Electromagnetics and Numerical Calculation of Fields

Lecture: Prof. Dr. rer nat Olaf Dössel

Examination 2011

Question 1

Write down the Maxwell's equations in differential and integral form (arbitrary materials). [8 pt]

Question 2

$\mathbf{E}_1$ and $\mathbf{E}_2$ are given by

$$\text{a)} \qquad \mathbf{E}_1 = k \cdot \begin{pmatrix} yz \\ xy \\ 2xz \end{pmatrix}$$

$$\text{b)} \qquad \mathbf{E}_2 = k \cdot \begin{pmatrix} y^2 \\ 2xy + z \\ y \end{pmatrix}$$

where k is a constant with appropriate units.

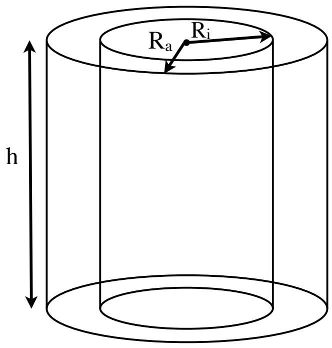
(a) Which of these is an electrostatic field. *(What was that again about the curl...)* [4 pt]

(b) For the electrostatic field, find the potential $\phi(x_0, y_0, z_0)$. Use the origin as your reference point. *Hint: You must integrate along an arbitrary path from* $(0,0,0)$ *to* (x_0, y_0, z_0). [8 pt]

(c) Check the results by calculating $\mathbf{E} = -\nabla\phi$. [2 pt]

Question 3

Find the capacitance of a cylindrical capacitor with inner radius R_i, outer radius R_a, Length L where $L >> R_a$ and the dielectric constant $\epsilon_r = 1$ (Vacuum) [6 pt]



Question 4

Find the electrostatic force between the plates of a parallel plate capacitor (area A, distance s and dielectric constant $\epsilon_r = 1$) via the electric field energy. [4 pt]

Question 5

(a) Give the "lumped element" circuit of one piece of transmission line (Transmission line model). [4 pt]

(b) Give the formula for the characteristic impedance of a lossless transmission line ? [2 pt]

(c) Give the formula for the phase velocity of a wave propagating in a lossless transmission line ? [2 pt]

(d) What do you have to do to avoid reflection at the end of a terminated transmission line. [2 pt]

Question 6

(a) What is characteristic for TM waves in a waveguide and what is characteristic for TE waves in a waveguide ? [4 pt]

(b) What is the meaning of "cut-off frequency" for a specific mode ? [2 pt]

Question 7

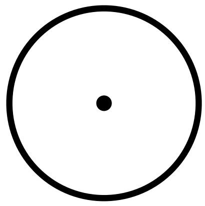
(a) What is the meaning of the Poynting vector **S** ? [2 pt]

(b) Give the formular to determine **S** from a given electromagnetic field. [2 pt]

(c) Give a sketch of the electric and magnetic field lines of a wave propagating in a coaxial cable (Sketch of the cross section of the coaxial cable is sufficient !). [4 pt]

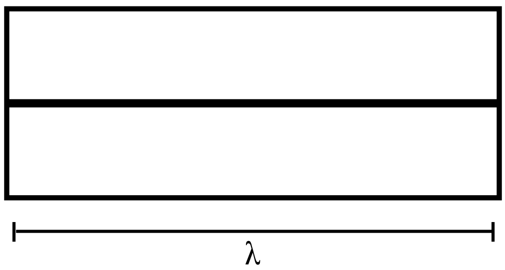
(d) Give a sketch of the Poynting vector along a piece of one wavelength of a coaxial line (Sketch of the side view of the coaxial cable) [2 pt]

Coaxial cable cross section



Give a sketch of **E** and **B**

Coaxial cable side view



Give a sketch of **S**

Question 8

Derive the general wave equation for conducting, dielectric, paramagnetic, isotropic and linear materials with charge density $\rho = 0$ from the Maxwell equations. [10 pt]
Hint: $rot\, rot\, \mathbf{F} = grad\, div\, \mathbf{F} - \Delta \mathbf{F}$

Question 9

Give a sketch of Yee's lattice (FDTD). [6 pt]

Question 10

(a) Describe the basic idea and equations of the Finite Element Method using Weighted Residuals with Poisson's Equation as an example. [8 pt]
- What is the Residuum?
- Show how to find the best approximation.
- Show how to apply the Green's 2nd identity to obtain the weak form without second derivatives.
- What was Galerkin's idea?
- Give a formula, how to approximate a scalar function ϕ using linear node shape functions.

