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Lecture: Electromagnetics and Numerical Calculation of Fields

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Exam

3rd of March 2016
Start: 08:00 am

Last Name:

First Name:

Student ID:

Question	Max. Points	Achieved Points
1	14	
2	3	
3	6	
4	6	
5	6	
6	4	
7	13	
8	10	
9	9	
Total:	71	

Grade:________

Question 1 **(14 Points)**

(a) Write down Maxwell equations in differential form. Explain briefly the meaning of each equation. (8 Points)

(b) Derive the wave equation for $\mathbf{E}$ and $\mathbf{H}$ with charge density $\rho = 0$ directly from the Maxwell equations for vacuum. *Hint:* $\Delta\mathbf{F} := \nabla(\nabla\cdot\mathbf{F}) - \nabla\times(\nabla\times\mathbf{F})$ (6 Points)

Solution:

(a) **Maxwell equations:**

$$\nabla \cdot \mathbf{D} = \rho$$
$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{E} = -\tfrac{\partial}{\partial t}\mathbf{B}$$
$$\nabla \times \mathbf{H} = \mathbf{J} + \tfrac{\partial}{\partial t}\mathbf{D}$$

Explanations:

1. Free charges are the sources of D (ρ : density of free charges)
2. The field of magnetic induction does not have any sources
3. Faraday's law of Induction: Time varying magnetic induction causes circulation of $\mathbf{E}$
4. Ampère's law: Free currents or displacement currents cause circulation of $\mathbf{H}$

(b) **Electric field:**

$$\nabla \times \mathbf{E} = -\mu_0 \frac{\partial}{\partial t}\mathbf{H}$$
$$\nabla \times \nabla \times \mathbf{E} = -\mu_0 \nabla \times \frac{\partial}{\partial t}\mathbf{H}$$
$$\nabla(\nabla \cdot \mathbf{E}) - \Delta\mathbf{E} = -\mu_0 \frac{\partial}{\partial t}\nabla \times \mathbf{H}$$
$$\nabla \cdot \mathbf{E} = 0 \quad \text{and} \quad \nabla \times \mathbf{H} = \epsilon_0 \frac{\partial}{\partial t}\mathbf{E}$$
$$\Rightarrow \Delta\mathbf{E} - \mu_0\epsilon_0 \frac{\partial^2}{\partial t^2}\mathbf{E} = 0$$

magnetic field:

$$\nabla \times \mathbf{H} = \epsilon_0 \frac{\partial}{\partial t}\mathbf{E}$$
$$\nabla \times \nabla \times \mathbf{H} = \epsilon_0 \nabla \times \frac{\partial}{\partial t}\mathbf{E}$$
$$\nabla(\nabla \cdot \mathbf{H}) - \Delta\mathbf{H} = \epsilon_0 \frac{\partial}{\partial t}\nabla \times \mathbf{E}$$
$$\nabla \cdot \mathbf{H} = 0 \quad \text{and} \quad \nabla \times \mathbf{E} = -\mu_0 \frac{\partial}{\partial t}\mathbf{H}$$
$$\Rightarrow \Delta\mathbf{H} - \mu_0\epsilon_0 \frac{\partial^2}{\partial t^2}\mathbf{H} = 0$$

Question 2 **(3 Points)**

The vector fields $\mathbf{B}_1$ and $\mathbf{B}_2$ are given by:

$$\mathbf{B}_1 = k \cdot \begin{pmatrix} yz \\ xy \\ -2xz \end{pmatrix}$$

$$\mathbf{B}_2 = k \cdot \begin{pmatrix} -y^2 + z \\ x - 2yz \\ -x + z^2 \end{pmatrix}$$

(a) Which of the two vector fields could be a magnetic field? Why? (3 Points)

Solution:

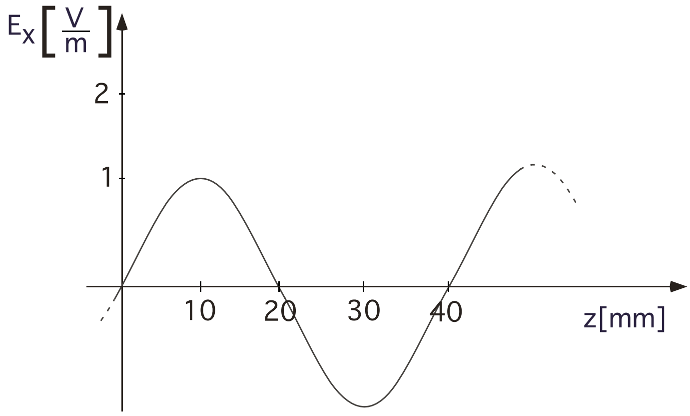
(a) Magnetic fields are divergence-free, i.e. $\nabla \cdot \mathbf{B} = 0$.

