Family name	First name	Matriculation number

☐ KSOP, M.Sc. Optics and Photonics
☐ other (please specify) ....

- 120 min to work on the questions
- Use **only** the sheets provided for your calcuations and answers
- Put your name on every sheet
- Use separate sheets for each question
- No calculators, no mobile phones, no electronic devices allowed
- No books, no personal notes, or other documents allowed
- The way taken to derive a solution has to be clear and comprehensive
- If you introduce new variables, define them well (e.g. use a sketch)
- Generally, derive first a final equation before plugging in values.

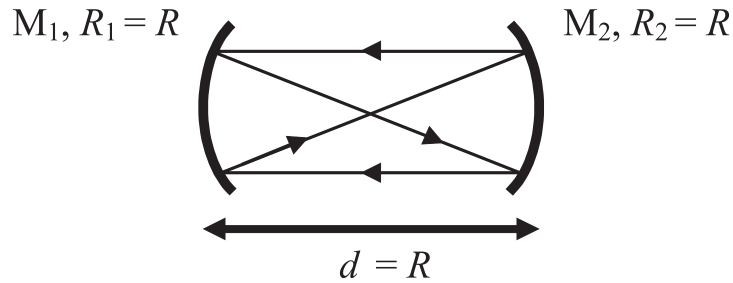
No	Points possible	Points achieved
1	10	
2	10	
3	10	
4	10	
5	10	
Sum	50	

Mark:	Signature:
Final mark:	Signature:

1. **Geometrical and matrix optics** (10 points)

a) Show that for a spherical mirror with curvature radius R, $2f = R$ holds (f is the focal length). Start with a sketch of the reflection of a ray parallel to the optical axis (paraxial approximation).,

b) A confocal resonator consists of two identical, concave sperical mirrors M_1 and M_2 with curvature $R_1 = R_2 = R$ (see drawing). The distance d between both mirrors is as large as their curvature ($d = R$).



Using matrix optics, show that a ray moving under an arbitrary angle to the optical axis from the left mirror to the right reaches its intial state after four reflections. This means that the ray does not leave the resonator.

The matrix of a mirror in the basis $\begin{pmatrix} \mathrm{r} \\ \alpha \end{pmatrix}$ is given by $\begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}$.[1]

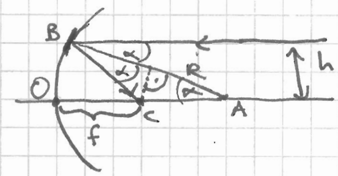
c) Metal mirrors typically have a high reflectivity in the visible spectral region. Sketch the reflectivity as function of frequency $R(\omega)$ under normal incidence for a typical metal like aluminum. Indicate the plasma frequency in your graph.

In which spectral range lies the plasma frequency ω_p for metals like aluminum?

Sketch the real part of the dielectric function $\epsilon_1(\omega)$. Why do metals become transparent for $\omega \geq \omega_p$?

[1] possible matrices $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$: $\begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 0 & \frac{n_1}{n_2} \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ -\frac{1}{R}\frac{n_1-n_2}{n_2} & \frac{n_1}{n_2} \end{pmatrix}$

a) Spherical mirror:



construct radius R normal to surface of mirror

beam parallel to optical axis goes through focal point f.

→ equal-sided triangle ABC

$\rightarrow \cos\alpha = \frac{R/2}{\overline{AC}} \rightarrow \overline{AC} = (R/2)/\cos\alpha$

$\rightarrow \overline{OC} = R\left(1 - \frac{1}{2\cos\alpha}\right)$

choose h small: → α small, $\cos\alpha \approx 1$

$\rightarrow \overline{OC} = f = \frac{R}{2}$ qed

b) Matrix optics:

propagation, reflection right, propagation, reflection left (one round-trip):

$$\begin{pmatrix} 1 & 0 \\ -2/R & 1 \end{pmatrix}\begin{pmatrix} 1 & R \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -2/R & 1 \end{pmatrix}\begin{pmatrix} 1 & R \\ 0 & 1 \end{pmatrix}$$

