial n} = 0$ on the boundary (1P)

From Green's first law it follows:

$$\int (\Phi \Delta G_N - G_N \Delta\Phi) dv = \oint \left(\Phi \frac{\partial G_N}{\partial n} - G_N \frac{\partial \Phi}{\partial n} \right) d\vec{a}$$

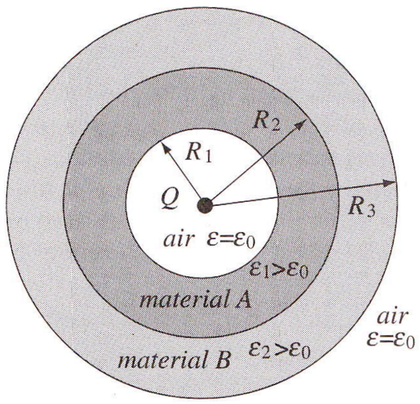
$$\int \left[-\Phi \delta(\vec{r} - \vec{r}_0) - G_N \cdot \left(-\frac{\rho}{\varepsilon} \right) \right] dv = -\oint G_N \frac{\partial \Phi}{\partial n} d\vec{a}$$

(3P)

$$-\Phi(\vec{r}_0) + \int G_N \cdot \frac{\rho}{\varepsilon} dv = -\oint G_N \frac{\partial \Phi}{\partial n} d\vec{a}$$

$$\Phi(\vec{r}_0) = \int G_N(\vec{r},\vec{r}_0) \cdot \frac{\rho}{\varepsilon} dv + \oint G_N \frac{\partial \Phi}{\partial n} d\vec{a}$$

4. A point charge Q is located at a point in space. The charge is surrounded by two spherical layers of materials as shown in the figure below. Both materials have permittivities different than free space, as indicated.
(a) Give the electric flux density and field intensity everywhere in space (with Unit). (6P)
(b) Plot the electric flux density and field intensity everywhere in space schematically. (4P)



Answer

(a) The electric flux density in all materials is radially oriented and equals

$$D = \frac{Q}{4\pi R^2} \left[\frac{C}{m^2} \right], \quad 0 < R < \infty \qquad \text{(2P)}$$

where R is the distance from the point charge Q.
The electric field intensity in any material is

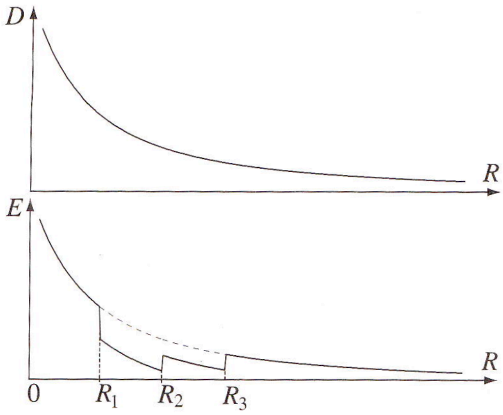
$$E = \frac{D}{\varepsilon} = \frac{Q}{4\pi\varepsilon R^2}\left[\frac{N}{C}\right] \quad \text{(2P)}$$

where ε is the permittivity of the corresponding material. For each material, we have

$$E = \frac{D}{\varepsilon_0} = \frac{Q}{4\pi\varepsilon_0 R^2}, \quad 0 < R \le R_1$$
$$E = \frac{D}{\varepsilon_1} = \frac{Q}{4\pi\varepsilon_1 R^2}, \quad R_1 < R \le R_2$$
$$E = \frac{D}{\varepsilon_2} = \frac{Q}{4\pi\varepsilon_2 R^2}, \quad R_2 < R \le R_3$$
$$E = \frac{D}{\varepsilon_0} = \frac{Q}{4\pi\varepsilon_0 R^2}, \quad R_3 < R \le \infty$$

(2P)

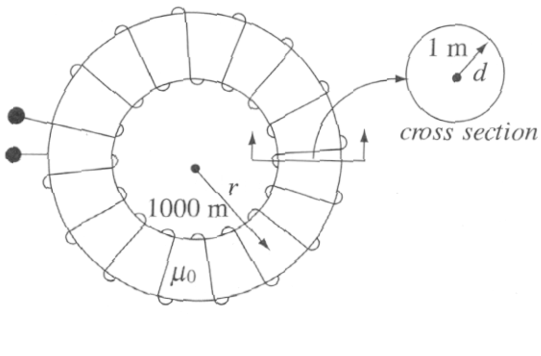
(b)



(2P)

(2P)

