1. 1942 B. Rossi and D.B. Hall carried out an experiment to test special relativity. They installed at 2000 m height a detector, which counted only muons with a velocity of $v = 0.98\ c$. A similar detector was installed at sea-level. The muon has a mean lifetime of 2.2 µs and decays into an electron and neutrinos.
a) Which fraction of incoming muons will reach the ground without special relativity?
b) Which fraction of incoming muons will actually reach the ground?
c) What happens in a frame of reference travelling with one of the muons?

2.
a) Sketch the spectral energy density of a black body as a function of frequency.
b) Explain the Stefan-Boltzmann law!
c) Explain Wien's displacement law!
d) The spectrum of a star has a maximum at a wavelength of 966 nm. What is the temperature of the star?
e) How large is the radius of the star with respect to the Sun, when it irradiates 100-times more power than the Sun. Note, that the spectral maximum of the Sun is at 550 nm.
f) How can the radiative power of a star be determined?

3. A photon with the energy 200 keV is scattered by a free electron at rest. The scattering angle is 60°.
a) How large is the momentum of the photon before the collision?
b) How large is the momentum of the photon after the collision?
c) How large is the angle between the paths of the electron and photon?
d) How large is the kinetic energy in eV of the electron after the collision?
e) How large is the velocity of the electron after the collision?

4.
a) Write down the Schrödinger-equation for a particle with the mass m moving in a the potential $V(\vec{r})$.
b) Write down the Schrödinger-equation and the wave function $\psi(\vec{r},t)$ of a particle moving with the constant energy E in the potential $V(\vec{r})$.
c) A ^{4}He-atom is inclosed in a cube with the edge length of L=0,38 nm. The wavefunction is completely restricted to the cube (infinite potential well). Calculate the energy of the groundstate.
d) At which temperature equals the internal energy of an ideal gas the groundstate energy of task c)?

5.
a) Write down all the quantum number of the 1s, 2s und 2p orbitals of the hydrogen atom.
b) Sketch the radial wavefunctions of the three orbitals.
c) Sketch the dependence of the wavefunction on the polar angle.
d) What is the influence of a magnetic field on the groundstate of the hydrogen atom?

6. The D-line is the dominant spectral line in the visible spectrum of sodium. This line is caused by the transition of electrons between the 3p excited state and the 3s groundstate.

a) Write down all quantum numbers of the 3s and 3p orbitals.

b) The D-line is composed of two spectral lines with the wavelength λ_1=589,5925 nm and λ_2=588,9950 nm. Explain, why the D-line is split into two components.

c) Attribute the wavelengths to the quantum states of subtask a)

d) It is possible to estimate the lifetime of the excited 3p state due to the fact that two spectral lines are observable. How long is the lifetime at least?

7. The electronic configuration of zinc is [Ar]$3d^{10}4s^2$.

a) Calculate the Fermi energy of zinc.

b) Calculate the Fermi temperature of zinc.

c) The density of electronic states in a metal is $D(E) = \frac{\partial N}{\partial E} = AE^{1/2}$. Calculate the constant A in terms of the Fermi energy and the total number of electrons.

d) Calculate $D(E_F)$ in terms of the Fermi energy and the total number of electrons.

8. The radioactive isotope ^{14}C is continously produced in the atmosphere by the neutron capture of ^{14}N. ^{14}C enters the biosphere via photosynthesis, so that living matter contains a certain concentration of ^{14}C atoms. In dead matter the ^{14}C concentration decreases with time. The half-life period of ^{14}C is 5730 years.

a) Write down the equation of the ^{14}C production process.

b) ^{14}C carries out a β-decay. Write down the decay equation.

c) The β-activity of a fossil bone is only 1/8 of the activity of a reference probe with the same amount of carbon. What is the age of the bone?

Required physical constants:

Wien's displacement constant:	b = 2,9 mm·K
Stefan-Boltzmann constant:	$\sigma = 5{,}67\cdot10^{-8}$ W/m^{2}K^4
Planck's constant:	$h = 4{,}14\cdot10^{-15}$ eVs
Compton wavelength:	$\lambda_C = 2{,}42\cdot10^{-12}$ m
Velocity of light:	$c = 3\cdot10^{+8}$ m/s
Elementary charge:	$e = 1{,}6\cdot10^{-19}$ As
Rest energy of the electron:	m_ec^2 = 0,5 MeV
Moar mass of Helium:	4 g/mol
Avogadro constant:	$N_A = 6\cdot10^{23}$ mol^{-1}
Boltzmann constant:	$k_B = 1{,}38\cdot10^{-23}$ J/K
Density of zinc:	7,14 g/cm^3
Molar mass of zinc:	65,39 g/mol