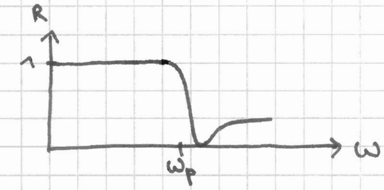
with $f = \frac{R}{2}$

$$= \begin{pmatrix} 1 & R \\ -2/R & -1 \end{pmatrix}\begin{pmatrix} 1 & R \\ -2/R & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

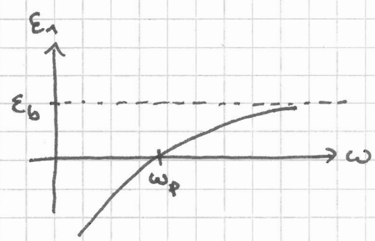
After two round trips (4 reflections):

$$\begin{pmatrix} r' \\ \alpha' \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}^2 \begin{pmatrix} r \\ \alpha \end{pmatrix} = \begin{pmatrix} r \\ \alpha \end{pmatrix}$$ qed

c) Reflectivity



ω_p typically in the UV range



for $\omega \geq \omega_p$: $\varepsilon_1 > 0$

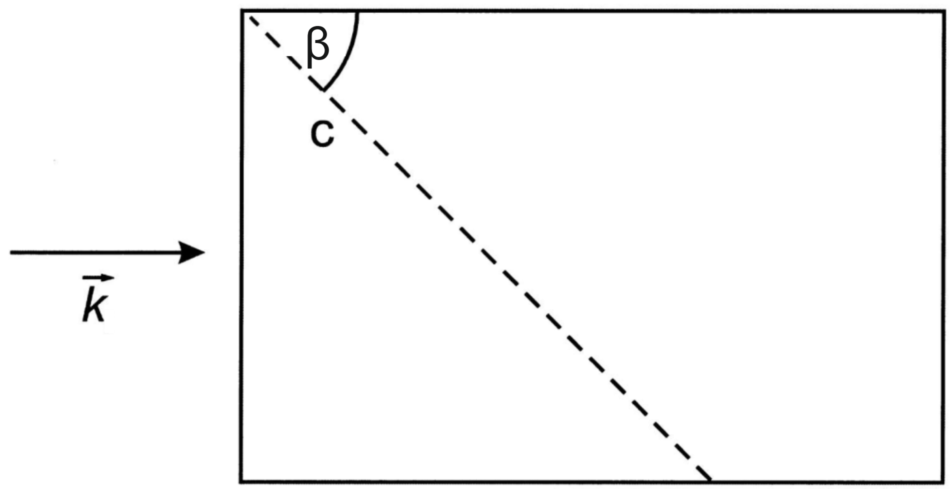
→ n is real

→ metal is (partly) transparent

(plasma oszillations cannot follow driving oscillations of light field)

2. **Birefringence** (10 points)

a) An unpolarized, plane wave (propagating from the left to the right side) with wave vector $\vec{k}$ impinges under normal incidence on a birefringent crystal. Complete this sketch by drawing the ordinary and extraordinary beam inside the crystal and behind the crystal (it is not necessary to take into account the correct deflection direction, yet).



