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Lecture: Electromagnetics and Numerical Calculations of Fields

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Exam

21th of February 2012
Start: 11:00 am

Last Name:	
First Name:	
Student ID:	

Exercise	Max. Points	Achieved Points
1	6	
2	4	
3	2	
4	6	
5	4	
6	6	
7	6	
8	6	
9	8	
10	8	
11	4	
Total:	60	

Grade:________

Question 1 **(6 Points)**

Derive the wave equation for **E** and **H** directly from the Maxwell's equations for vacuum.
Hint: $\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \Delta \mathbf{A}$

Solution:

$$\nabla \times \mathbf{E} = -\mu_0 \frac{d}{dt}\mathbf{H}$$
$$\nabla \times \ \nabla \times \mathbf{E} = -\mu_0 \nabla \times \frac{d}{dt}\mathbf{H}$$
$$\nabla(\nabla \cdot \mathbf{E}) - \Delta \mathbf{E} = -\mu_0 \frac{d}{dt} \nabla \times \mathbf{H}$$
$$\nabla \cdot \mathbf{E} = 0 \quad \text{and} \quad \nabla \times \mathbf{H} = \epsilon_0 \frac{d^2}{dt^2}\mathbf{E}$$
$$\rightarrow \Delta \mathbf{E} - \mu_0 \epsilon_0 \frac{d^2}{dt^2}\mathbf{E} = 0$$

$$\nabla \times \mathbf{H} = \epsilon_0 \frac{d}{dt}\mathbf{E}$$
$$\nabla \times \ \nabla \times \mathbf{H} = \epsilon_0 \nabla \times \frac{d}{dt}\mathbf{E}$$
$$\nabla(\nabla \cdot \mathbf{H}) - \Delta \mathbf{H} = \epsilon_0 \frac{d}{dt} \nabla \times \mathbf{E}$$
$$\nabla \cdot \mathbf{H} = 0 \quad \text{and} \quad \nabla \times \mathbf{E} = -\mu_0 \frac{d^2}{dt^2}\mathbf{H}$$
$$\rightarrow \Delta \mathbf{H} - \mu_0 \epsilon_0 \frac{d^2}{dt^2}\mathbf{H} = 0$$

Question 2 **(4 Points)**

(a) What type of equation has to be solved for finding the electric potential ϕ of an arbitrary charge distribution $\rho(x, y, z)$? (1 Points)

(b) What type of equation has to be solved for finding the electric potential ϕ if equipotential surfaces are given ? (1 Points)

(c) What are Dirichlet boundary conditions and Neumann boundary conditions ? (2 Points)

Solution:

(a) Coulomb integral: $\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon} \int \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} dV'$

(b) Poisson equation: $\Delta\phi = -\frac{\rho}{\epsilon_0}$ or Laplace equation: $\Delta\phi = 0$

(c) Dirichlet: On the boundaries the potential is given: $\phi|_{boundary} = \hat{\phi}$. Neumann: On the boundaries the normal derivative of the potential is given: $\frac{\partial\phi}{\partial n}|_{boundary} = \hat{\phi}_n$

Question 3 **(2 Points)**

What is the force on a moving charge in an **E** and **B** field (simultaneously)

Solution:

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

Question 4 **(6 Points)**

(a) What is characteristic for TM waves in a waveguide and what is characteristic for TE waves in a waveguide ? (4 Points)

(b) What is the meaning of "cut-off frequency" for a specific mode ? (2 Points)

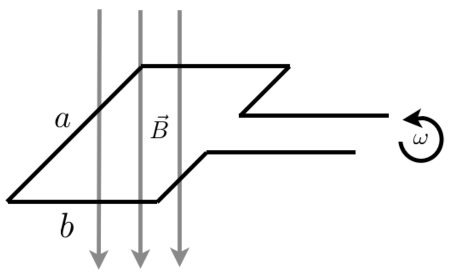
Solution:

(a) For TE waves the electric field is completely transverse to the direction of propagation, $E_z = 0$ for propagation in z-direction.
For TM waves the magnetic field is completely transverse to the direction of propagation, $H_z = 0$ for propagation in z-direction.

(b) For every mode, the cut-off frequency is the lowest frequency that allows undamped propagation.

Question 5 **(4 Points)**

A rectangular coil of length a and width b is turning with constant angular velocity ω inside a homogenous field of magnetic induction **B**. Calculate the induced voltage.



Solution:

$$U_{ind} = -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{a} = -\frac{d}{dt}(Bab\cos(\omega t)) = Bab\omega\sin(\omega t)$$

Question 6 **(6 Points)**

A magnetic field

$$\mathbf{H} = \begin{pmatrix} 0 \\ 0 \\ a\cos(by)e^{bx^2} \end{pmatrix}$$

is produced by an electric current ($\frac{d}{dt}\mathbf{D} = 0$). Calculate the corresponding current density $\mathbf{J}$.