Problem	1	2	3	4	5	6	7	8
Points	4	4	4	3	5	4	4	4

1. 1942 B. Rossi and D.B. Hall carried out an experiment to test special relativity. They installed at 2000 m height a detector, which counted only muons with a velocity of $v = 0.98\ c$. A similar detector was installed at sea-level. The moun has a mean lifetime of $\tau = 2.2$ µs and decays into an electron and neutrinos.

a) Which fraction of incoming muons will reach the ground without special relativity?
b) Which fraction of incoming muons will actually reach the ground?
c) What happens in a frame of reference travelling with one of the muons?

Answer

a) Decay law $N(t) = N_0 \exp\left(-\frac{t}{\tau}\right)$

$$t = \frac{s}{v} = \frac{2000\ \text{m}}{0.98 \cdot 3 \cdot 10^8\ \text{m/s}} = 6.8 \cdot 10^{-6}\ \text{s}$$

$$\frac{N(t)}{N_0} = \exp\left(-\frac{6.8}{2.2}\right) = 0{,}0454$$

b) Due to the motion the clock of the muon is slowed down: time dilation
The time in the frame of the muon shorter than for the external observer

$$t' = t \cdot \sqrt{1 - \left(\frac{v}{c}\right)^2} = 6.8\ \mu\text{s}\ \sqrt{1 - (0.98)^2} = 6.8\ \mu\text{s} \cdot 0.2 = 1.35\ \mu\text{s}$$

$$\frac{N(t)}{N_0} = \exp\left(-\frac{1.35}{2.2}\right) = 0{,}54$$

c) In the frame of the muon, the traveled distance becomes smaller: length contraction

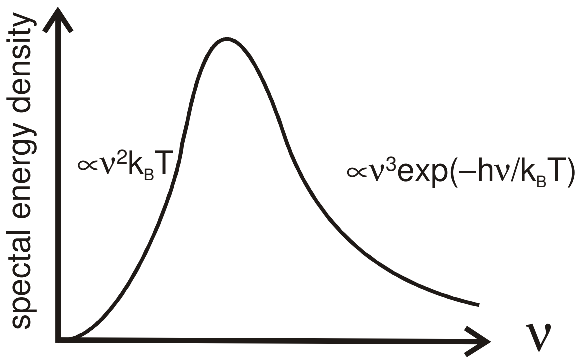
$$h' = h \cdot \sqrt{1 - \left(\frac{v}{c}\right)^2} = 2000\ \text{m}\ \sqrt{1 - 0{,}98^2} = 398\ \text{m}$$

2.

a) Sketch the spectral energy density of a black body as a function of frequency.
b) Explain the Stefan-Boltzmann law!
c) Explain Wien's displacement law!
d) The spectrum of a star has a maximum at a wave length of 966 nm. How large is the temperature of the star?
e) How large is the radius of the star with respect to the sun, when it irradiates 100-times more power than the Sun. Note, that the spectral maximum of the Sun is at 550 nm.
f) How can the radiative power of a star be determined?

Answer

a)



b) The total emitted power of a black body is

$$P = \sigma T^4 A$$

A : surface of the black body
T : temperature of the black body

c) Wien's displacement law : the maximum of the emitted power appears at the wavelength

$$\lambda_{\text{Max}} = \frac{2{,}9 \text{ mmK}}{T}$$

d) The temperature of the star is: $T = 2{,}9 \text{ mmK}/966 \cdot 10^{-9} \text{ m} = 3002 \text{ K}$

e) $P_{\text{Stern}} = 100 P_{\odot}$

$$P_{\odot} = \sigma T_{\odot}^4 4\pi r_{\odot}^2 = \sigma \frac{b^4}{\lambda_{\max,\odot}^4} 4\pi r_{\odot}^2$$

$$P_{\text{Stern}} = \sigma T_{\text{Stern}}^4 4\pi r_{\text{Stern}}^2 = \sigma \frac{b^4}{\lambda_{\max,\text{ Stern}}^4} 4\pi r_{\text{Stern}}^2$$

$$P_{\text{Stern}} = \sigma \frac{b^4}{\lambda_{\max,\text{ Stern}}^4} 4\pi r_{\text{Stern}}^2 = 100\sigma \frac{b^4}{\lambda_{\max,\odot}^4} 4\pi r_{\odot}^2$$

