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# Lecture: Electromagnetics and Numerical Calculation of Fields

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# Exam

3rd of March 2017 Start: 14:00 am

Last Name	TERE
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Student PUT	SILIO

Question	Max. Points	Achieved Points
1	16	
2	9	
3	6	
4	8	
5	8	
6	5	
7	2	
8	4	
9	10	
Total:	68	

Grade:\_\_\_\_\_

(16 Points)

- (a) Write down Maxwell equations in integral form. Explain briefly the meaning (8 Points) of each equation.
- (b) Write down the material equations that combine E with D, B with H and (6 Points) J with E for the most general case.
- (c) How does the normal component of **D** behave at boundaries? How does the (2 Points) tangential component of **H** behave at boundaries?

#### Solution:

(a) Maxwell equations:

$$\oint \mathbf{D} \, d\mathbf{f} = \int \varrho \, dv$$

$$\oint \mathbf{B} \, d\mathbf{f} = 0$$

$$\oint \mathbf{H} \, d\mathbf{s} = \int (\mathbf{J} + \dot{\mathbf{D}}) d\mathbf{f}$$

$$\oint \mathbf{E} \, d\mathbf{s} = -\frac{d}{dt} \int \mathbf{B} \, d\mathbf{f}$$

#### **Explanations:**

- 1. Free charges are the sources of D ( $\rho$  : density of free charges).
- 2. The field of magnetic induction does not have any sources.
- 3. Faraday's law of Induction: Time varying magnetic induction causes circulation of **E**.
- 4. Ampère's law: Free currents or displacement currents cause circulation of  ${\bf H}.$

# (b) Dielectrics:

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}$$
  $\varepsilon_0 = \frac{10^7}{c^2 4 \pi}$  unit:  $\frac{\mathrm{As}}{\mathrm{Vm}}$ 

linear relation between  $\mathbf{E}$  and  $\mathbf{P} \Rightarrow \mathbf{D} = \underline{\varepsilon_r} \varepsilon_0 \mathbf{E}$ linear relation between  $\mathbf{E}$  and  $\mathbf{P}$  and isotropic  $\Rightarrow \mathbf{D} = \varepsilon_r \varepsilon_0 \mathbf{E}$ Magnetic materials:

$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$$
  $\mu_0 = 4\pi \cdot 10^{-7}$  unit:  $\frac{\text{Vs}}{\text{Am}}$ 

linear relation between **H** and  $\mathbf{M} \Rightarrow \mathbf{B} = \mu_0 \underline{\mu_r} \mathbf{H}$ linear relation between **H** and **M** and isotropic  $\Rightarrow \mathbf{B} = \mu_0 \mu_r \mathbf{H}$ Conducting materials:

$$\mathbf{J} = \kappa \mathbf{E} + \mathbf{J}^i$$
  $\mathbf{J}^i$ : impressed current density

(c) The normal component of **D** makes a step of  $\sigma$  at the boundary. The tangential component of **H** makes a step of  $\mathbf{J}_F = \frac{dI}{dl}$ . It is continuous if there are no free surface currents.

The two vector fields  $\mathbf{F}_1$  and  $\mathbf{F}_2$  are given by:

$$\mathbf{F}_{1} = k \cdot \begin{pmatrix} yz \\ xy \\ -2xz \end{pmatrix}$$
$$\mathbf{F}_{2} = k \cdot \begin{pmatrix} -y^{2} + z \\ x - 2yz \\ -x + z^{2} \end{pmatrix}$$

- (a) Which of the vector fields could be a magnetic field? Why? (3 Points)
- (b) That magnetic field was produced by an electric current  $(\frac{d}{dt}D = 0)$ . Calculate the corresponding current density **J**.