Solution:

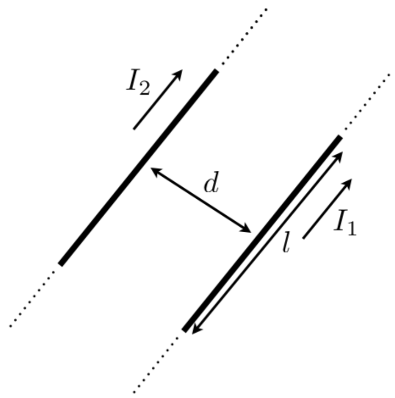
$$\nabla \times \mathbf{H} = \mathbf{J}$$

$$\nabla \times \mathbf{H} = \begin{pmatrix} \frac{\partial H_z}{\partial y} \\ -\frac{\partial H_z}{\partial x} \\ 0 \end{pmatrix}$$

$$\Rightarrow \mathbf{J} = -abe^{bx^2} \begin{pmatrix} \sin(by) \\ 2x\cos(by) \\ 0 \end{pmatrix}$$

Question 7 **(6 Points)**

Two infinitely long straight wires are each carrying the same current in the same direction ($I_1 = I_2$). The distance between the wires is $d = 4\,m$. The force per meter which acts on wire 1 is $F_1 = \frac{\hat{\mu}_0}{2\pi}[N/m]$ where $\hat{\mu}_0$ is the value of μ_0 without the unit ($\hat{\mu}_0 = 4\pi \cdot 10^-7$).
Calculate I_1 !



Solution: The magnitude of the magnetic field at wire 1 is given by:

$$B = \frac{\mu_0 I_2}{2\pi d}$$

The force F_1 per length l acting on wire 1 is:

$$F/l = \frac{\hat{\mu}_0}{2\pi}[N/m] = I_1 B$$

Since $I_1 = I_2$ we obtain for I_1

$$I_1 \cdot I_2 = I_1^2 = \frac{\hat{\mu}_0}{2\pi}\frac{4\ \cdot 2\pi}{\mu_0} = 4\,\mathrm{A}$$
$$\rightarrow I_1 = I_2 = 2\,\mathrm{A}$$

Question 8 **(6 Points)**

Find the capacitance of a cylindrical capacitor with inner radius R_i, outer radius R_a, height h where $h >> R_a$ and the dielectric constant $\epsilon_r = 1$ (vacuum). [6 pt]

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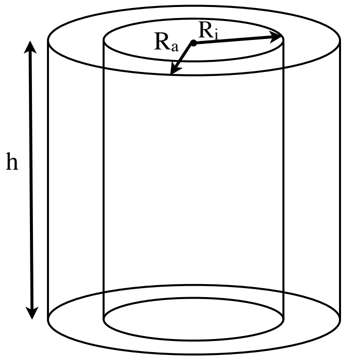


Solution:

$$\int \mathbf{E}d\mathbf{a} = \frac{Q}{\epsilon_0}$$

$$\rightarrow E_R = \frac{Q}{2\pi\epsilon_0 h}\frac{1}{R}$$

$$U = \int_{R_i}^{R_a} E_R dR = \frac{Q}{2\pi\epsilon_0 h} ln\left(\frac{R_a}{R_i}\right)$$

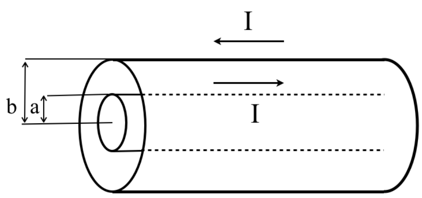
$$C = \frac{Q}{U}$$

$$\rightarrow C = \frac{2\pi\epsilon_0 h}{ln\left(\frac{R_a}{R_i}\right)}$$

Question 9 **(8 Points)**

A long coaxial cable carries a current I . The current flows down the surface(!) of the inner cylinder with radius a, and back along the outer cylinder with radius b (Fig 9).

Hint: $dV = R\,dRd\phi\,dz$



(a) Find the magnetic energy stored in a section of length l (6 Points)

(b) Find the self-inductance of the cable using the result of part (a). (2 Points)

Solution:

(a)

$$\int \mathbf{B}d\mathbf{l} = \mu_0 \int \mathbf{J}d\mathbf{a}$$
$$\Rightarrow 2\pi B_\phi R = \mu_0 I_{enc}$$
$$R < a : I_{enc} = 0 \Rightarrow B_\phi = 0$$
$$R > a < b : I_{enc} = I \Rightarrow B_\phi = \frac{\mu_0 I}{2\pi R}$$
$$R > b : I_{enc} = 0 \Rightarrow B_\phi = 0$$
$$W_{mag} = \frac{1}{2\mu_0}\int B_\phi^2 dV = \int_a^b \frac{1}{2\mu_0}\left(\frac{\mu_0 I}{2\pi R}\right)^2 2\pi l R dR$$
$$= \frac{\mu_0 I^2 l}{4\pi}\int_a^b \frac{1}{R}dR = \frac{\mu_0 I^2 l}{4\pi} ln(\frac{b}{a})$$