$$r_{\text{Stern}}^2 = 100 \frac{\lambda_{\max,\text{ Stern}}^4}{\lambda_{\max,\odot}^4} r_{\odot}^2$$

$$r_{\text{Stern}} = \frac{\lambda_{\max,\text{ Stern}}^2}{\lambda_{\max,\odot}^2} 10 \cdot r_{\odot} = \left(\frac{966}{550}\right)^2 10 \cdot r_{\odot} = 31 \cdot r_{\odot}$$

f) The radiation intensity of a star is

$$I = \frac{P_{\text{Stern}}}{4\pi r^2}$$

r: distance star-observer

The radiative power in the detector of the observer is

$$P_{\text{Detektor}} = I A_{\text{Detektor}}$$

A_{Detektor}: area of the detector.

By means of Planck's radiation law and corrections due to possible losses of radiation it is possible to determine the total radiative power of a star when its distance is known.

3. A photon with the energy 200 keV is scattered by a free electron at rest. The scattering angle is 60°.

a) How large is the momentum of the photon before the collision?
b) How large is the momentum of the photon after the collision?
c) How large is the angle between the paths of the electron and photon?
d) How large is the kinetic energy in eV of the electron after the collision?
e) How large is the velocity of the electron after the collision?

Answer

a) De Broglie $p = \frac{h}{\lambda}$

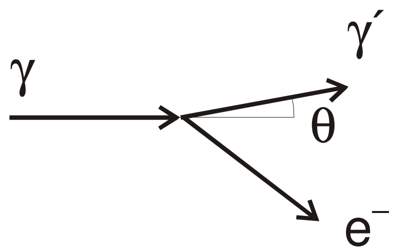
Planck $E = h\nu = h\frac{c}{\lambda} = pc$

Momentum of the photon $p = E / c = 200\ \text{keV}/3 \cdot 10^8 \frac{\text{m}}{\text{s}} = 1{,}067 \cdot 10^{-22}\ \text{kgm/s}$

b) Wavelength before the collison $\lambda = \frac{h}{p} = \frac{4{,}14 \cdot 10^{-15}\ \text{eVs}}{1{,}067 \cdot 10^{-22}\ \text{kgm/s}} = 6{,}2 \cdot 10^{-12}\ \text{m}$

Wavelength after the collison
Compton-scattering $\lambda' - \lambda = \lambda_C (1 - \cos\theta)$

$\lambda' = \lambda + \lambda_C (1 - \cos\theta) = 6{,}2 \cdot 10^{-12}\ \text{m} + 2{,}42 \cdot 10^{-12}\ \text{m}(1 - \cos 60°) = 7{,}41 \cdot 10^{-12}\ \text{m}$



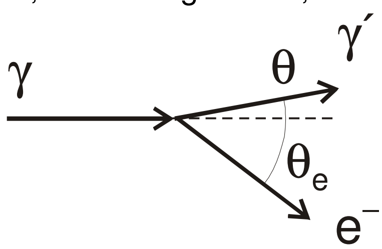
Momentum of the photon after the collison:

$p' = \frac{h}{\lambda'} = \frac{4{,}14 \cdot 10^{-15}\text{eVs}}{7{,}41 \cdot 10^{-12}\ \text{m}} = 8{,}94 \cdot 10^{-23}\ \text{kgm/s}$

c) Conservation of momentum

$\vec{p} = \vec{p}' + \vec{p}_e \rightarrow p_e^{\ 2} = (\vec{p} - \vec{p}')^2 = \vec{p}^2 + \vec{p}'^2 - 2\vec{p}\vec{p}'\cos\theta$

$p_e = \sqrt{10{,}7^2 + 8{,}94^2 - 10{,}7 \cdot 8{,}94} \cdot 10^{23}\ \text{kgm/s} = 9{,}94 \cdot 10^{23}\ \text{kgm/s}$



The transverse momenta cancel each other

$p' \sin\theta = p_e \sin\theta_e \rightarrow \sin\theta_e = \frac{p'}{p_e}\sin\theta = \frac{8{,}94}{9{,}94}\sin 60° \rightarrow \theta_e = 51°$

