Mock Examination for "Theoretical Optics" 2019

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Drop point: 0th floor, Bldg.30.23
Due Date: Jul 17 2019, 16:00

In order to make the return of the corrected exercises easier, please indicate to which tutorial (by the name of the tutor) you normally attend:

Please note that this particular exercise (called a mock examination) is expected to give you an impression of how the examination could look. Also note that 50 points are assigned to this set of exercise in contrast to the other exercises that have 20 points.

Problem 1. Pulse propagation in a dispersive medium (10 points)

(a) Consider a medium where the dispersion relation $k(\omega)$ can be approximated up to the second order around a central frequency ω_0 as (2.5 points)

$$k(\omega) = k(\omega_0) + \frac{dk}{d\omega}\Big|_{\omega=\omega_0} (\omega - \omega_0) + \frac{1}{2}\frac{d^2k}{d\omega^2}\Big|_{\omega=\omega_0} (\omega - \omega_0)^2 . \tag{1}$$

Discuss the physical meanings of the three coefficients, $k(\omega_0)$, $\frac{dk}{d\omega}\big|_{\omega=\omega_0}$, and $\frac{d^2k}{d\omega^2}\big|_{\omega=\omega_0}$.

(b) In scalar approximation, where the electric field is expressed using the scalar quantity $u(z,t)$, assume that the field has a spectrum with a Gaussian envelope that reads as

$$U(z=0,\omega) = \frac{\tau u_0}{\sqrt{2\pi}} e^{-\frac{\tau^2(\omega-\omega_0)^2}{2}} .$$

Here, u_0 is the pulse amplitude at $z=0$ and τ is the pulse duration. Such pulse shall propagate through a slab of thickness d made from the medium considered in problem 3(a). By calculating the output field at $z=d$, find the expression of the pulse duration of the output pulse. (5 points)

(*Hints*: The spectral pulse $U(z,\omega)$ propagates with a phase factor $e^{ik(\omega)z}$, which eventually affects the pulse shape. The pulse duration can be found by looking at $u(z=d,t) = \mathcal{F}.\mathcal{T}.\{U(z=d,\omega)\}$, where $\mathcal{F}.\mathcal{T}.$ represents the Fourier transformation. Useful formula: $\int_{-\infty}^{\infty} e^{-AX^2+BX} dX = \sqrt{\frac{\pi}{A}} e^{\frac{B^2}{4A}}$, where A, B are complex values, and $\Re[A] > 0$.)

(c) The pulse duration after propagation through a medium will either be the same or changes depending on the dispersion relation. Discuss the effect of the constant terms $k(\omega_0)$, $\frac{dk}{d\omega}\big|_{\omega=\omega_0}$, and $\frac{d^2k}{d\omega^2}\big|_{\omega=\omega_0}$, and the distance d on the output pulse duration. (2.5 points)

Problem 2. Kramers-Kronig relation (8 points)

Given that the imaginary part of the permittivity of a medium is

$$\Im[\varepsilon(\omega)] = \frac{\gamma\omega_p^2}{\omega(\gamma^2+\omega^2)},$$

find the real part of the permittivity $\Re[\varepsilon(\omega)]$ by using the Kramers-Kronig relation.

(*Hint*: Use the formula $\Re[\varepsilon(\omega)] - 1 = \frac{1}{\pi}\mathrm{PV}\int_{-\infty}^{+\infty} \frac{\Im[\varepsilon(\bar{\omega})]}{\bar{\omega}-\omega} d\bar{\omega}$, where 'PV' denotes the Cauchy's principal value.)

Problem 3. Talbot Effect (10 points)

Consider that a scalar field directly behind a one-dimensional amplitude mask at z_0 is given by $u_0(x, z_0) = A\left[1 + \cos\left(\frac{2\pi}{G}x\right)\right]$, where G is a real constant, and this field propagates in free space in the direction of z-axis. Show that the field $u(x, z)$ will reproduce itself except for a constant phase factor $e^{i\phi}$ for $\phi \in \mathbb{R}$ at certain propagation distances $z = z_T$, where $z_T = \lambda m / \left(\sqrt{1 - (\lambda/G)^2} - 1\right)$ for a wavelength λ and positive integer m. This phenomenon is called *Talbot effect* and z_T is called *Talbot distance*.

Problem 4. Double refraction (12 points)

Let us consider an uniaxial crystal with $\epsilon_1 = \epsilon_2 = \epsilon_{or}$, $\epsilon_3 = \epsilon_e$.