5. A superconducting storage ring used for powering a city is made as a toroidal coil (see the figure below). The coil's cross-sectional radius is $d = 1\ \text{m}$, the radius of the torus is $r = 1\ \text{km}$ and $\mu_r = 1$ in the torus. The torus is wound with $N = 150{,}000$ turns of superconducting wire and can carry a current $I = 100{,}000\ \text{A}$.
(a) Calculate the magnetic flux density in the torus. (2P)
(b) What is the total amount of energy stored in this torus? (5P)
(c) A city requires $100\ \text{MW}$ of power. How long can a storage ring of this type power the city in case of a blackout in power generation? Assume no losses in the conversion and the transportation of energy. (3P)



(Hint: the magnetic field strength in the torus can be considered constant along the radial direction approximately, since $r >> d$.)

Answer

(a) The flux density in the torus is

$$B = \frac{\mu_0 NI}{2\pi r} = \frac{4\times\pi\times10^{-7}\times150{,}000\times100{,}000}{2\times\pi\times1000} = 3\ [\mathrm{T}]\quad \text{(2P)}$$

(b) The total flux in the torus is

$$\Phi = BS = \frac{\pi d^2\mu_0 NI}{2\pi r} = \frac{d^2\mu_0 NI}{2r}\ [\mathrm{Wb}]\quad \text{(1P)}$$

The total flux linkage is

$$\Lambda = N\Phi = \frac{d^2\mu_0 N^2 I}{2r}\ [\mathrm{Wbt}]\quad \text{(1P)}$$

The self-inductance of the storage ring is

$$L = \frac{\Lambda}{I} = \frac{d^2\mu_0 N^2}{2r}\ [\mathrm{H}]\quad \text{(1P)}$$

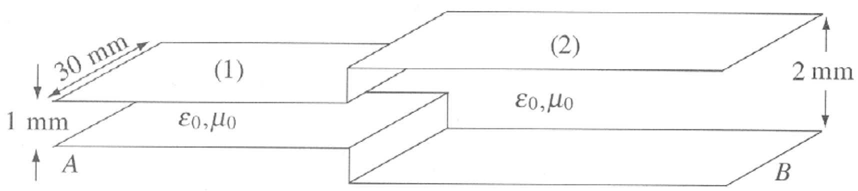
The stored energy in the ring is

$$W = \frac{LI^2}{2} = \frac{d^2\mu_0 N^2 I^2}{4r} = \frac{1^2\times4\times\pi\times10^{-7}\times2.25\times10^{10}\times10^{10}}{4\times1000} = 7.07\times10^{10}\ [\mathrm{J}]\quad \text{(2P)}$$

(c) Since energy is power integrated over time and the city requires $100\ \mathrm{MW}$, the energy needs of the city may be met for

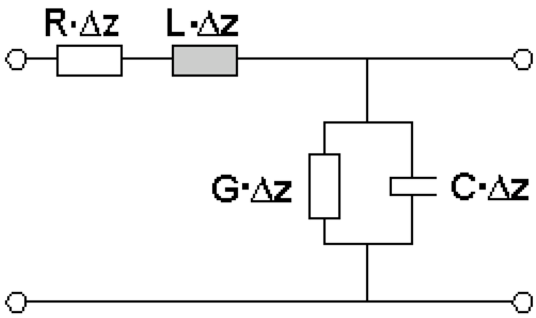
$$t = \frac{W}{P} = \frac{7.07\times10^{10}}{1\times10^{8}} = 707\ [\mathrm{s}]\quad \text{(3P)}$$

6. (a) Give the equivalent circuit of a two-conductor lossy transmission line for a differential length Δz. (3P)
(b) Write down the wave equation for lossy transmission lines. (4P)
(c) Give a sketch of fields, charge and current distribution along a coaxial transmission line. (4P)
(d) Two striplines are connected as shown in the figure below. Both striplines are 30 mm wide. Assuming TM propagation, calculate the lowest frequency that can be propagated if the source is connect on side *A* of the structure and the lowest frequency that can be propagated if the source is connected on side *B* of the structure. (4P)



Answer

(a)



(3P)