The angle between the elektron and photon is 111°

d) The kinetic energy of the electron after the collison is

$$E_{\mathrm{kin}} = c\left(p - p'\right) = 3 \cdot 10^{8}\,\mathrm{m/s}\left(1{,}07 - 0{,}9\right) \cdot 10^{-22}\ \mathrm{kgm/s} = 32\ \mathrm{keV}$$

d) The velocity of the electron is

$$mc^2 = E_{kin} + m_e c^2 = \gamma m_e c^2$$

$$\frac{1}{\gamma} = \frac{m_e c^2}{E_{kin} + m_e c^2} = \sqrt{1 - \left(\frac{v}{c}\right)^2} = \frac{0{,}5}{0{,}532} = 0{,}94 \rightarrow v = c\sqrt{1 - 0{,}94^2} = 0{,}34c$$

4.

a) Write down the Schrödinger-equation for a particle with the mass *m* moving in a the potential $V(\vec{r})$.

b) Write down the Schrödinger-equation and the wave function $\psi(\vec{r},t)$ of a particle moving with the constant energy *E* in the potential $V(\vec{r})$.

c) A ^{4}He-atom is inclosed in a cube with the edge length of *L*=0,38 nm. The wavefunction is completely restricted to the cube (infinite potential well). Calculate the energy of the groundstate.

d) At which temperature equals the internal energy of an ideal gas the groundstate energy of task c)?

Answer

a) Schrödinger-equation $\quad i\hbar\frac{\partial}{\partial t}\psi(\vec{r},t)=-\frac{\hbar^2}{2m}\nabla^2\psi(\vec{r},t)+V(\vec{r})\psi(\vec{r},t)$

b) Time-independent Schrödinger-equation

$$E\varphi(\vec{r})=-\frac{\hbar^2}{2m}\nabla^2\varphi(\vec{r})+V(\vec{r})\varphi(\vec{r})$$

und

$$\psi(\vec{r},t)=\varphi(\vec{r})\exp\left(-\frac{iEt}{\hbar}\right)$$

c) The wavefunction is a standing wave

$\varphi(x,y,z)=A\sin(k_x x)\sin(k_y y)\sin(k_z z)$

for $0\le x,y,z\le L$, else $\varphi(x,y,z)=0$

with the boundary condition

$k_x L=\pi n_x,\ k_y L=\pi n_y,\ k_z L=\pi n_z$

$k_{x,y,z}=\frac{\pi}{L}n_{x,y,z}$

results the energy of the groundstate

$$E_{n_x=1,n_y=1,n_z=1}=\frac{\hbar^2}{2m}3\left(\frac{\pi}{L}\right)^2=\frac{3h^2}{8mL^2}=\frac{3\cdot\left(4{,}14\cdot10^{-15}\text{ eVs}\right)^2 6\cdot10^{23}\text{ mol}^{-1}}{8\cdot4\cdot10^{-3}\text{ kg/mol}\cdot\left(3{,}8\cdot10^{-10}\text{ m}\right)^2}$$

$=1{,}71\cdot10^{-22}\text{ Nm }=1{,}07\text{ meV}$

d) Ideal gas $\bar{E}=\frac{3}{2}k_B T$

$$T=\frac{2}{3}\frac{1{,}71\cdot10^{-22}\text{ J}}{1{,}38\cdot10^{-23}\text{ J/K}}=8{,}3\text{ K}$$

5.

a) Write down all the quantum number of the 1s, 2s und 2p orbitals of the hydrogen atom.
b) Sketch the radial wavefunctions of the three orbitals.
c) Sketch the dependence of the wavefunction on the polar angle.
d) What is the influence of a magnetic field on the groundstate of the hydrogen atom?

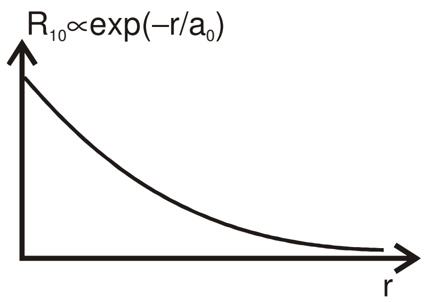
Answer

a)

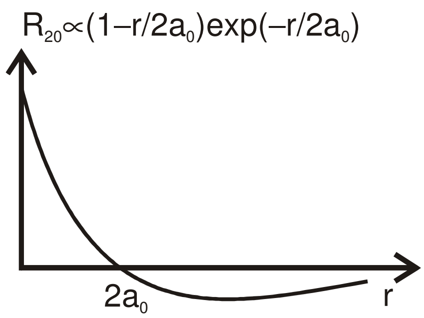
	n	ℓ	m_ℓ	m_s
1s	1	0	0	±1/2
2s	2	0	0	±1/2
2p	2	1	±1,0	±1/2

b)