Add the directions of the vectors listed below for the ordinary and for the extraordinary beam for both inside and behind the crystal:

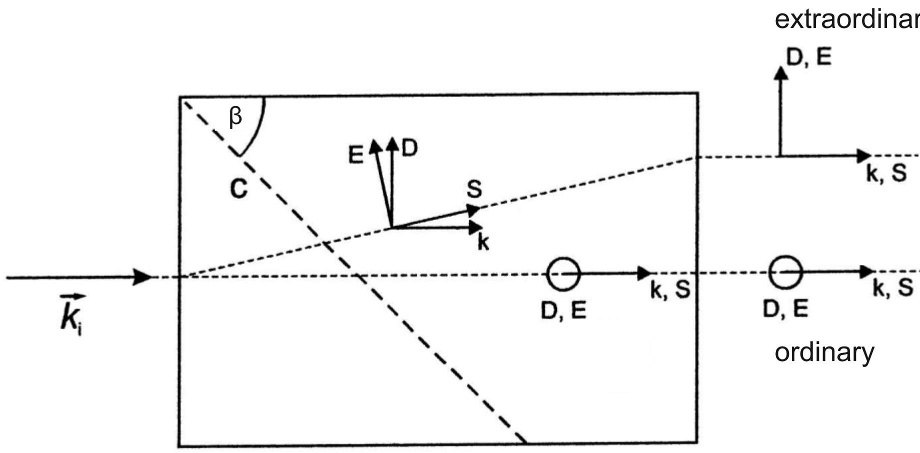
- wave vector $\vec{k}$
- dielectric displacement $\vec{D}$
- electric field vector $\vec{E}$
- Poynting vector $\vec{S}$

b) Justify briefly for **each** vector of the above list why you choose its specific direction for both the ordinary and extraordinary beam (only inside the crystal).

c) Deduce qualitatively, for the case of TiO_2 ($n_\perp = 2.6$, $n_{||} = 2.9$ and $\beta = 45°$) in which direction the extraordinary beam is actually deflected.

Hint: Consider the parallel and orthogonal components of $\vec{E}$ and $\vec{D}$ with respect to the optical axis c.

a)



b)

dielectric displacement $\vec{D}$: The normal component of the dielectric displacement $\vec{D}$ is continuous. This component is zero at the interface where the beam enters the crystal and is thus zero inside the crystal as well. The vector $\vec{D}$ therefore does not change its direction. This is true for the ordinary and extraordinary beam.

wave vector $\vec{k}$: The equation $\vec{k} \cdot \vec{D} = 0$ holds before, inside and after the crystal, as div($\vec{D}$)= 0. Consequently, the wave vector $\vec{k}$ is always orthogonal to $\vec{D}$. As the direction of $\vec{D}$ remains unchanged, the same is true for the wave vector $\vec{k}$. This is true for both beams.

electric field vector $\vec{E}$:
For the ordinary beam, $\vec{E}$ is normal to the optical axis. Thus $\vec{E} \parallel \vec{D}$.
For the extraordinary beam: $\vec{E}$ generally changes its direction due to the anisotropy of the refractive index of the material (and thus of the dielectric constant (tensor) $\hat{\epsilon}$).
$\vec{E} = \frac{1}{\epsilon_0}\hat{\epsilon}^{-1} \cdot \vec{D}$

Poynting vector $\vec{S}$: The Poynting vector$\vec{S}$ is defined as $\vec{S} = \vec{E} \times \vec{H}$. The vector $\vec{H}$ remains unaltered. Thus $\vec{S}$ is always normal to $\vec{E}$. $\vec{S} \perp \vec{E}$. This is true for both beams.

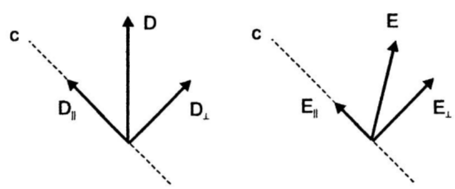
c)

direction of deflection:

