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Lecture: Electromagnetics and Numerical Calculations of Fields

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Exam

31th of March 2014
Start: 08:00 am

Last Name:

First Name:

Student ID:

Question	Max. Points	Achieved Points
1	3	
2	2	
3	9	
4	8	
5	10	
6	6	
7	3	
8	4	
9	6	
10	4	
Total:	55	

Grade:_______

Question 1 **(3 Points)**

Given is the electric flux density $\mathbf{D} = a\frac{1}{r^3}\mathbf{e}_r$. Using the Gauss' law find the total charge Q inside the volume $r \leq r_1$.

Solution:

$$\nabla \cdot \mathbf{D} = \rho$$

$$Q = \rightarrow \int \rho dV = \int \nabla \cdot \mathbf{D} dV = \oint D \cdot d\mathbf{S} = \mathbf{D}_r \cdot 4\pi r^2\big|_{r=r_1} =$$

$$= a\frac{4\pi r^2}{r^3}\bigg|_{r=r_1} = a\frac{4\pi}{r_1}$$

Question 2 **(2 Points)**

Given is a point charge Q_1 at the origin. Find the energy you need to move a second point charge Q_2 from infinity to the point $\mathbf{r}_0$.

Solution: The work to be done:

$$\Delta W_e = Q_2 \left(\phi(\mathbf{r}_1) - \phi(\mathbf{r}_2)\right)$$

$$\phi(\mathbf{r}) = \frac{Q_1}{4\pi\epsilon r}$$

$$\Delta W_e = \frac{Q_1 Q_2}{4\pi\epsilon r_0}$$

Question 3 **(9 Points)**

(a) Given is the first Green's identity: (2 Points)

$$\int_V (\phi\Delta\psi) + (\nabla\phi)(\nabla\psi)dv = \oint_S \phi(\nabla\psi)d\mathbf{S}$$

Prove the second Green's identity:

$$\int_V \phi\Delta\psi - \psi\Delta\phi dv = \oint_S (\phi\nabla\psi - \psi\nabla\phi)d\mathbf{S}$$

Given is the Poisson equation $\Delta\phi = -\frac{\varrho}{\epsilon}$

(b) What is the Dirichlet problem, when a charge distribution ρ is given? (1 Points)

(c) What is the Neumann problem, when a charge distribution ρ is given? (1 Points)

(d) What is the Dirichlet Green's function? (1 Points)

(e) What is the Neumann Green's function? (1 Points)

(f) How can you find the solution of the Dirichlet problem when its Green's function is known? (3 Points)
Hint: Use the second Green's identity.

Solution:

(a) From the first Green's identity:

$$\int_V (\phi\Delta\psi) + (\nabla\phi)(\nabla\psi)dv = \oint_S \phi(\nabla\psi)d\mathbf{S}$$
$$\int_V (\psi\Delta\phi) + (\nabla\psi)(\nabla\phi)dv = \oint_S \psi(\nabla\phi)d\mathbf{S}$$

Subtracting the second formula from the first one we obtain the second Green's identity.

(b) Dirichlet problem:

$$\Delta\Phi = -\frac{\rho}{\epsilon}$$
$$\Phi|_{\text{boundary}} = \tilde{\Phi}$$

(c) Neumann problem:

$$\Delta\Phi = -\frac{\rho}{\epsilon}$$
$$\frac{\partial\Phi}{\partial n}|_{\text{boundary}} = \tilde{\Phi}_n$$

(d) The Dirichlet Green's function is defined as:

$$\Delta G_D(\mathbf{r}, \mathbf{r}_0) = -\delta(\mathbf{r} - \mathbf{r}_0)$$
$$G_D|_{\text{boundary}} = 0$$

(e) The Neumann Green's function is defined as:

$$\Delta G_N(\mathbf{r}, \mathbf{r}_0) = -\delta(\mathbf{r} - \mathbf{r}_0)$$
$$\frac{\partial G_N}{\partial n}|_{\text{boundary}} = 0$$

(f) When G_D is known, substituting to the second Green's law for the ψ

$$\int_V \phi\Delta G_D - G_D\Delta\phi dv = \oint_S (\phi\nabla G_D - G_D\nabla\phi)d\mathbf{S}$$
$$\int_V -\phi\delta(\mathbf{r} - \mathbf{r}_0) - G_D \cdot (-\frac{\rho}{\epsilon})dv = \oint_S \phi\nabla G_D d\mathbf{S}$$
$$-\phi(\mathbf{r}_0) + \int_V G_D \cdot \frac{\rho}{\epsilon}dv = \oint \phi\frac{\partial G_D}{\partial n}d\mathbf{S}$$

and the final formula becomes

$$\phi(\mathbf{r}_0) = \int G_D(\mathbf{r}, \mathbf{r}_0)\frac{\rho}{\epsilon}dV - \oint \phi\frac{\partial G_D}{\partial n}d\mathbf{S}$$

Question 4 **(8 Points)**

(a) Provide a formula for the Poynting vector? (1 Points)

(b) What is the meaning of the Poynting vector? (1 Points)

(c) What three components of energy change inside a volume contribute to the power flux through the surface of a volume? Provide formulae for them. (6 Points)

Solution:

(a)

$$\mathbf{S} = \mathbf{E} \times \mathbf{H}$$

(b) The Poynting vector is the power per area that is transmitted through a given area via electro- magnetic fields.