Hint: $\int (\psi \Delta \phi + (\nabla \psi)(\nabla \phi)) dv = \int \psi (\nabla \phi) d\mathbf{a}$

(b) Give a sketch of the linear shape functions in 1D case. [4 pt]

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Examination 2011: Solutions

Question 1

$$\nabla \cdot \mathbf{D} = \rho \leftrightarrow \int_S \mathbf{D} d\mathbf{a} = \int \rho dV \tag{1}$$

$$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}\mathbf{B} \leftrightarrow \int_C \mathbf{E} d\mathbf{l} = -\frac{\partial}{\partial t}\int_S \mathbf{B} d\mathbf{a} \tag{2}$$

$$\nabla \cdot \mathbf{B} = 0 \leftrightarrow \int_S \mathbf{B} d\mathbf{a} = 0 \tag{3}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial}{\partial t}\mathbf{D} \leftrightarrow \int_C \mathbf{H} d\mathbf{l} = \int_S (\mathbf{J} + \frac{\partial}{\partial t}\mathbf{D}) d\mathbf{a} \tag{4}$$

Question 2

(a)

$$\nabla \times \mathbf{E}_1 = k \begin{pmatrix} 0-0 \\ y-2z \\ y-z \end{pmatrix} \neq 0 \rightarrow \text{Impossible electrostatic field !}$$

$$\nabla \times \mathbf{E}_2 = k \begin{pmatrix} 1-1 \\ 0-0 \\ 2y-2y \end{pmatrix} = 0 \rightarrow \text{Possible electrostatic field !}$$

(b) Proposed Path:

$$\text{Step I: } y = z = 0 \rightarrow \int \mathbf{E}_2\, d\mathbf{l} = \int ky^2 dx = 0 (\text{since y=0})$$

$$\text{Step II: } x = x_0, y: 0 \rightarrow y_0, z = 0 \rightarrow \int \mathbf{E}_2\, d\mathbf{l} = 2kx_0 \int_0^{y_0} ydy = kx_0 y_0^2$$

$$\text{Step III: } x = x_0, y = y_0, z: 0 \rightarrow z_0 \rightarrow \int \mathbf{E}_2\, d\mathbf{l} = ky_0 \int_0^{z_0} 1dz = ky_0 z_0$$

$$\rightarrow V(x_0, y_0, z_0) = -\int \mathbf{E}_2 d\mathbf{l} = -\sum_{\text{Step I - III}} = -k(x_0 y_0^2 + y_0 z_0)$$

$$\text{or } V(x, y, z) = -k(xy^2 + yz)$$

(c)

$$-\nabla \cdot \mathbf{V} = k \begin{pmatrix} y^2 \\ 2xy + z \\ y \end{pmatrix} = \mathbf{E}_2 \surd$$

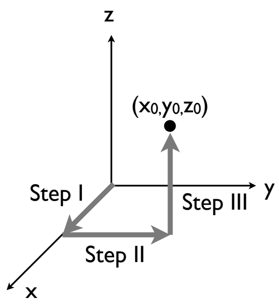


Figure 1: Proposed Path

Question 3

$$\int \mathbf{E} d\mathbf{a} = \frac{Q}{\epsilon_0} \tag{5}$$

$$\rightarrow E_r = \frac{Q}{2\pi\epsilon_0 L}\frac{1}{r} \tag{6}$$

$$U = \int_{R_i}^{R_a} E_r dr = \frac{Q}{2\pi\epsilon_0 L} ln\left(\frac{R_a}{R_i}\right) \tag{7}$$

$$C = \frac{Q}{U} \tag{8}$$

$$\rightarrow C = \frac{2\pi\epsilon_0 L}{ln\left(\frac{R_a}{R_i}\right)} \tag{9}$$

$$\tag{10}$$

Question 4

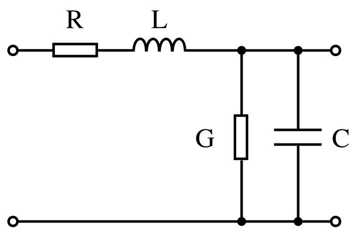
$$W_e = \frac{1}{2}CU^2 = \frac{1}{2}\frac{Q^2}{C} \tag{11}$$

$$C = \frac{\epsilon_0 A}{s} \tag{12}$$

$$F_Q = \left|-\frac{\partial}{\partial s}\left(\frac{1}{2}\frac{s}{\epsilon_0 A}Q^2\right)\right| = \frac{1}{2}\frac{1}{\epsilon_0 A}Q^2 \tag{13}$$