$$\nabla \cdot \mathbf{B}_1 = k\,(\nabla \cdot \begin{pmatrix} yz \\ xy \\ -2xz \end{pmatrix}) = k(0 + x - 2x) = -kx \neq 0$$

$$\nabla \cdot \mathbf{B}_2 = k\,(\nabla \cdot \begin{pmatrix} -y^2 + z \\ x - 2yz \\ -x + z^2 \end{pmatrix}) = k(0 - 2z + 2z) = 0 \quad \Rightarrow \text{magnetic field}$$

Question 3 **(6 Points)**

In seawater the electric field of a plane electromagnetic wave is the following:



seawater around 1 GHz: $T = 20°\text{C}$, $\varepsilon_r = 64$, $\mu_r = 1$, $\sigma = 3\,\Omega/\text{m}$, $\Gamma_{\text{vacuum}} = 337\,\Omega$

(a) Find the corresponding frequency. (2 Points)

(b) Find the corresponding magnetic field. (2 Points)

(c) Find the penetration depth. (2 Points)

Solution:

(a) corresponding frequency

$$c_{\text{water}} = \frac{3 \cdot 10^8}{\sqrt{64}} = \frac{3 \cdot 10^8}{8} = 37.5 \cdot 10^6\,\text{m/s}$$

$$c = \lambda f, \quad \lambda = 0.04\,\text{m} \quad \Rightarrow \quad f = \frac{c}{\lambda} = \frac{37.5 \cdot 10^6}{0.04} = 937\,\text{MHz}$$

(b) magnetic field

$$E_x = \Gamma \cdot H_y \quad \Rightarrow \quad H_y = \frac{E_x}{\Gamma}$$

$$\Gamma = \sqrt{\frac{\mu}{\varepsilon}} \cdot \Gamma_0 = 337\,\Omega \quad \Rightarrow \quad \Gamma_{\text{water}} = 42.1\,\Omega$$

$$H_y = \frac{E_x}{\Gamma} = \frac{1\,\mathrm{V/m}}{42.1\,\Omega} = 23.7 \cdot 10^{-3}\,\mathrm{A/m}$$

(c) penetration depth

$$d = \sqrt{\frac{2}{\sigma \mu \omega}} = \sqrt{\frac{2}{3 \cdot 4\pi\, 10^{-7}\, 2\pi\, 937 \cdot 10^{6}}} = 9.5\,\mathrm{mm}$$

Question 4 **(6 Points)**

(a) What type of differential equation is valid inside a rectangular waveguide? (1 Points)

(b) What is the recommended approach to solve this differential equation? (1 Points)

(c) Which types of mode can occur in waveguides? Describe their characteristics shortly. (3 Points)

(d) What is the meaning of the “cutoff frequency”? (1 Points)

Solution:

(a) Laplace equation.

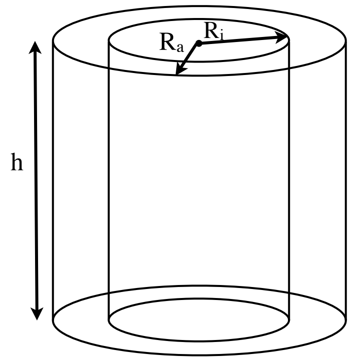
(b) Solve by separation of variables: $\Phi(x, y, z) = U(x) \cdot V(y) \cdot W(z)$

(c)
- TE: Transverse electric, no electric field in propagation direction $E_z = 0$
- TM: Transverse magnetic, no magnetic field in propagation direction $B_z = 0$
- TEM: transverse electromagnetic, magnetic and electric field are zero in propagation direction $E_z = B_z = 0$. These are not possible in hollow waveguides.

(d) For every mode this is the lowest frequency allowing undamped propagation.

