Karlsruhe Institute of Technology

Examination 2010

„Electromagnetics and Numerical Calculation of Fields“

March 9, 2010

Name: ______________________________

Student number: ______________________________

Exercise 1:

a.) Given is the potential $\phi = \frac{1}{4\pi\epsilon_0} \frac{q}{\sqrt{x^2 + y^2 + z^2}}$

Calculate the electric Field $\vec{E}$. (3 pt)

b.) Given is the electric flux density $\vec{D} = \begin{pmatrix} 4(x-y) \\ 3(x+2y) \\ 2z \end{pmatrix}$.

Calculate the charge density ρ. (3 pt)

Exercise 2:

(a) What type of equation has to be solved for finding the electric potential ϕ of an arbitrary charge distribution $\rho(x, y, z)$? (2 pt)

(b) What are Dirichlet boundary conditions and Neumann boundary conditions? (4 pt)

Exercise 3:

a.) Write down Gauss‘ law and Stokes‘ law and and show how to translate the Maxwell‘s equations from differential form to integral form. (6 pt)

b.) Use the Gauss‘ law to prove the

first: $$\int_S \phi(\nabla\psi)d\vec{a} = \int_V (\phi\Delta\psi) + (\nabla\phi)(\nabla\psi)dv$$

and second Green‘s identity: $$\int_S \psi\nabla\phi - \phi\nabla\psi d\vec{a} = \int_V \psi\Delta\phi - \phi\Delta\psi dv$$. (6 pt)

Hint: $\nabla(\phi\nabla\psi) = \phi\Delta\psi + (\nabla\phi)(\nabla\psi)$

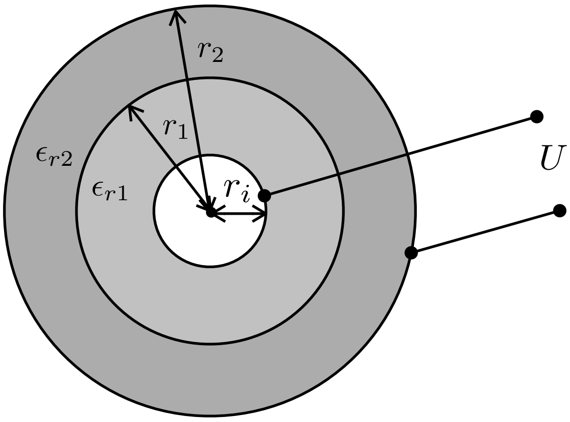
Exercise 4:

Derive the wave equation for $\vec{E}$ and $\vec{H}$ directly from the Maxwell‘s equations for vacuum ($\rho = 0,\ \vec{j} = 0$). (6 pt)

Hint: $\text{rot rot } \vec{F} = \text{grad div } \vec{F} - \Delta\vec{F}$

Exercise 5:

Find the capacitance of a spherical capacitor (inner radius r_i, the outer radii r_1, r_2 and the relative dielectric constants ϵ_{r1}, ϵ_{r2}). (8 pt)



Exercise 6:

a.) Write down the sinusoidal plane-wave solutions of the electromagnetic wave equation for $\vec{E}$ and $\vec{H}$ (two types of solutions). (4 pt)

b.) How can you find E_0 if H_0 is given ? (2 pt)

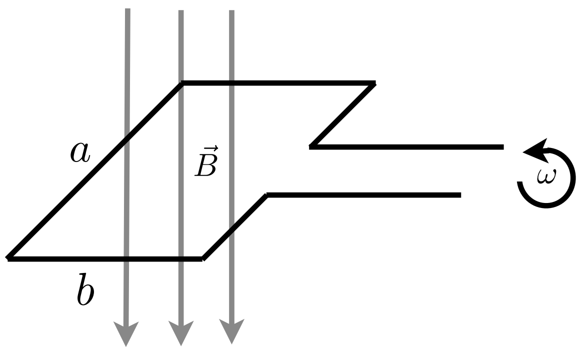
Exercise 7:

a.) What is the definition of the magnetic flux ϕ_{mag} and how can it be found directly from the magnetic vector potential (use Stokes‘ law !) (4 pt)

b.) Assume a thin wire of arbitrary shape is given together with the current I in the wire. How can you find the magnetic vector potential ? (4 pt)

Exercise 8:

