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Lecture: Electromagnetics and Numerical Calculations of Fields

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Exam

19th of March 2015
Start: 08:00 am

Last Name:

First Name:

Student ID:

Question	Max. Points	Achieved Points
1	3	
2	7	
3	6	
4	2	
5	3	
6	8	
7	4	
8	5	
9	6	
Total:	44	

Grade:________

Question 1 **(3 Points)**

Given is the charge density $\rho = k\frac{1}{r}$ (spherical coordinate system). Find the electric flux density $\mathbf{D}$.
Hint: Volume element in spherical coordinate system is $r^2 \sin\theta dr d\theta d\phi$.

Solution:

$$\oint \mathbf{D} d\mathbf{a} = 4\pi r^2 D_r$$

$$\int \rho dV = \int \nabla \cdot \mathbf{D} dV = \oint \mathbf{D} \cdot d\mathbf{a} =$$

$$k \int_0^{2\pi} \int_0^{\pi} \int_0^{r} \frac{1}{r'} r'^2 \sin\theta dr' d\theta d\phi = k4\pi \int_0^r r' dr' = k2\pi r^2$$

$$\implies \mathbf{D} = k\frac{1}{2}\mathbf{e}_r$$

Question 2 **(7 Points)**

Given is a parallel-plate capacitor with the distance d between the plates, the plate area A and the charge Q.

(a) Derive the expression for the voltage U. (1 Points)

(b) What is the capacitance C? (1 Points)

(c) What is the energy stored in the capacitor? (express it in terms of capacitance and voltage) (3 Points)

(d) What is the force between the two plates (constant charge case)? (2 Points)

Solution:

(a)

$$U = \int \mathbf{E} dl = Ed$$

$$\epsilon E = D = \frac{Q}{A}$$

$$\implies U = \frac{Q}{\epsilon A} d$$

(b)

$$C = \frac{Q}{U}$$

$$\implies C = \frac{\epsilon A}{d}$$

(c)

$$W_e = \frac{1}{2}\int \mathbf{E}\mathbf{D} dV = \frac{1}{2}\int \epsilon \frac{Q^2}{\epsilon^2 A^2} dV =$$

$$= \frac{\epsilon Q^2 A d}{2\epsilon^2 A^2} = \frac{Q^2}{2}\left(\frac{d}{\epsilon A}\right) =$$

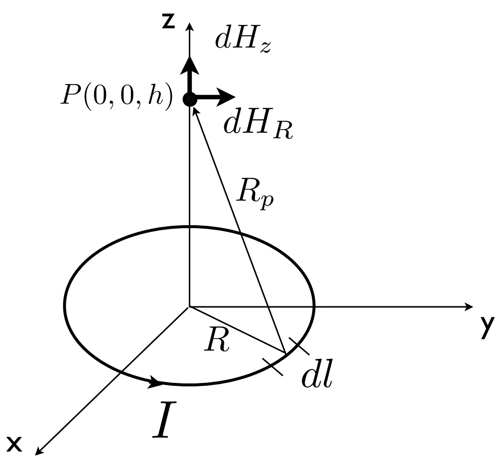
$$= \frac{Q^2}{2C} = \frac{1}{2} C V^2$$

(d)

$$F = -\text{grad}\, W_e = -\text{grad}\left(\frac{Q^2}{2C}\right) = -\text{grad}\left(\frac{Q^2 d}{2\epsilon A}\right) = -\frac{Q^2}{2\epsilon A}$$

Question 3 **(6 Points)**

A circular loop located on $z = 0$ carries a current I along $\mathbf{e}_\phi$. Determine the magnetic field $\mathbf{H}$ at the point $P(0, 0, h)$.
Hint: differential length element in cylindrical coordinates $d\mathbf{l} = Rd\phi\mathbf{e}_\phi$.