Question 5 **(6 Points)**

(a) Find the capacitance of a cylindrical capacitor with inner radius R_i, outer radius R_a and length L where $L \ll R_a$ and the dielectric constant $\varepsilon_r = 1$ (vacuum). (6 Points)



Solution:

$$\int \mathbf{E} d\mathbf{a} = \frac{Q}{\epsilon_0} \Rightarrow E_r = \frac{Q}{2\pi\epsilon_0 L}\frac{1}{r}$$

$$U = \int_{R_i}^{R_a} E_r dr = \frac{Q}{2\pi\epsilon_0 L} \ln\left(\frac{R_a}{R_i}\right)$$

$$C = \frac{Q}{U} \Rightarrow C = \frac{2\pi\epsilon_0 L}{\ln\left(\frac{R_a}{R_i}\right)}$$

Question 6 **(4 Points)**

(a) Write down the “Coulomb potential” that finds the potential Φ arising from a distribution of free charges ρ. (2 Points)

(b) How does this potential has to be modified if the time that a change in the charge density needs to reach the measurement point cannot be neglected? How are these potentials called? (2 Points)

Solution:

(a)

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon} \int \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dV'$$

(b)

$$\phi(\mathbf{r}, t) = \frac{1}{4\pi\epsilon} \int \frac{\rho\left(\mathbf{r}', t - \frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)}{|\mathbf{r} - \mathbf{r}'|} dV'$$

The time shift is equal to the time, light would need to come from the source position to the measurement point. These potentials are called "retarded potentials".

Question 7 **(13 Points)**

(a) What are the 5 golden rules for accuracy in numerical field calculation? (5 Points)

(b) Give two reasons to use the finite element method over the method of finite differences. (2 Points)

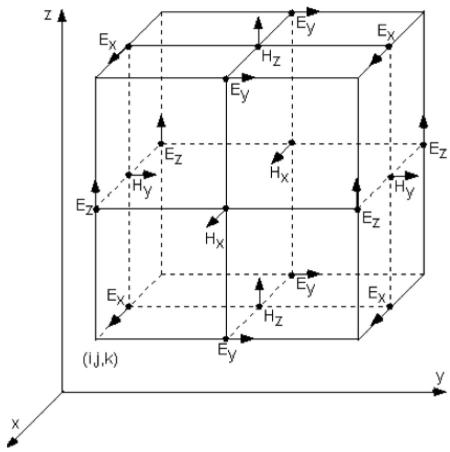
(c) Give a sketch of Yee's lattice (FDTD). What is the ingenious idea of Yee's lattice? (6 Points)

Solution:

(a)
1. Mesh must fit to the geometry of the probem.
2. Mesh must fit to the interpolation function.
3. time step must fit to the mesh. Example: $\Delta t \le \frac{\Delta x}{c} \quad c = \frac{c_{vac}}{\sqrt{\varepsilon\mu}}$
4. numerical accuracy has to be checked.
5. "physics accuracy" has to be checked (e.g. $\nabla \cdot \mathbf{B} = 0$).

(b)
- FEM allows to use irregular and problem-specific grids with elements of variable size.
- For FEM exist higher order methods, i.e. better numerical accuracy is possible.
- FEM allows for weak solutions, i.e. no 2nd derivative is needed.

(c) The idea is to use interlaced lattices for **E** and **H** which allows for an easy translation of Maxwell's equations into an algorithm.



Question 8 **(10 Points)**

Given is the following one dimensional partial differential equation with boundary conditions:

$$-\Delta u = 2, \qquad u(0) = 1, \qquad \frac{d}{dt}u(1) = -1$$

(a) What type of boundary condition is prescribed at the point $x_l = 0$? What type at the point $x_r = 1$? (2 Points)

(b) Discretize the given partial differential equation using central difference approximation on three equally spaced subintervals of the interval $[0, 1]$. Write down the equations that has to be fulfilled at the points x_0, x_1, x_2, x_3. *Hint: Don't forget the boundaries.* (5 Points)

(c) Write down the resulting equations in the form $\mathbf{Lx} = \mathbf{b}$. (3 Points)

Solution:

(a) $x_l: \quad u(0) = 1$ absolute value prescribed $\Rightarrow$ Dirichlet boundary
$x_r: \quad \frac{d}{dt}u(1) = -2$ derivative prescribed $\Rightarrow$ Neumann boundary