A rectangular coil of length a and width b is turning with constant angular velocity ω inside a homogenous field of magnetic induction $\vec{B}$ (see figure) . How large is the induced voltage? What is the time course of the voltage? (4 pt)



Exercise 9:

a.) Write down the central difference approximation used in the Finite Difference Method (FDM) for:

$$\frac{\partial \Phi}{\partial x} \quad \text{and} \quad \frac{\partial^2 \Phi}{\partial x^2}. \text{ (4 pt)}$$

b.) Give a sketch of the computing molecule for the Laplace/Poisson equation (FDM).
Caption the axes.
Mark the values that are used to calculate the next value.
What does the computing molecule deliver ?
What do you do at the boundaries ? (8 pt)

Exercise 10:

a.) Describe the basic idea and equations of the Finite Element Method using Weighted Residuals with Poisson‘s Equation as an example.
What is the Residuum? What was Galerkin's idea? (8 pt)

b). Describe how a scalar function ϕ can be approximated using linear node shape functions.
Give a sketch of the linear shape functions in 1D case. (6 pt)

Examination Result:

Ex 1	Ex 2	Ex 3	Ex 4	Ex 5	Ex 6	Ex 7	Ex 8	Ex 9	Ex 10	SUM
6	6	12	6	8	6	8	4	12	14	**82**

GRADE: ________

Solutions for Examination 2010

Electromagnetics and Numerical Calculations of Field

1a)
$$\phi = \frac{1}{4\pi\epsilon_0}\frac{q}{\sqrt{x^2+y+^2+z^2}}$$

$$\vec{E} = -\nabla\phi$$

$$\frac{\partial\phi}{\partial x} = -\frac{1}{4\pi\epsilon_0}\frac{2xq}{2(\sqrt{x^2+y^2+z^2})^3}$$

$$\frac{\partial\phi}{\partial y} = -\frac{1}{4\pi\epsilon_0}\frac{2yq}{2(\sqrt{x^2+y^2+z^2})^3}$$

$$\frac{\partial\phi}{\partial z} = -\frac{1}{4\pi\epsilon_0}\frac{2zq}{2(\sqrt{x^2+y^2+z^2})^3}$$

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{(\sqrt{x^2+y^2+z^2})^3}\begin{pmatrix}x\\y\\z\end{pmatrix}$$

1b)

$$\nabla\vec{D} = \partial_x D_x + \partial_y D_y + \partial_z D_z = 4+6+2 = 12$$

2a) Poisson equation: $\Delta\phi = -\frac{\rho}{\epsilon_0}$

or Coulomb Integral $\Phi(\vec{r}) = \frac{1}{4\pi\varepsilon}\iiint\frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}dv'$

2b) Neumann: $\frac{\partial\phi}{\partial n}|_{boundary} = \hat{\phi}_n$

On the boundaries the normal derivative of the potential is given.

Dirichlet: $\phi|_{boundary} = \hat{\phi}$

On the boundaries the potential is given.

3a)

Gauss: $\int_V div\vec{A}dv = \int_S \vec{A}d\vec{a}$

Stokes: $\int_S rot\vec{A}d\vec{a} = \int_s \vec{A}d\vec{s}$

Maxwell:

$$div\vec{D} = \rho \rightarrow \int div\vec{D}dv = \int \vec{D}d\vec{a} = \int \rho dv$$

$$div\vec{B} = 0 \rightarrow \int div\vec{B}dv = \int \vec{B}d\vec{a} = 0$$

$$rot\vec{E} = -\frac{\partial}{\partial t}\vec{B} \rightarrow \int rot\vec{E}d\vec{a} = \int \vec{E}d\vec{s} = -\frac{\partial}{\partial t}\int \vec{B}d\vec{a}$$

$$rot\vec{H} = \vec{J} + \frac{\partial}{\partial t}\vec{D} \rightarrow \int rot\vec{H}d\vec{a} = \int \vec{H}d\vec{s} = \int (\vec{J} + \frac{\partial}{\partial t}\vec{D})d\vec{a}$$

3b)

First Green‘s Identity

$$\int_V (\phi\Delta\psi + (\nabla\phi)(\nabla\psi)dv = \int_V \nabla(\phi\nabla\psi)dv = \int_S \phi\nabla\psi d\vec{a}$$