(b)

$$W = \frac{1}{2}LI^2 \Rightarrow L = \frac{\mu_0 l}{2\pi} ln(\frac{b}{a})$$

Question 10 **(8 Points)**

Given is a one-dimensional Poisson equation with Dirichlet boundary conditions:

$$\frac{d^2}{dx^2}u(x) = -64x$$
$$u(0) = 0, \quad u(1) = 1$$

The interval [0,1] is divided into 5 equal parts with the points:

$$x_0 = 0,\ x_1 = 0.25,\ x_2 = 0.5,\ x_3 = 0.75,\ x_4 = 1$$

(a) Write down the system of linear equations which you obtain if you replace $\frac{d^2}{dx^2}$ by the centered finite differences approximation at the points x_i. (2 Points)

(b) This system can be written in matrix form: (6 Points)

$$\mathbf{Au} = \mathbf{b}$$

where $\mathbf{u} = [u_1, u_2, u_3]^T$, $u_0 = 0$ and $u_4 = 1$. Determine $\mathbf{A}$ and $\mathbf{b}$!

Solution:

(a)

$$\frac{u_{i-1} - 2u_i + u_{i+1}}{(0.25)^2} = -64x_i$$

(b) x_1 :

$$\frac{0 - 2u_1 + u_2}{0.0625} = -16$$
$$2u_1 - u_2 = 1$$
$$A_{11} = 2,\ A_{12} = -1,\ A_{13} = 0,\ b_1 = 1$$

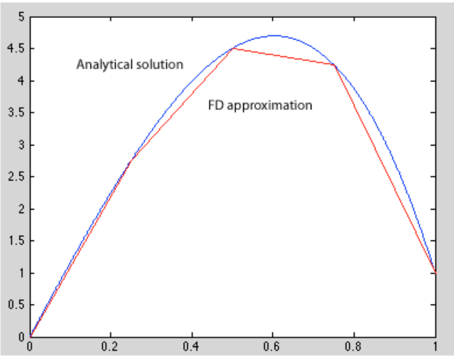
x_2 :

$$\frac{u_1 - 2u_2 + u_3}{0.0625} = -32$$
$$-1u_1 + 2u_2 - u_3 = 2$$
$$A_{21} = -1,\ A_{22} = 2,\ A_{23} = -1,\ b_2 = 2$$

x_3 :

$$\frac{u_2 - 2u_3 + 1}{0.0625} = -48$$
$$-u_2 + 2u_3 = 4$$
$$A_{31} = 0,\ A_{32} = -1,\ A_{33} = 2,\ b_3 = 4$$

$$\mathbf{A} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$$

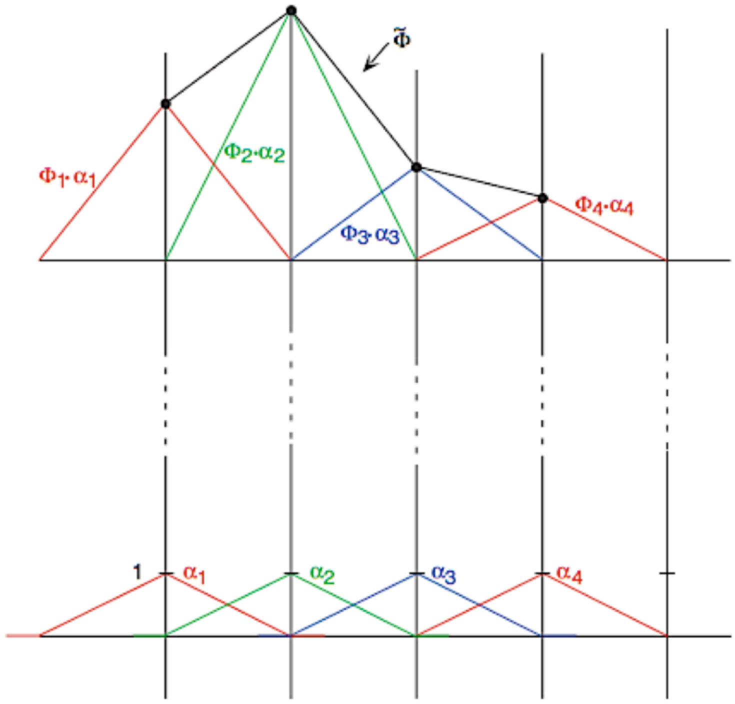


Question 11 **(4 Points)**

(a) Give a sketch of the linear node shape functions in 1D case. (3 Points)

(b) How can an approximated solution for e.g. the electric potential be expressed in terms of the node shape functions. (1 Points)

Solution: (a)



(b)

$$\tilde{\phi} = \sum_{k=1}^{n} \alpha_k \cdot \phi_k$$

where α_k are the node shape functions and ϕ_k are the node voltages.