(b)

$$-\frac{dV(z)}{dz} = (R + j\omega L)I(z) \qquad \text{(1P)}$$

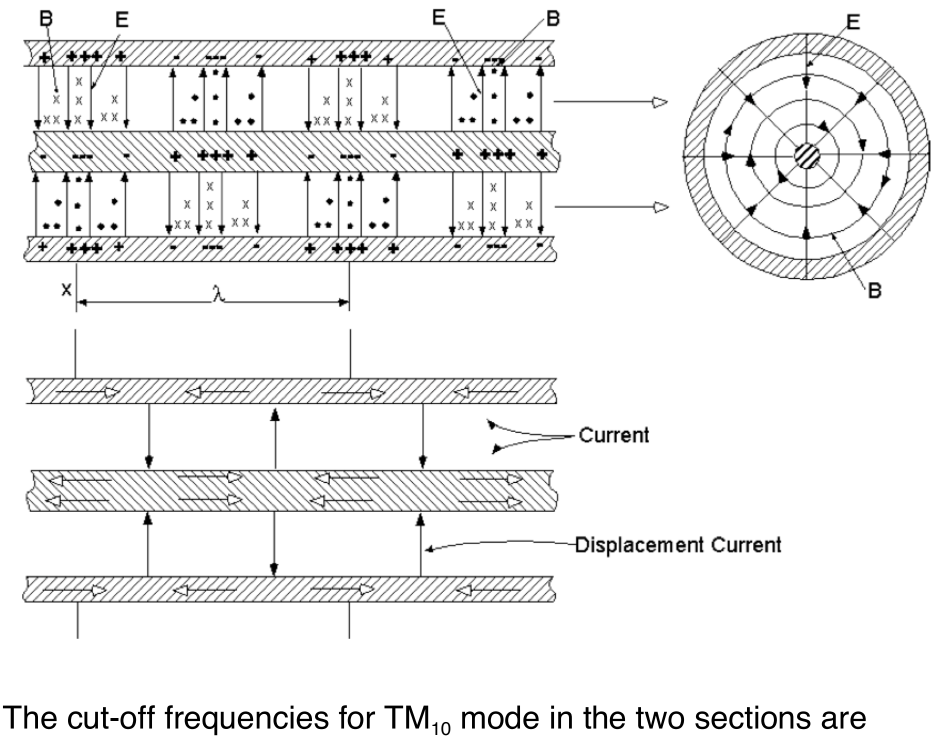
$$-\frac{dI(z)}{dz} = (G + j\omega C)V(z) \qquad \text{(1P)}$$

$$\frac{d^2V(z)}{dz^2} = \gamma^2 V(z) \qquad \text{(1P)}$$

$$\frac{d^2I(z)}{dz^2} = \gamma^2 I(z) \qquad \text{(1P)}$$

where $\gamma = \alpha + j\beta = \sqrt{(R + j\omega L)(G + j\omega C)}$

(c)



(4P)

(d) The cut-off frequencies for TM_{10} mode in the two sections are in the small guide:

$$f_{c1}^{(1)} = \frac{1}{2d\sqrt{\mu_0\varepsilon_0}} = \frac{3\times10^8}{2\times0.001} = 150\ \text{[GHz]} \quad \text{(1P)}$$

in the large guide:

$$f_{c1}^{(2)} = \frac{1}{2d\sqrt{\mu_0\varepsilon_0}} = \frac{3\times10^8}{2\times0.002} = 75\ \text{[GHz]}$$ (1P)

Since the small guide can only propagate above 150 GHz and the large guide above 75 GHz, the combined structure can only propagate above 150 GHz regardless of which side the source is connected on. (2P)

7. Write down the diffusion equation of a scalar function $\Phi(x,t)$ in the general case and translate it into a finite difference approximation (2D). Give the sketch of the "computing molecule" for this case. (7P)

Answer

Diffusion equation: $k\frac{\partial\Phi}{\partial t} = \frac{\partial^2\Phi}{\partial x^2}$ (1P)

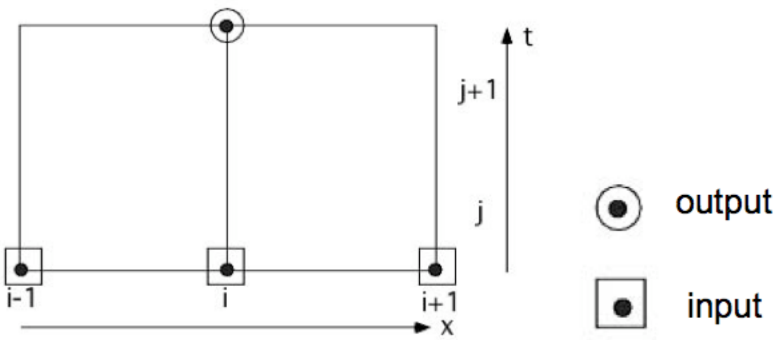
Translate it in a finite difference equation:

$$k\frac{\Phi(i,j+1)-\Phi(i,j)}{\Delta t} = \frac{\Phi(i+1,j)-2\Phi(i,j)+\Phi(i-1,j)}{(\Delta x)^2}$$ (2P)

$$\Phi(i,j+1) = r\Phi(i+1,j)+(1-2r)\Phi(i,j)+r\Phi(i-1,j)$$

where $r = \frac{\Delta t}{k(\Delta x)^2}$ (1P)