Question 5

(a) Transmission line:



(b)

$$Z_0 = \sqrt{\frac{L}{C}} \tag{14}$$

(c)

$$v = \frac{1}{\sqrt{LC}} \tag{15}$$

(d) The load impedance has to be equal to the characteristic impedance of the transmission line.

Question 6

(a) For TE waves the electric field is completely transverse to the direction of propagation, $E_z = 0$. For TM waves the magnetic field is completely transverse to the direction of propagation, $H_z = 0$.

(b) For every mode, the cut-off frequency is the lowest frequency that allows undamped propagation.

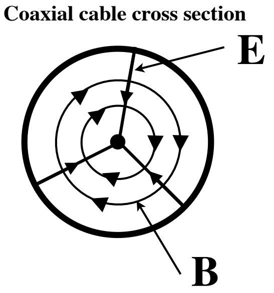
Question 7

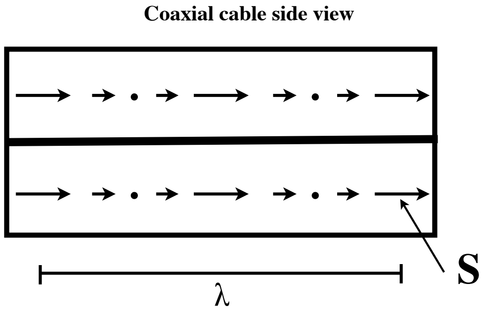
(a) The Poynting vector is the power per area that is transmitted through a given area via electromagnetic fields.

(b) Power flux density:

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} \tag{16}$$

(c) and (d)



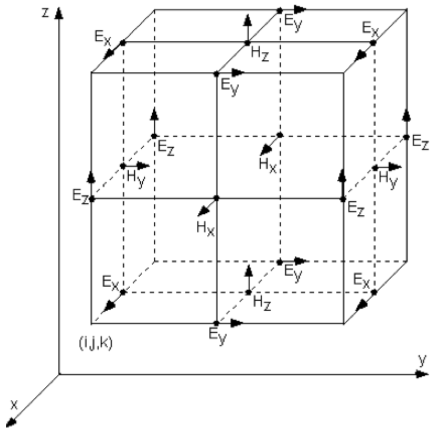


Question 8

$$\mathbf{D} = \epsilon_0 \epsilon \mathbf{E} = \epsilon \mathbf{E} \tag{17}$$
$$\nabla \times \mathbf{E} = -\mu \frac{\partial}{\partial t}\mathbf{H} \tag{18}$$
$$\nabla \times \nabla \times \mathbf{E} = -\mu \nabla \times \frac{\partial}{\partial t}\mathbf{H} \tag{19}$$
$$\nabla\nabla\mathbf{E} - \Delta\mathbf{E} = -\mu \frac{\partial}{\partial t}\nabla \times \mathbf{H} \tag{20}$$
$$\nabla\mathbf{E} = 0 \tag{21}$$
$$\rightarrow \Delta\mathbf{E} - \mu \frac{\partial}{\partial t}(\mathbf{J} + \epsilon \frac{\partial}{\partial t}\mathbf{E}) = 0 \tag{22}$$
$$\Delta\mathbf{E} - \mu\kappa \frac{\partial}{\partial t}\mathbf{E} - \mu\epsilon \frac{\partial^2}{\partial t^2}\mathbf{E} = 0 \tag{23}$$