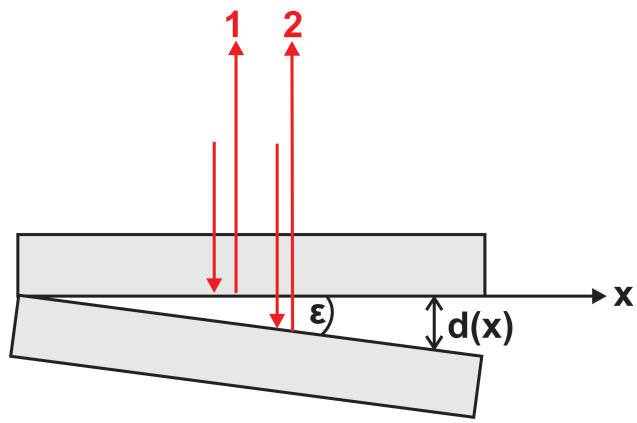
For the normal and parallel components of $\vec{E}$ and $\vec{D}$, respectively, hold:

$$\begin{aligned} D_{||} &= \epsilon_0 \epsilon_{||} \vec{E}_{||} = \epsilon_0 n_{||}^2 E_{||} \\ D_{\perp} &= \epsilon_0 \epsilon_{\perp} E_{\perp} = \epsilon_0 n_{\perp}^2 E_{\perp} \\ \Rightarrow \frac{E_{||}}{E_{\perp}} &= \frac{n_{\perp}^2}{n_{||}^2} \frac{D_{||}}{D_{\perp}} \end{aligned}$$

As $n_{\perp} < n_{||} \Rightarrow \frac{n_{\perp}^2}{n_{||}^2} < 1$. That means the ratio $E_{||}$ to $E_{\perp}$ is smaller than the ratio $D_{||}$ to $D_{\perp}$ (which is equal to 1 due to $\beta = 45°$). Consequently, the electric field vector $\vec{E}$ is rotated clockwise compared to $\vec{D}$ (compare to drawing). The beam is thus deflected toward the bottom of the crystal.



3. **Interference** (10 points)
Two planparallel glass plates ($n_{glass} > 1$) are stacked on top of each other. At one edge, a hair as a spacer is used, so that a wedge-shaped air gap ($n_{air} = 1$) is formed between the glass plates (compare figure below).



Parallel light ($\lambda = 500\,\text{nm}$) under normal incidence leads to 10 interference stripes per cm due to the interference between ray 1 and 2 denoted in the figure (assume that ϵ is so small that ray 2 exits parallel to the incident light).

a) Find the function $d(x, \epsilon)$.
b) Explain the optical path of the rays 1 and 2 concidering possible phase shifts at the interfaces in maximal 4 sentences.
c) Consider two neighbouring interference maxima: How large is the angle ϵ between both glass plates?
d) If the plates have a length of $l = 20\,\text{cm}$, calculate $d(l)$.
e) If white light is used, describe qualitatively the appearence of two neighbouring interference stripes.

Solution Distance between the plates as a function of x:
$d(x) = x \cdot \tan(x)$
The beam 2 is reflected at the interface (plate 2) at an angle 2ϵ, for very small angle $2\epsilon \approx 0$
Phase shift:

- beam 1: $n_1 > n_2$: 0
- beam 2: $n_1 < n_2$: π

Condition for constructive interference: $\Delta\phi = 2m \cdot \pi$
$\Delta\phi = 2\pi\frac{\Delta s}{\lambda} - \pi$ $\Delta s = 2d(x) = 2x\tan(\epsilon)$
$\Rightarrow 2x\tan(\epsilon) = (m + \frac{1}{2}\lambda)$ The distance between two adjacent stripes of the interference pattern: Δx use $m_1 = m$, $m_2 = m_1 + 1$

$$\text{I.) } 2x_1\tan(\epsilon) = (m_1 + \frac{1}{2}\lambda)$$

$$\text{II.) } 2x_2\tan(\epsilon) = (m_2 + \frac{1}{2}\lambda) = (m_1 + \frac{3}{2}\lambda)$$

Substract I.) - II.):