(b) $h := \frac{1}{3}$ = width of the three sub-intervals. Then

$$\begin{aligned}
x_0 &: & u_0 &= 1 \\
x_1 &: & -\frac{u_0 - 2u_1 + u_2}{h^2} &= 2 \\
x_2 &: & -\frac{u_1 - 2u_2 + u_3}{h^2} &= 2 \\
x_3 &: & \frac{u_3 - u_2}{h} &= -1
\end{aligned}$$

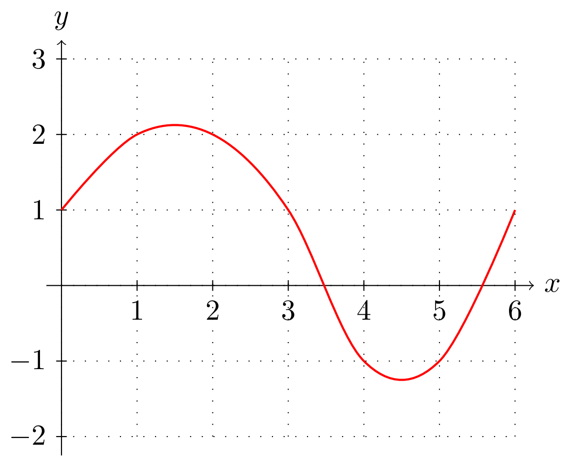
$$\begin{aligned}
x_0 &: & u_0 &= 1 \\
x_1 &: & u_0 - 2u_1 + u_2 &= -2h^2 \\
x_2 &: & u_1 - 2u_2 + u_3 &= -2h^2 \\
x_3 &: & u_3 - u_2 &= -h
\end{aligned}$$

(c)

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix} \qquad b = \begin{pmatrix} 1 \\ -2/9 \\ -2/9 \\ -1/3 \end{pmatrix}$$

Question 9 **(9 Points)**

(a) Given is the following graph of a function: (3 Points)

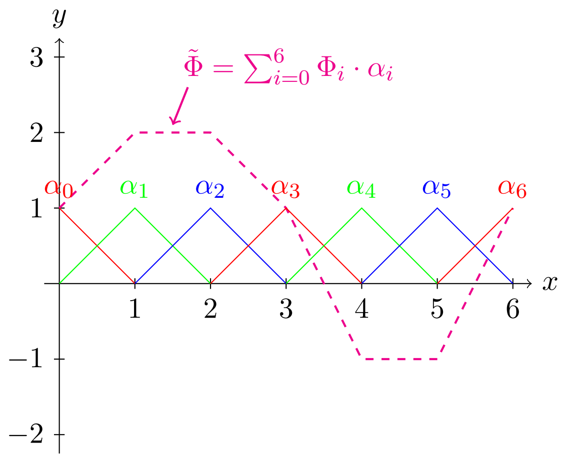


Discretize the domain into 6 equally spaced intervals and sketch the node shape functions if elements of first order are applied.

(b) Describe the basic idea and equations of the finite element method cooperating with the method of weighted residual in the case of solving the Poisson's equation for piecewise homogenous materials. If the Galerkin's choice is applied here, how should the weighting functions look like? Derive the equation until the second derivative of Φ is removed. (6 Points)
Hint: $\int [\Phi \Delta \Psi + (\nabla \Phi)(\nabla \Psi)] dV = \oint \Phi \nabla \Psi \, d\mathbf{a}$

Solution:

(a)



(b) The basic idea is as follows:

Poisson's equation:	$\Delta\Phi = -\frac{\rho}{\varepsilon}$
Ideal solution:	$\Delta\Phi + \frac{\rho}{\varepsilon} = 0$
Approximated solution:	$\Delta\tilde{\Phi} + \frac{\rho}{\varepsilon} = R$
Best approximation:	$\int w \cdot R\, dv = \int \left(\Delta\tilde{\Phi} + \frac{\rho}{\varepsilon}\right) dv = 0$ where w is the weighting function.

Using Green's second law:

$$\int (\nabla\tilde{\Phi}) \cdot (\nabla w)\, dv - \oint \left(w \frac{\partial\tilde{\Phi}}{\partial n}\right) da - \int \frac{\rho}{\varepsilon} w\, dv = 0$$

In the Garkin method, the weighting functions are chosen to be the basis functions: $w_l(x, y, z) = \alpha_l(x, y, z)$