Solution:

$$H = \int \frac{Id\mathbf{l} \times \mathbf{R}_p}{4\pi R_p^3}$$

where $d\mathbf{l} = Rd\phi\mathbf{e}_\phi$, $\mathbf{R}_p = (0, 0, h) - (x, y, 0) = -R\mathbf{e}_R + h\mathbf{e}_z$. Hence,

$$d\mathbf{l} \times \mathbf{R}_p = \begin{vmatrix} \mathbf{e}_R & \mathbf{e}_\phi & \mathbf{e}_z \\ 0 & \rho d\phi & 0 \\ -\rho & 0 & h \end{vmatrix} = Rhd\phi\mathbf{e}_R + \rho^2 d\phi\mathbf{e}_z$$

Due to rotational symmetry the radial components cancel, which results in

$$\mathbf{H} = \int_0^{2\pi} \frac{IR^2 d\phi\mathbf{e}_z}{4\pi[R^2 + h^2]^{3/2}} = \frac{IR^2 2\pi\mathbf{e}_z}{4\pi[R^2 + h^2]^{3/2}}$$

Question 4 **(2 Points)**

(a) What is the cut-off frequency for a rectangular waveguide? (1 Points)

(b) Write down the formula for finding the cut-off frequency for a TM_{mn} mode. (1 Points)

Solution:

(a) For every mode it is the lowest frequency that allows undamped propagation.

(b) For the mode TM_{mn} the cut-off frequency is

$$f_c = \frac{1}{2\pi\sqrt{\mu\epsilon}}\sqrt{\left(\frac{m\pi}{a}\right)+\left(\frac{n\pi}{b}\right)}$$

Question 5 **(3 Points)**

Given the magnetic vector potential $\mathbf{A} = -R^2/4\mathbf{e}_z$, calculate the total magnetic flux Φ_m crossing the surface $\phi = \pi/2, 1 \le R \le 2, 0 \le z \le 5$ (you are free to choose the method of calculation).

Hint: $\mathrm{rot}\mathbf{A} = \mathbf{e}_R\left(\frac{1}{R}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}\right) + \mathbf{e}_\phi\left(\frac{\partial A_R}{\partial z} - \frac{\partial A_z}{\partial R}\right) + \mathbf{e}_z\left(\frac{1}{R}\frac{\partial}{\partial R}(RA_\phi) - \frac{1}{R}\frac{\partial A_R}{\partial\phi}\right)$.

Solution:

$$\mathbf{B} = \nabla\times\mathbf{A} = -\frac{\partial A_z}{\partial R}\mathbf{e}_\phi = \frac{\rho}{2}\mathbf{e}_\phi$$
$$d\mathbf{a} = dRdz\mathbf{e}_{phi}$$

Therefore,

$$\Phi_m = \int \mathbf{B}\cdot d\mathbf{a} = \frac{1}{2}\int_{z=0}^{5}\int_{R=1}^{2} RdRdz = \frac{15}{4}$$

Question 6 **(8 Points)**

(a) Write down the general wave equation for conducting, dielectric and magnetic materials (linear material equations). (2 Points)

(b) When can we reduce the equation to the case of good conductors? Write down the resulting equation. What type of equation is it? (3 Points)

(c) When can we reduce the equation to the lossless case? Write down the resulting equation. What type of equation is it? (3 Points)

Solution:

(a) For conductivity σ

$$\Delta\mathbf{H} = \sigma\mu\frac{\partial\mathbf{H}}{\partial t} + \epsilon\mu\frac{\partial^2\mathbf{H}}{\partial t^2}$$
$$\Delta\mathbf{E} = \sigma\mu\frac{\partial\mathbf{E}}{\partial t} + \epsilon\mu\frac{\partial^2\mathbf{E}}{\partial t^2}$$

(b) $\frac{\sigma}{\omega\epsilon} \gg 1 \rightarrow$ good conductors and the diffusion equation:

$$\Delta\mathbf{H} = \sigma\mu\frac{\partial\mathbf{H}}{\partial t}$$

(c) $\frac{\sigma}{\omega\epsilon} \ll 1 \rightarrow$ lossless media and the wave equation:

$$\Delta\mathbf{H} = \epsilon\mu\frac{\partial^2\mathbf{H}}{\partial t^2}$$

Question 7 **(4 Points)**

(a) Write down the "Coulomb potential" that finds the potential ϕ arising from a distribution of free charges ρ. (2 Points)

(b) In what way the Coulomb potential has to be modified if you can not neglect the time it takes for any change in the charge density to reach the point where we want to know the electric potential? How do you call these potentials? (2 Points)

Solution:

(a)

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon} \int \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dV'$$

(b)

$$\phi(\mathbf{r}, t) = \frac{1}{4\pi\epsilon} \int \frac{\rho\left(\mathbf{r}', t - \frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)}{|\mathbf{r} - \mathbf{r}'|} dV'$$

The time shift is equal to the time, light would need to come from the source position to the measurement point. These potentials are called "retarded potentials".