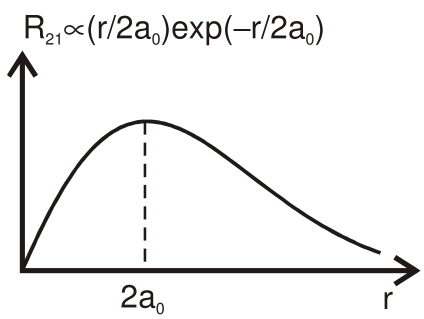
1s:



2s:



2p:

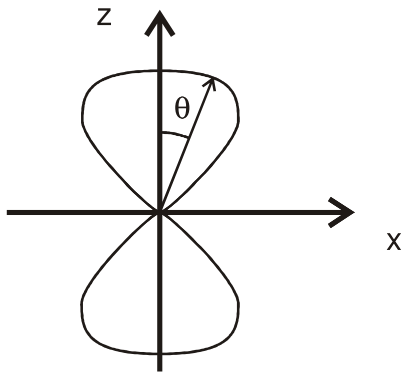


c)

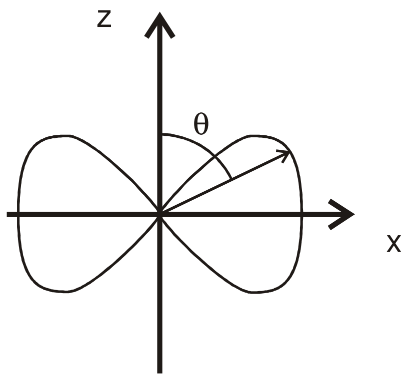
s-orbitals have no angular dependence $Y_{\ell=0,m_\ell=0} = \sqrt{\frac{1}{4\pi}}$

p-orbitals

$Y_{\ell=1,m=0} \propto \cos\theta$



$Y_{\ell=1,m=\pm1} \propto \sin\theta$



d) The groundstate is splitted into two states.

6. The D-line is the dominant spectral line in the visible spectrum of sodium. This line is caused by the transition of electrons between the 3p excited state and the 3s groundstate.

a) Write down all quantum numbers of the 3s and 3p orbitals.

b) The D-line is composed of two spectral lines with the wavelength λ_1=589,5925 nm and λ_2=588,9950 nm. Explain, why the D-line is split into two components.

c) Attribute the wavelengths to the quantum states of subtask a)

d) It is possible to estimate the lifetime of the excited 3p state due to the fact that two spectral lines are observable. How long is the lifetime at least?

Answer

a) 3s: $3\,^2S_{1/2}$, main quantum number n=3, orbital angular momentum ℓ=0, magnetic quantum number er m_ℓ=0, spin s=1/2, m_s=±1/2, total angular momentum j=1/2, m_j=±1/2.

3p: $3\,^2P_{1/2}$, main quantum number n=3, orbital angular momentum ℓ=1, magnetic quantum number m_ℓ=±1,0, spin s=1/2, m_s=±1/2, total angular momentum j=1/2, m_j=±1/2.

$3\,^2P_{3/2}$, main quantum number n=3, orbital angular momentum ℓ=1, magnetic quantum number m_ℓ=±1,0, spin s=1/2, m_s=±1/2, total angular momentum j=3/2, m_j=±1/2, ±3/2.

b) spin-orbit coupling

c) The magnetic moments of the orbital motion and of the spin of the electron couple antiparallel. Therefore the potential energy of $3\,^2P_{1/2}$ is smaller than the potential energy of $3\,^2P_{3/2}$.

λ_1 results of the transition $3\,^2P_{1/2} \rightarrow 3\,^2S_{1/2}$ and

λ_2 results of the transition $3\,^2P_{3/2} \rightarrow 3\,^2S_{1/2}$

d)

$$E = h\frac{c}{\lambda}$$

$$\Delta E = hc\left(\frac{1}{\lambda_2} - \frac{1}{\lambda_1}\right)$$

$$= 4{,}14 \cdot 10^{-15}\,\text{eVs} \cdot 3 \cdot 10^{8}\ \text{m/s}\left(\frac{1}{588{,}995\ \text{m}} - \frac{1}{589{,}5925\ \text{m}}\right) \cdot 10^{9}$$

$$= 2{,}14\ \text{meV}$$

uncertainty relation

$$\Delta E \Delta t \geq \frac{\hbar}{2}$$

$$\Delta t \geq \frac{h}{4\pi}\frac{1}{\Delta E} = \frac{4{,}14 \cdot 10^{-15}\ \text{eVs}}{4\pi \cdot 2{,}14 \cdot 10^{-3}\ \text{eV}} = 1{,}5 \cdot 10^{-13}\ \text{s}$$

7. The electronic configuration of zinc is [Ar]$3d^{10}4s^2$.

a) Calculate the Fermi energy of zinc.

b) Calculate the Fermi temperature of zinc.

c) The density of electronic states in a metal is $D(E) = \dfrac{\partial N}{\partial E} = AE^{1/2}$. Calculate the constant A in terms of the Fermi energy and the total number of electrons.

d) Calculate $D(E_F)$ in terms of the Fermi energy and the total number of electrons.