#### Solution:

(a) Magnetic fields are divergence-free, i.e.  $\nabla \cdot \mathbf{B} = 0$ .

$$\nabla \cdot \mathbf{F}_1 = k \left( \nabla \cdot \begin{pmatrix} yz \\ xy \\ -2xz \end{pmatrix} \right) = k(0 + x - 2x) = -kx \neq 0$$
$$\nabla \cdot \mathbf{F}_2 = k \left( \nabla \cdot \begin{pmatrix} -y^2 + z \\ x - 2yz \\ -x + z^2 \end{pmatrix} \right) = k(0 - 2z + 2z) = 0 \quad \Rightarrow \text{ magnetic field}$$

(b) Current density can be computed using curl operator:

$$\mathbf{J} = \nabla \times \mathbf{H} = \nabla \times \mathbf{F}_{2}$$
$$\mathbf{J} = \begin{pmatrix} \partial_{y}H_{z} - \partial_{z}H_{y} \\ \partial_{z}H_{x} - \partial_{x}H_{z} \\ \partial_{x}H_{y} - \partial_{y}H_{x} \end{pmatrix} = k \begin{pmatrix} 0+2y \\ 1+1 \\ 1+2y \end{pmatrix}$$
$$\Rightarrow \mathbf{J} = k \begin{pmatrix} 2y \\ 2 \\ 1+2y \end{pmatrix}$$

#### Question 3

In seawater the electric field of a plane electromagnetic wave is the following:



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seawater around 1 GHz:  $T = 20^{\circ}$ C,  $\varepsilon_r = 64$ ,  $\mu_r = 1$ ,  $\sigma = 3 \frac{1}{\Omega m}$ ,  $\Gamma_{\text{vacuum}} = 377 \Omega$ (a) Find the corresponding frequency.

- (b) Find the corresponding magnetic field.
- (c) Find the penetration depth.

# Solution:

(a) corresponding frequency

$$c_{\text{water}} = \frac{3 \cdot 10^8}{\sqrt{64}} = \frac{3 \cdot 10^8}{8} = 37.5 \cdot 10^6 \text{ m/s}$$
$$c = \lambda f, \quad \lambda = 0.04 \text{ m} \quad \Rightarrow \quad f = \frac{c}{\lambda} = \frac{37.5 \cdot 10^6}{0.04} = 937 \text{ MHz}$$

(b) magnetic field

$$E_x = \Gamma \cdot H_y \quad \Rightarrow \quad H_y = \frac{E_x}{\Gamma}$$
$$\Gamma = \sqrt{\frac{\mu}{\varepsilon}} \cdot \Gamma_0 = 377 \,\Omega \quad \Rightarrow \quad \Gamma_{\text{water}} = 47.1 \,\Omega$$
$$H_y = \frac{E_x}{\Gamma} = \frac{1 \,\text{V/m}}{47.1 \,\Omega} = 21.2 \cdot 10^{-3} \,\text{A/m}$$

(c) penetration depth

$$d = \sqrt{\frac{2}{\sigma\mu\omega}} = \sqrt{\frac{2}{3 \cdot 4\pi \, 10^{-7} \, 2\pi \, 937 \cdot 10^6}} = 9.5 \,\mathrm{mm}$$

#### Question 4

- (a) Suggest three meaningful combinations of T, E and M and explain.
- (b) Give a sketch of the electric and magnetic field of the  $TM_{11}$  mode inside a rectangular waveguide. Hint: Draw electric field lines using — and magnetic field lines using  $\cdots$  or the corresponding symbols  $(\odot, \otimes)$ . Direction indicators and axis labels are important, too!

#### Solution:

- TE: Transverse electric, no electric field in propagation direction (a) $E_z = 0$ 
  - TM: Transverse magnetic, no magnetic field in propagation direction  $B_z = 0$
  - TEM: transverse electromagnetic, magnetic and electric field are zero in propagation direction  $E_z = B_z = 0$ . These are not possible in hollow waveguides.

(b)  $TM_{11}$  mode:

# (8 Points)

- (3 Points)
- (5 Points)

# (2 Points)

- (2 Points)
- (2 Points)



- (a) Provide a formula for the Poynting vector.
- (b) What is the meaning of the Poynting vector?
- (c) What three components of energy change inside a volume contribute to the power flux through the surface of a volume? Provide formulae for them.