(c) The power flux out of a volume ($\nabla \cdot \mathbf{S}$) is equal to the sum of ohmic losses, change of electrostatic field energy, change of magnetic field energy:

$$\int \mathbf{JE} dV dt \quad \text{- ohmic losses}$$

$$\frac{1}{2}\int \mathbf{ED} dV \quad \text{- electrostatic field energy}$$

$$\frac{1}{2}\int \mathbf{BH} dV \quad \text{- magnetostatic field energy}$$

Question 5 **(10 Points)**

(a) What is the vector potential **A** (formula). (1 Points)

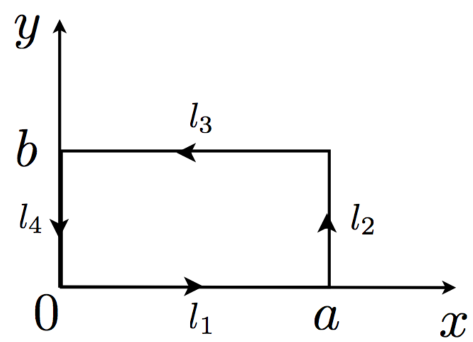
(b) In what respect it is not well defined? (1 Points)

(c) Give the formula for the magnetic flux and the relation to the vector potential. (2 Points)

(d) Given a vector potential (6 Points)

$$A = x^2 y\, \mathbf{e}_x + y^2 x\, \mathbf{e}_y + xyz\, \mathbf{e}_z$$

Find the magnetic induction **B**. Calculate the magnetic flux through a rectangular surface ($z = 0, 0 \le x \le a, 0 \le y \le b$) with 2 methods: first, using **A** and solving a line integral, then using **B** and solving a surface integral. *Hint:* Use Stokes' theorem



Solution:

(a)

$$\mathbf{B} = \nabla \times \mathbf{A}$$

(b) We can add the gradient of any scalar function without changing $\mathbf{B}$.

(c)

$$\phi_m = \int \mathbf{B} d\mathbf{S} = \oint A \cdot d\mathbf{l}$$

(d)

$$\mathbf{B} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 y & y^2 x & xyz \end{vmatrix} = xz\,\mathbf{e}_x - yz\,\mathbf{e}_y + (y^2 - x^2)\,\mathbf{e}_z$$

When using $\mathbf{A} = x^2 y\,\mathbf{e}_x + y^2 x\,\mathbf{e}_y$ (as $z = 0$) we must calculate the sum of 4 line integrals, 2 of them vanish: l_1 $(0 \le x \le a, y = 0)$, and l_4 $(x = 0, 0 \le y \le b)$. For the remaining 2 we have $y = b$ for l_3 and $x = a$ for l_2.

$$\phi_m = \int_0^b a y^2 dy - \int_0^a b x^2 dx = \frac{ab}{3}(b^2 - a^2)$$

When using $\mathbf{B} = (y^2 - x^2)\,\mathbf{e}_z$ (as $z = 0$) the formula is straightforward:

$$\phi_m = \int_0^b \int_0^a (y^2 - x^2) dx dy = \frac{ab}{3}(b^2 - a^2).$$

Question 6 **(6 Points)**

(a) Write down the general "plane wave" solutions of the wave equation for $\mathbf{E}$ and $\mathbf{H}$ (two types of solutions in 1D). (2 Points)

(b) What is the direction of $\mathbf{E}$ with respect to $\mathbf{H}$? Provide the direction of $\mathbf{E} \times \mathbf{H}$ for each type of the solution. (2 Points)

(c) What is the factor between $|\mathbf{E}|$ and $|\mathbf{H}|$ in vacuum? (1 Points)

(d) What is the unit of this factor? (1 Points)

Solution:

(a)

$$E_y = f(t - x/c) + g(t + x/c)$$
$$\Gamma H_z = f(t - x/c) - g(t + x/c)$$

(b) $\mathbf{E}$ is always perpendicular to $\mathbf{H}$. For the f-type solution $\mathbf{E} \times \mathbf{H}$ is positive, for the g-type is negative.

(c)

$$\Gamma = \sqrt{\frac{\mu}{\epsilon}} = \frac{E_y}{H_z} \quad \text{the impedance of vacuum}$$

(d) The unit is $\frac{V}{m} \cdot \frac{m}{A} = \frac{V}{A} = \Omega$ ohm.

Question 7 **(3 Points)**

(a) What type of differential equation is valid inside a rectangular waveguide? (1 Points)

(b) What is the recommended approach to solve this differential equation? (1 Points)

(c) What is the "mode" of a propagating wave in a waveguide? (1 Points)

Solution:

(a) The Laplace equation.

(b) The method of separation of variables.

(c) The separation of variables delivers 3 sets of orthogonal basis functions. A "mode" is a specific selection of basis functions. If two basis functions have been chosen, the third one is not free anymore.