Second Green‘s Identitiy

$$\begin{aligned}\int_S \psi\nabla\phi - \phi\nabla\psi d\vec{a} &= \int_V \nabla(\psi\nabla\phi - \phi\nabla\psi)dv \\ &= \int_V \psi\Delta\phi + (\nabla\psi)(\nabla\phi) - \phi\Delta\psi - (\nabla\phi)(\nabla\psi)dv \\ &= \int_V \psi\Delta\phi - \phi\Delta\psi dv\end{aligned}$$

4)

$$rot\vec{E} = -\mu \frac{\partial}{\partial t}\vec{H}$$

$$rot\, rot\vec{E} = -\mu\, rot \frac{\partial}{\partial t}\vec{H}$$

$$grad\, div\vec{E} - \Delta\vec{E} = -\mu \frac{\partial}{\partial t} rot\vec{H}$$

since $div\vec{E} = 0$

$$\rightarrow \Delta\vec{E} - \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} = 0$$

$$rot\vec{H} = \varepsilon \frac{\partial}{\partial t}\vec{E}$$

$$rot\, rot\vec{H} = \varepsilon\, rot \frac{\partial}{\partial t}\vec{E}$$

$$grad\, div\vec{H} - \Delta\vec{H} = \varepsilon \frac{\partial}{\partial t} rot\vec{E}$$

since $div\vec{H} = 0$

$$\rightarrow \Delta\vec{H} - \varepsilon\mu \frac{\partial^2 \vec{H}}{\partial t^2} = 0$$

5)

The spherical capacitor can be treated as two partial spherical capacitors in series.

$$\frac{1}{C_{total}} = \frac{1}{C_1} + \frac{1}{C_2}$$

$$U_1 = \int_{r_i}^{r_1} E dr = \int_{r_i}^{r_1} \frac{Q}{4\pi\epsilon_{r1}\epsilon_0}\frac{1}{r^2} dr = \frac{Q}{4\pi\epsilon_{r1}\epsilon_0}\left(\frac{1}{r_i} - \frac{1}{r_1}\right)$$

$$C_1 = \frac{4\pi\epsilon_{r1}\epsilon_0}{\frac{1}{r_i} - \frac{1}{r_1}}$$

$$U_2 = \int_{r_1}^{r_2} E dr = \int_{r_1}^{r_2} \frac{Q}{4\pi\epsilon_{r2}\epsilon_0}\frac{1}{r^2} dr = \frac{Q}{4\pi\epsilon_{r2}\epsilon_0}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

$$C_2 = \frac{4\pi\epsilon_{r2}\epsilon_0}{\frac{1}{r_1} - \frac{1}{r_2}}$$

$$C_{total} = \frac{4\pi\epsilon_0\epsilon_{r1}\epsilon_{r2}}{\epsilon_{r2}(\frac{1}{r_i} - \frac{1}{r_1}) + \epsilon_{r1}(\frac{1}{r_1} - \frac{1}{r_2})}$$

6)

$$\begin{aligned} E_y &= E_0 e^{j(wt-kx)} \\ H_z &= H_0 e^{j(wt-kx)} \end{aligned}$$

or

$$\begin{aligned} E_y &= E_0 e^{j(wt+kx)} \\ H_z &= -H_0 e^{j(wt+kx)} \end{aligned}$$

6b) $$E_0 = \Gamma H_0 = \sqrt{\frac{\mu}{\epsilon}} H_0$$

7a)

Magnetic Flux:

$$\begin{aligned} \phi_m &= \int_S \vec{B} d\vec{a} \\ \vec{B} &= rot \vec{A} \\ \Rightarrow \phi_m &= \int_S rot \vec{A} d\vec{a} = \int_S \vec{A} d\vec{s} \end{aligned}$$

7b)

Bio-Savart-Law:

$$\vec{A}(r) = \frac{\mu I}{4\pi} \int_S \frac{1}{r - r'} dv'$$

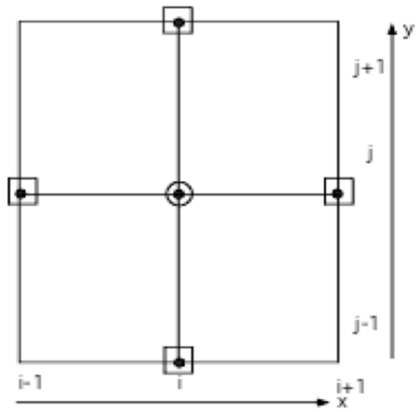
8)

$$U_{ind} = -\frac{d}{dt} \int_S \vec{B} d\vec{a} = -\frac{d}{dt}(Babcos(\omega)) = Bab\omega sin(\omega t)$$