$$\mathbf{H} = \frac{1}{\mu_0 \mu_r}\mathbf{B} = \frac{1}{\mu}\mathbf{B} \tag{24}$$
$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial}{\partial t}\mathbf{D} \tag{25}$$
$$\nabla \times \nabla \times \mathbf{H} = \kappa \nabla \times \mathbf{E} + \epsilon \nabla \times \frac{\partial}{\partial t}\mathbf{E} \tag{26}$$
$$\nabla\nabla\mathbf{H} - \Delta\mathbf{H} = -\mu\kappa \frac{\partial}{\partial t}\mathbf{H} + \epsilon \frac{\partial}{\partial t}\nabla \times \mathbf{E} \tag{27}$$
$$\nabla\nabla\mathbf{H} - \Delta\mathbf{H} = -\mu\kappa \frac{\partial}{\partial t}\mathbf{H} - \mu\epsilon \frac{\partial^2}{\partial t^2}\mathbf{H} \tag{28}$$
$$\nabla\mathbf{H} = 0 \tag{29}$$
$$\rightarrow \Delta\mathbf{H} - \mu\kappa \frac{\partial}{\partial t}\mathbf{H} - \mu\epsilon \frac{\partial^2}{\partial t^2}\mathbf{H} = 0 \tag{30}$$
$$\tag{31}$$

Question 9



Question 10

(a) Poisson's equation:

$$\Delta\phi = -\frac{\rho}{\epsilon} \tag{32}$$

Ideal solution:

$$\Delta\phi + \frac{\rho}{\epsilon} = 0 \tag{33}$$

Approximated solution:

$$\Delta\tilde{\phi} + \frac{\rho}{\epsilon} = R \tag{34}$$

where R is the Residuum
Best approximation:

$$\int \omega \cdot R dv = \int \left(\Delta\tilde{\phi} + \frac{\rho}{\epsilon}\right)\omega dv = 0 \tag{35}$$

where ω is the weighting function
Using Green's 2nd law:

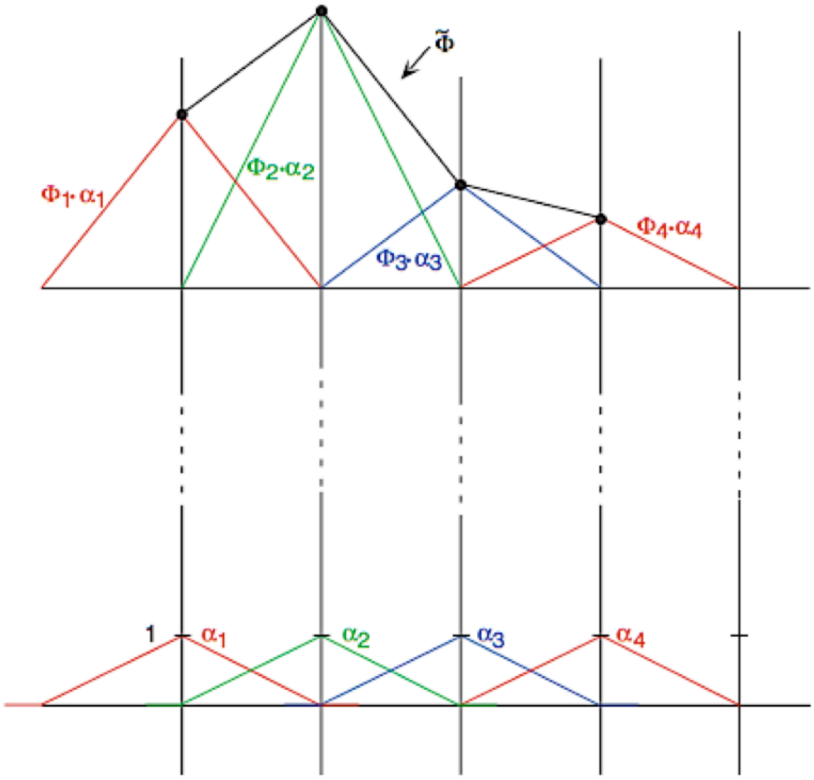
$$\int (\nabla\,\tilde{\phi})(\nabla\,\omega) dv - \int \omega \frac{\partial\tilde{\phi}}{\partial n} da - \int \frac{\rho}{\epsilon}\omega dv = 0 \tag{36}$$

In the Galerkin's method, the weighting functions are chosen to be the basis functions (node shape functions):

$$\tilde{\phi} = \sum_{l=1}^{nodes} \alpha_l(x,y,z)\phi_l \tag{37}$$

$$\omega_l(x,y,z) = \alpha_l(x,y,z) \tag{38}$$

(b) Linear 1D shape functions:



$\tilde{\Phi} = \sum_{k=1}^{nodes} \alpha_k(x,y,z) \cdot \Phi_k$, where Φ_k is a node potential.