Answer

a) Fermi wavenumber $k_F = \sqrt[3]{3\pi^2 n}$

$n = N_0/V$ density of the conduction electrons

atomic density

$$n_{\text{Atom}} = \frac{7{,}14\ \text{g/cm}^3}{65{,}39\ \text{g/mol}} = 0{,}11\ \text{mol/cm}^3 \rightarrow n = 2n_{\text{Atom}} = 0{,}22\ \text{mol/cm}^3$$

$$k_F = \sqrt[3]{3\pi^2 n} = \sqrt[3]{3\pi^2 0{,}22\ \text{mol/cm}^3 \cdot 6\cdot 10^{23}\ \text{mol}^{-1}} = 1{,}58\cdot 10^{8}\ \text{cm}^{-1}$$

Fermi energy

$$E_F = \frac{\hbar^2 k_F^2}{2m_e} = \frac{\left(4{,}14\cdot 10^{-15}\ \text{eVs}\cdot 1{,}58\cdot 10^{10}\ \text{m}^{-1}\cdot 3\cdot 10^{8}\ \text{m/s}\right)^2}{(2\pi)^2\, 2\cdot 0{,}5\cdot 10^{6}\ \text{eV}} = 9{,}7\ \text{eV}$$

b) Fermi temperature

$$T_F = E_F / k_B = \frac{9{,}7\ \text{eV}}{1{,}38\cdot 10^{-23}\ \text{J/K}} = 113094\ \text{K}$$

c) $$D(E) = \frac{dN}{dE} = AE^{1/2} \quad \rightarrow N_0 = \int_0^{E_F} AE^{1/2}dE = \frac{2}{3}AE_F^{3/2} \rightarrow A = \frac{3}{2}\frac{N_0}{E_F^{3/2}}$$

d) $$D(E) = \frac{3}{2}\frac{N_0}{E_F^{3/2}}\sqrt{E} \rightarrow D(E_F) = \frac{3}{2}\frac{N_0}{E_F}$$

8. The radioactive isotope ^{14}C is continously produced in the atmosphere by the neutron capture of ^{14}N. ^{14}C enters the biosphere via photosynthesis, so that living matter contains a certain concentration of ^{14}C atoms. In dead matter the ^{14}C concentration decreases with time. The half-life period of ^{14}C is 5730 years.

a) Write down the equation of the ^{14}C production.

b) ^{14}C carries out a β-decay. Write down the decay equation.

c) The β-activity of a fossil bone is only 1/8 of the activity of a reference probe with the same amount of carbon. What is the age of the bone?

Answer

a) $^{14}\mathrm{N} + {}^{1}\mathrm{n} \rightarrow {}^{14}\mathrm{C} + {}^{1}\mathrm{p}$

b) $^{14}\mathrm{C} \rightarrow {}^{14}\mathrm{N} + \mathrm{e}^{-} + \bar{\nu}_e$

$\bar{\nu}_e$: electron anti-neutrino

c) Decay law $N(t) = N_0 \exp\left(-\frac{t}{\tau}\right)$

half-life period

$$N\left(t_{1/2}\right) = \frac{N_0}{2} = N_0 \exp\left(-\frac{t_{1/2}}{\tau}\right) \rightarrow t_{1/2} = \tau \cdot \ln 2 \rightarrow \tau = \frac{5730\ \mathrm{a}}{\ln 2} = 8267\ \mathrm{a}$$

activity

$$A = -\frac{\partial N(t)}{\partial t} = \frac{N_0}{\tau} \exp\left(-\frac{t}{\tau}\right)$$

$$\frac{A}{A_0} = \frac{1}{8} = \exp\left(-\frac{t}{\tau}\right) \rightarrow t = \tau \cdot \ln 8 = 8267\ \mathrm{a} \cdot \ln 8 = 17190\ \mathrm{a}$$