Question 8 **(5 Points)**

Given a one-dimensional Poisson equation with two sets of boundary conditions

$$u''(x) = -6x + 2$$

$$u(0) = 0 \quad u(1) = 1 \quad \text{Dirichlet boundary conditions}$$

$$u'(0) = 0 \quad u'(1) = 1 \quad \text{Neumann boundary conditions}$$

(a) Write down the central difference approximation of the Poisson equation. (1 Points)

(b) Write down 3 linear equations for the 3 unknown potentials $u_1 = u(0.25), u_2 = u(0.5), u_3 = u(0.75)$ for Dirichlet boundary conditions $u_0 = u(0) = 0, u_4 = u(1) = 1$. (2 Points)

(c) Write down 5 linear equations for the 5 unknown potentials $u_0 = u(0), u_1 = u(0.25), u_2 = u(0.5), u_3 = u(0.75), u_4 = u(1)$ for Neumann boundary condotions $u'(0) = 0, u'(1) = 1$. (2 Points)

Solution:

(a) The central difference approximation of a Poisson equation with the discretization step h is given by:

$$\frac{u_{i-1} - 2u_i + u_{i+1}}{h^2} = f(x_i)$$

For the given problem the discretization step $h = 0.25$ and discretization points $x_0 = 0, x_1 = 0.25, x_2 = 0.5, x_3 = 0.75, x_4 = 1$, the source function $f(x) = -6x + 2$.

(b)

$$u_0 - 2u_1 + u_2 = (0.25)^2 \cdot (-6 \cdot 0.25 + 2) = 0.03125$$

$$u_1 - 2u_2 + u_3 = (0.25)^2 \cdot (-6 \cdot 0.5 + 2) = -0.0625$$

$$u_2 - 2u_3 + u_4 = (0.25)^2 \cdot (-6 \cdot 0.75 + 2) = -0.15625$$

with $u_0 = 0$ and $u_4 = 1$.

(c)

$$u_0 - 2u_1 + u_2 = (0.25)^2 \cdot (-6 \cdot 0.25 + 2) = 0.03125$$
$$u_1 - 2u_2 + u_3 = (0.25)^2 \cdot (-6 \cdot 0.5 + 2) = -0.0625$$
$$u_2 - 2u_3 + u_4 = (0.25)^2 \cdot (-6 \cdot 0.75 + 2)) = -0.15625$$

with $\frac{u_0 - u_1}{0.25} = 0$ and $\frac{u_4 - u_3}{0.25} = 1$.

Question 9 **(6 Points)**

Describe the basic idea and equations of the Boundary Element Method for a Laplace equation. What is BEMs choice for the weighting functions? Derive the formula for the approximated potential as a function of the boundary values.
Hint: Green's second law: $\int (\Phi \Delta \Psi + \nabla \Phi \nabla \Psi)\, dV = \oint \Phi \nabla \Psi d\mathbf{a}$.

Solution: For a Laplace equation the residual we want to minimize is

$$\Delta \tilde{\Phi} = R$$

and in the weak sense (weighted residuals):

$$\int wRdV = 0$$

Using Green's second law, we obtain

$$\int \Delta w \tilde{\Phi} dV = \oint \left(\tilde{\Phi} \frac{\partial w}{\partial n} - w \frac{\partial \tilde{\Phi}}{\partial n} \right) d\mathbf{S}$$

BEMs choice for the weighting function is Green's function for open space

$$\Delta w_i = -\delta_i$$

So, the solution approximation gets the following form

$$\tilde{\Phi} = \oint \left(\tilde{\Phi} \frac{\partial w}{\partial n} - w \frac{\partial \tilde{\Phi}}{\partial n} \right) d\mathbf{S}$$