# Solution:

(a)

#### $\mathbf{S}=\mathbf{E}\times\mathbf{H}$

- (b) The Poynting vector is the power per area that is transmitted through a given area via electro-magnetic fields.
- (c) The power flux out of a volume  $(\nabla \cdot \mathbf{S})$  is equal to the sum of ohmic losses, change of electrostatic field energy, change of magnetic field energy:

$$\int \mathbf{JE} \, dV dt \quad \text{- ohmic losses}$$

$$\frac{1}{2} \int \mathbf{ED} \, dV \quad \text{- electrostatic field energy}$$

$$\frac{1}{2} \int \mathbf{BH} \, dV \quad \text{- magnetostatic field energy}$$

# (8 Points)

- (1 Points)
- (1 Points)
- (6 Points)

(a) Find the capacitance of a spherical capacitor with inner radius  $R_1$ , outer radius  $R_2$  and the dielectric constant  $\varepsilon_r$ .

# $R_2$

 $= \int_{R_1}^{R_2} \frac{Q}{4\pi\varepsilon r^2} \, dr$  $= \frac{Q}{4\pi\varepsilon} \int_{R_1}^{R_2} \frac{1}{r^2} \, dr$  $= \frac{Q}{4\pi\varepsilon} \left( -\frac{1}{r} \right) \Big|_{R_1}^{R_2}$  $= \frac{Q}{4\pi\varepsilon} (\frac{1}{R_1} - \frac{1}{R_2})$ 

# Question 7

(a) Assume a thin wire of arbitrary shape is given together with the current Iin the wire. How can you find the magnetic vector potential? What is the name for this?

 $C = \frac{4\pi\varepsilon}{\frac{1}{R_1} - \frac{1}{R_2}}$ 

#### Solution:

(a) It is the Biot-Savart-Law:

$$\mathbf{A} = \frac{\mu \cdot I}{4\pi} \int \frac{1}{|\mathbf{r} - \mathbf{r}'|} d\mathbf{s}$$

## Question 8

- (a) Give a sketch of the "computing molecule" for the Laplace/Poisson equation (finite difference method FDM).
- (b) Write down a finite difference approximation of the Poisson operator in 2d using central difference scheme for approximating second derivatives.
- (c) Based on the obtained formula, describe "mean value property" of the Laplace equation when the discretization steps in x and y directions are equal.

# (2 Points)

(2 Points)



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#### (4 Points)

- (1 Points)
- (2 Points)
- (1 Points)



- (a) Draw a figure of a linear node shape function. How can we approximate a potential  $\Phi$  using N node shape functions  $\phi_i$ ?
- (b) Give two reasons to use the finite element method over the method of finite differences.
- (c) What is the reason to use a weak formulation in FEM?
- (d) Describe the basic idea of *weighed residuals* and equations to get there from the strong formulation in the case of Poisson's equation for piecewise homogenous materials. Hint:  $\int [\Phi \Delta \Psi + (\nabla \Phi)(\nabla \Psi)] dV = \oint \Phi \nabla \Psi d\mathbf{a}$



#### (10 Points)

- (2 Points)
- (2 Points)
- (1 Points)
- (5 Points)

- FEM allows to use irregular and problem-specific grids with elements of variable size for a better approximation.
- FEM allows for weak solutions, i.e. no 2nd derivative is needed.
- (c) The weak formulation has less derivatives and allows to fulfill the differential equation with functions that are only piecewise differentiable.
- (d) The basic idea is as follows:

Using Green's second law:

$$\int (\nabla \tilde{\Phi}) \cdot (\nabla w) \, dv - \oint \left( w \frac{\partial \tilde{\Phi}}{\partial n} \right) \, da - \int \frac{\rho}{\varepsilon} w \, dv = 0$$

In the Garkin method, the weighting functions are chosen to be the basis functions:  $w_l(x, y, z) = \alpha_l(x, y, z)$