The computing molecule used here:

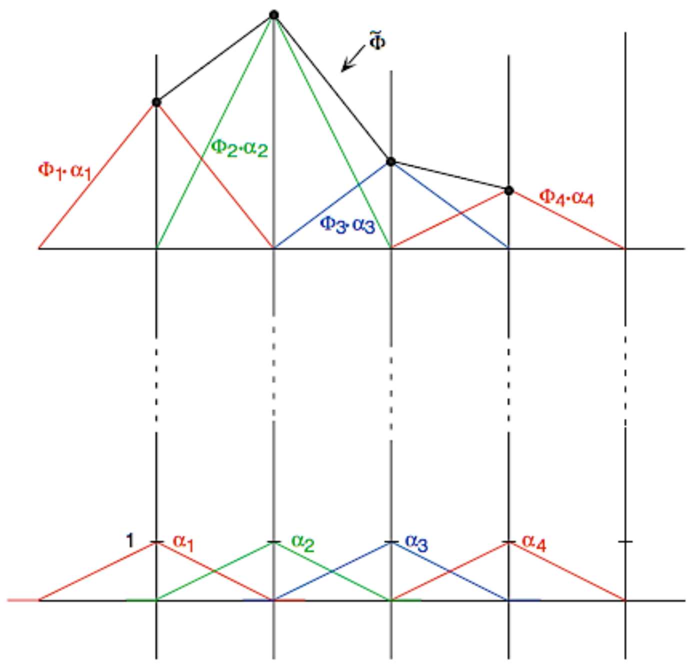


(3P)

8. (a) Describe how a scalar function (like $\Phi(x)$) can be approximated using linear node shape functions, with the aid of giving a sketch of linear node shape functions in 1D case. (4P)
(b) Describe the basic idea and equations of the finite element method cooperating with the method of weighted residual in the case of solving the Poisson's equation for piecewise homogenous materials. If the Galerkin's choice is applied here, how should the weighting function look like? Derive the equation until the second derivative of Φ is removed. (7P)

Answer

(a)



(3P)

$\tilde{\Phi} = \sum_{k=1}^{nodes} \alpha_k(x,y,z) \cdot \Phi_k$, where Φ_k is node potential. (1P)

(b) Poisson's equation: $\Delta\Phi = -\frac{\rho}{\varepsilon}$ (1P)

Ideal solution: $\Delta\Phi + \frac{\rho}{\varepsilon} = 0$ (1P)

Approximated solution: $\Delta\tilde{\Phi} + \frac{\rho}{\varepsilon} = R$ (1P)

Best approximation: $\int w \cdot R dv = \int \left(\Delta\tilde{\Phi} + \frac{\rho}{\varepsilon} \right) \cdot w dv = 0$ (1P)

where w is weighting function.
Using Green's 2[nd] law:

$\int \left(grad\tilde{\Phi} \right) \cdot (grad w) dv - \oint w \frac{\partial \tilde{\Phi}}{\partial n} da - \int \frac{\rho}{\varepsilon} w dv = 0$ (2P)

In the Galerkin's method, the weighting functions are chosen to be the basis functions: $w_l(x,y,z) = \alpha_l(x,y,z)$. (1P)

9. Comment on the typical number of nodes for the same type of problem but using FDM, FEM, BEM. Compare the matrices that describe the set of linear equations to be solved. (6P)

Answer
Since FDM uses a regular grid, the number of nodes is usually the largest, especially in case of very irregular geometries. (1P)
FEM can use e.g. tetrahedrons with variable size and is very flexible to approximate irregular geometries with a small number of nodes. But the complete volume has to be discretisized. (1P)
BEM only demands for a discretization of the surfaces and not the volumes. So the number of nodes is usually the smallest. (1P)

FDM usually ends up with a very regular and sparse. (1P)
FEM also ends up with a sparse matrix. Some tricks are needed to make the matrix diagonal dominant. (1P)
BEM delivers a matrix with the smallest rank, but it is not sparse at all. All elements are non-zero. (1P)

1	2	3	4	5	6	7	8	9	Sum
6	12	8	10	10	15	7	11	6	85