Question 8 **(4 Points)**

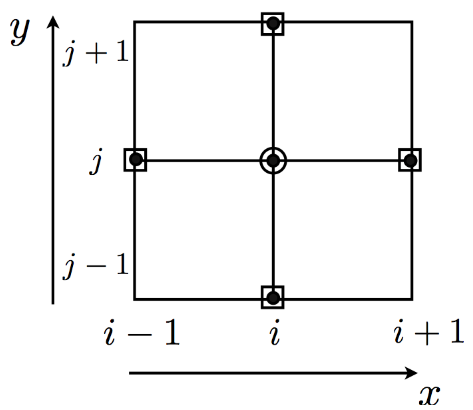
(a) Give a sketch of the "computing molecule" for the Laplace/Poisson equation (finite difference method FDM). (1 Points)

(b) Write down a finite difference approximation of the Poisson operator in 2-D using central difference scheme for approximating second derivatives. (2 Points)

(c) Based on the obtained formula, describe "mean value property" of the Laplace equation when the discretization steps in x and y directions are equal. (1 Points)

Solution:

(a) The computing molecule



(b) For the Poisson equation

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = g(x, y)$$

the difference approximation becomes

$$\frac{\Phi(i+1, j) - 2\Phi(i, j) + \Phi(i-1, j)}{(\Delta x)^2} + \frac{\Phi(i, j+1) - 2\Phi(i, j) + \Phi(i, j-1)}{(\Delta y)^2} = g(i, j)$$

(c) In case $g = 0$ the solution at point (i, j)

$$\Phi(i, j) = \frac{1}{4} [\Phi(i+1, j) + \Phi(i-1, j) + \Phi(i, j+1) + \Phi(i, j-1)]$$

Question 9 **(6 Points)**

For the given parabolic differential equation

$$\frac{\partial \phi}{\partial t} = \frac{\partial^2 \phi}{\partial x^2}$$

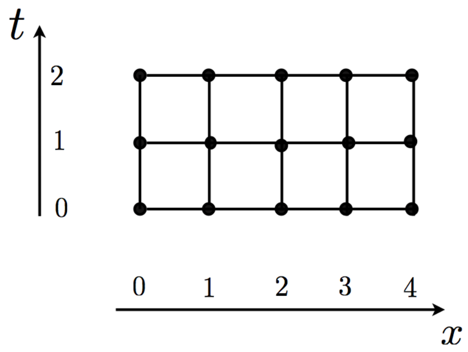
with the initial condition:

$$\phi|_{t=0} = 4 - (x-2)^2$$

and boundary conditions

$$\phi|_{x=0} = 0$$
$$\phi|_{x=4} = 0$$

find the values of potentials in all depicted grid points using FDM: $x \in [0 \dots 4]$, $t \in [0 \dots 2]$, $\Delta x = 1, \Delta t = 1$.



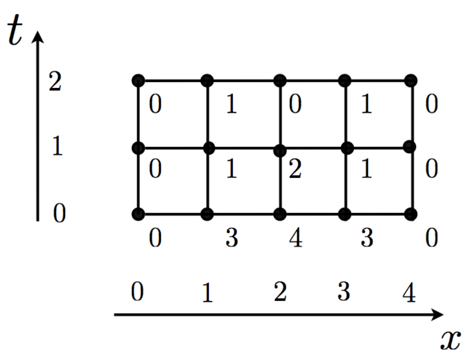
Solution: Using finite difference scheme

$$\frac{\Phi(i,j+1) - \Phi(i,j)}{\Delta t} = \frac{\Phi(i+1,j) - 2\Phi(i,j) + \Phi(i-1,j)}{(\Delta x)^2}$$

we come to the solution

$$\Phi(i,j+1) = \Phi(i+1,j) - \Phi(i,j) + \Phi(i-1,j)$$

The initial values $\Phi(i,0) = 4 - (i-2)^2$. The resulting grid with potentials is



Question 10 **(4 Points)**

Given is the equation

$$L\phi(x) = f(x)$$

where L is a linear differential operator.

(a) Describe the method of weighted residuals for solving this equation. (2 Points)

(b) What is the Galerkin method as a particular case of the method of weighted residuals? (2 Points)

Solution:

(a) An ideal solution would lead to

$$L\phi(x) - f(x) = 0$$

For any approximate solution the following holds:

$$L\tilde{\phi}(x) - f(x) = R$$

The idea of the method of weighted residuals is to minimize the residual R by multiplying it with weight functions $w(x)$ and integrating over the domain

$$\int Rw(x)d\Omega \stackrel{!}{=} 0 \tag{1}$$

If it is satisfied for all weight functions $w(x)$ then an approximate solution $\tilde{\phi}(x)$ approaches the exact solution $\phi(x)$.

(b) An approximate solution is sought in the form:

$$\tilde{\phi}(x) = \sum_{i=1}^{N} \phi_i \alpha_i(x)$$

where α_i are called trial or node shape functions. The idea of Galerkin was to use the same function space for both trial and weight functions.