$-2\Delta x\tan(\epsilon) = -1\lambda$

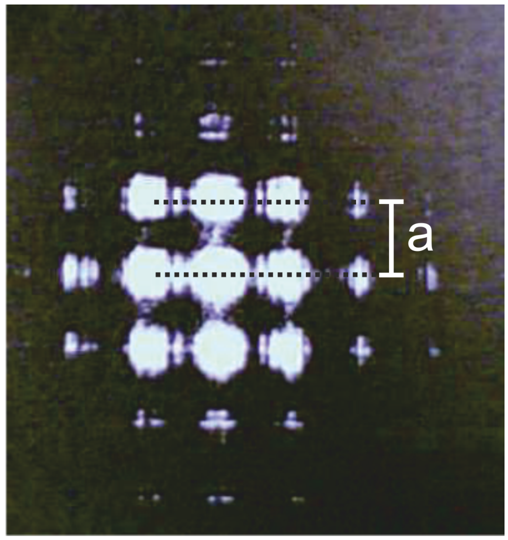
$\tan\epsilon = \epsilon = \frac{\lambda}{2 \cdot \Delta x} = 2.5 \cdot 10^{-4} deg,$

$\Delta x = \frac{1}{10} cm = 10^{-3} m$, $\tan\epsilon = \frac{d}{l}$

$\Rightarrow d = \tan\epsilon \cdot l = 50\mu m$

The stripes will be broader, increasing with higher m. The stipe will be a spectrum with blue light on the left side (smaller distance) and red on the other side, in direction to the spacer.

4. **Grating** (10 points)
Parallel light (an expandet laser beam with wavelength $\lambda = 600\,\text{nm}$) hits a woven cloth (or fabric). In the distance $D = 20\,\text{m}$ to the cloth one can observe the following pattern on a wall, where the bright spots are seperated with $a = 12\,\text{cm}$ from each other.



a) Which symmetry has the weaving of the cloth? Give a reason for your answer in one sentence.
b) How big is the aperture b between the adjacent threads of the cloth?
c) What would be the distance a' of the bright spots at the wall if one uses another laser with wavelength $\lambda = 400\,\text{nm}$?
d) State in one sentence what would happen to the pattern (of **a)** and **b)**) if you would exchange the cloth with another cloth with $b' = 2 \cdot b$. In which distance D' would you have to put the cloth in front of the same wall to observe the initial pattern again?
e) Upon close inspection of the interference pattern one notices, that the fifth maximum of the substructure of each bright spot is missing. Determine the width d of the threads.

Solution
a) quadratic/ four fold symmetry, Fourier transform and grating have the same symmetry.
b) $m\lambda = 2b \sin\theta$

$$\theta = \sin\theta = \tan\theta = \frac{D}{a} = 0.006deg$$

$$b = \frac{m\lambda}{2\sin\theta} = \frac{m\lambda D}{2a} = 50\mu m$$

c) $a' = \dfrac{m\lambda D}{2b} = 8cm.$

d) The pattern would be the same structure but the distance of the bright spots would be half the original distance, to get the same interference pattern, the distance has to be doubled.

e) maximum: $\sin\theta_m = \dfrac{m\lambda}{g}$, minimum: $\sin\theta_n = \dfrac{n\lambda}{b}$

$n = 1$, $m = 5 \Rightarrow 5d = g$

$d = g - b = 4b = 80\,\mu m$

5. **Double slit** (10 points)