9a)

$$\left.\frac{\partial \Phi}{\partial x}\right|_{ij} \cong \frac{\Phi(i+1,j) - \Phi(i-1,j)}{2\Delta x}$$

$$\left.\frac{\partial^2 \Phi}{\partial x^2}\right|_{ij} \cong \frac{\Phi(i+1,j) - 2\Phi(i,j) + \Phi(i-1,j)}{(\Delta x)^2}$$



The computing molecule delivers one linear equation for each node
=> A linear equation system Ax = g has to be solved for the whole system. Matrix A has size n*n for n nodes;

At the boundary, Dirichlet, Neumann or Mixed boundary conditions have to be taken into account

10a)

Poisson's equation: $\Delta\Phi = -\frac{\rho}{\varepsilon}$

Ideal solution: $\Delta\Phi + \frac{\rho}{\varepsilon} = 0$

Approximated solution: $\Delta\tilde{\Phi} + \frac{\rho}{\varepsilon} = R$ R is the Residuum.

Best approximation: $\int w \cdot R dv = \int \left(\Delta\tilde{\Phi} + \frac{\rho}{\varepsilon} \right) \cdot w dv = 0$

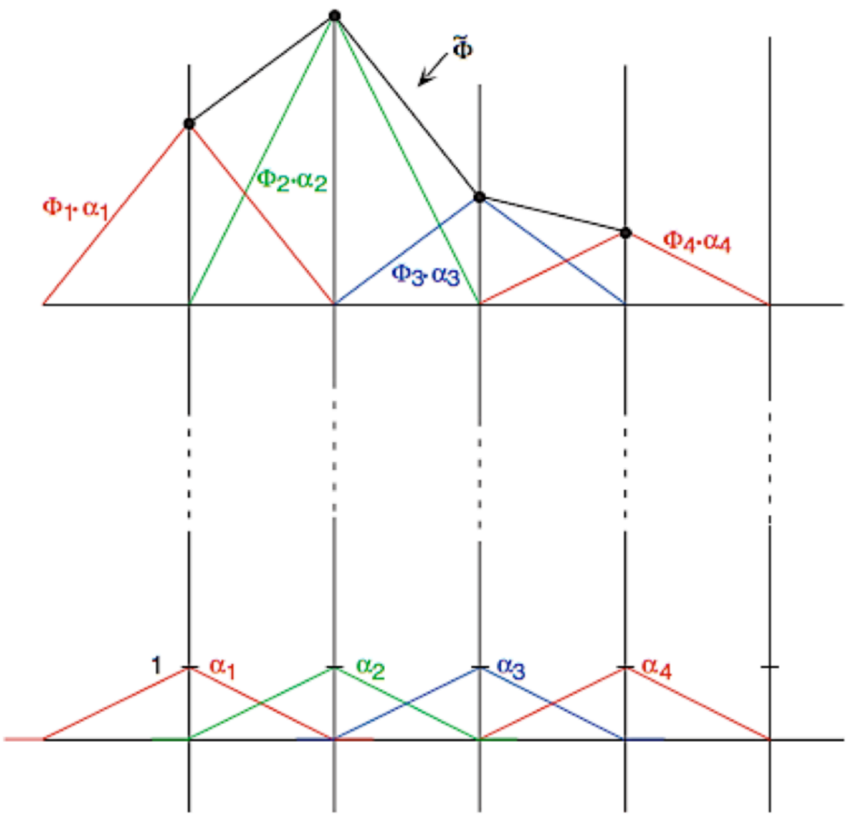
where w is weighting function.

Using Green's 2nd law:

$$\int \left(grad\tilde{\Phi} \right) \cdot (grad w) dv - \oint w \frac{\partial \tilde{\Phi}}{\partial n} da - \int \frac{\rho}{\varepsilon} w dv = 0$$

In the Galerkin's method, the weighting functions are chosen to be the basis functions (node shape functions): $w_l(x,y,z) = \alpha_l(x,y,z)$.

10b)



$$\tilde{\Phi} = \sum_{k=1}^{nodes} \alpha_k(x,y,z) \cdot \Phi_k$$, where Φ_k is a node potential.