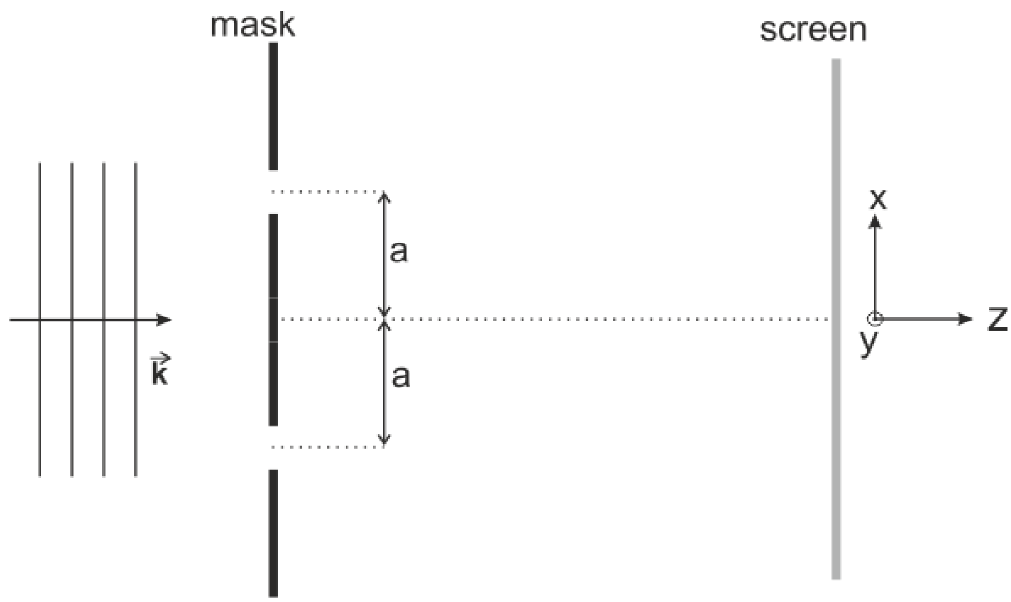
A plane wave with wave vector $\vec{k}$ (coming from the left)parallel to z hits a mask containing two parallel slits along the y-direction as shown below. Each slit can be approximated by a delta-function.

a) Derive an expression for the pattern of the diffracted intensity (in x-direction) as function of $k_x a$ in the Fraunhofer approximation for this double-slit mask using Fourier transformation.

b) Calculate the zeros and the maxima of the diffraction pattern using the expression derived in a). Then, sketch the diffraction pattern as function of $k_x a$ using these points.

c) Now a phase plate is put in front of one slit, retarding the wave by π.

Calculate again the diffraction pattern. How has it changed compared to the situation before (in **a)** + **b)**)?



Solution

a) The intensity of the diffraction pattern is given by:

$$\frac{I}{I_0} = \left| \int_{-\infty}^{+\infty} T(x) e^{ik_x x} \right|^2$$

The transmission function $T(x)$ of the mask is:

$$T(x) = \delta(x+a) + \delta(x-a)$$

Therefore:

$$\frac{I}{I_0} = \left| e^{-ik_x a} + e^{ik_x a} \right|^2 = \left| 2\cos k_x a \right|^2 .$$

b) The zeros of $\frac{I}{I_0}$ can be calculated as follows:

$$\begin{aligned} 2\cos k_x a &= 0 \Rightarrow \cos k_x a = 0 \\ \Rightarrow k_x a &= \pm 1/2\pi \pm n \cdot \pi; \text{ with } n = 0, 1, 2, .. \end{aligned}$$

and the maxima are at

$$\begin{aligned} \cos k_x a &= 1 \text{ and -1} \\ \Rightarrow \pm k_x a &= 0, , 1\pi, 2\pi, 3\pi, 4\pi, \ldots \end{aligned}$$

A drawing can be made using the position $k_x a$ and associated the values for $\frac{I}{I_0}$: missing...still to be done, but it is a $\cos^2$ function.

c) With a phase plate in front of the central slit the transmission function is modified:

$$T(x) = \delta(x + a) + \delta(x - a) \cdot e^{i\pi}$$

and the intensity of the diffraction pattern is now:

$$\begin{aligned} \frac{I}{I_0} &= \left| e^{-ik_x a} - e^{ik_x a} \right|^2 \\ &= |2i \sin k_x a|^2 \\ &= (2 \sin k_x a)^2 \end{aligned}$$

The diffraction pattern for this case is shifted by $\pi/2$ in respect to the diffraction pattern without the phase plate. That means a maximum is now a minimum and vice versa.