(a) Show that the Poynting vector $\boldsymbol{S}$ is perpendicular to the normal surfaces (see figure 1). Where ϕ is the angle between the crystal axis and the $\boldsymbol{k}$ vector of the wave inside the crystal. (8 points)

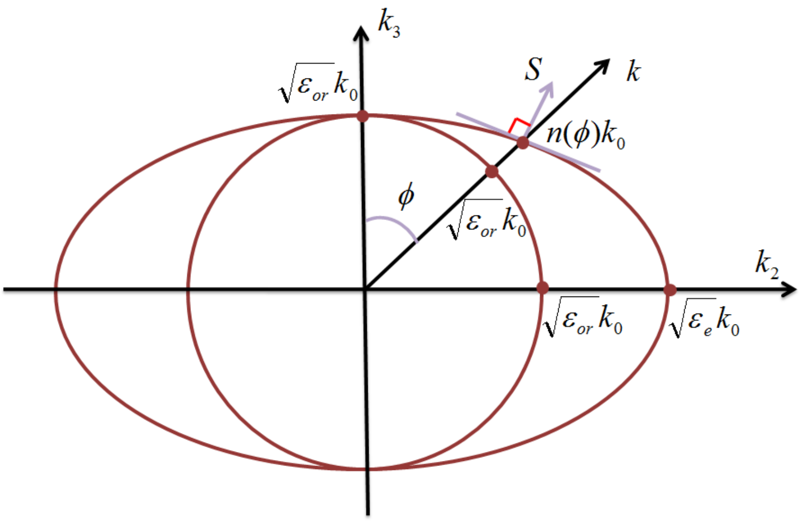

Figure 1: Intersection of the k surface with y-z plane for the uniaxial crystal

(b) Consider the interface between an isotropic medium (characterized by ϵ_i) and an uniaxial crystal (ϵ_{or}, ϵ_e) and light impinging at an angle ϕ_i to the surface normal. The optical axis and the surface normal form an angle α (see figure 2). Show that there will be two refracted rays with two different directions of wavevectors, $\boldsymbol{k}_o$ and $\boldsymbol{k}_e$: I. the *ordinary* one, which obeys ordinary Snell's law

$$\sqrt{\epsilon_i}\sin\phi_i = \sqrt{\epsilon_{or}}\sin\phi_o.$$

II. the *extraordinary* one, which obeys

$$\sqrt{\epsilon_i}\sin\phi_i = n_b(\alpha + \phi_e)\sin\phi_e,$$

where $n_b(\phi)$ is the refractive index of the extraordinary ray traveling at an angle ϕ to the optical axis of the crystal (i.e. the formula presented in the lecture). (4 points)
Hint: Use the fact that the tangential component of the wavevector $\mathbf{k}$ needs to be continuous at the interface.

Problem 5. Temporal coherence - Michelson Interferometer (10 points)

(a) Show that the intensity of light arriving at the photodetector in the Michelson interferometer (see Fig. 3) is written as

$$I_D(h) = 2I_0\left\{1 + \Re\left[g\left(\frac{2h}{c}\right)\right]\right\},$$

where h is the displacement of the mirror M1 (or equivalently, the relative distance between two optical paths), $g(\tau)$ is the complex degree of temporal coherence of the source signal $u(t)$, and the intensity of the

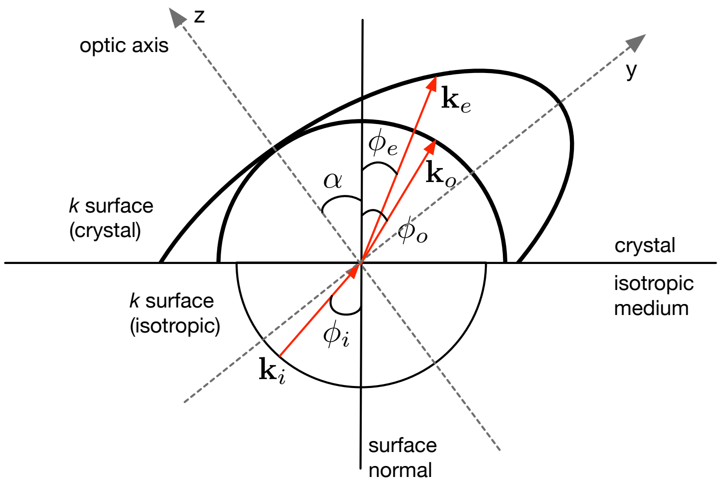

Figure 2: Intersection of the k surfaces with the plane of incidence

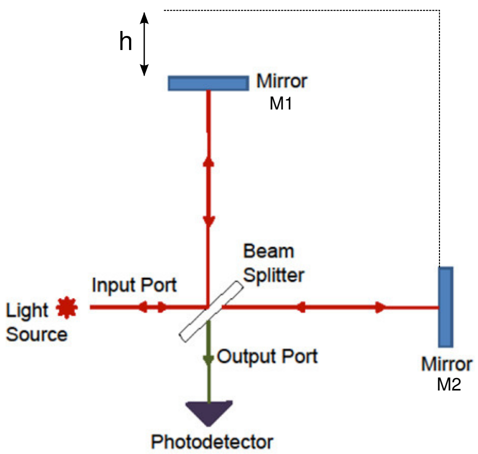

Figure 3: Schematic representation of Michelson interferometer in free space. M1 and M2 denote ideal mirrors (i.e., reflectances are 1), and consider a 50/50 beam splitter which induces a $\pi/2$-phase shift only for the reflected beam.

source signal is represented by $I_0 = \langle |u(t)|^2 \rangle$. Assume that there are no losses in the optical paths. Also consider the source produces a stationary wave, whose intensity is invariant under the translation in time. (5.5 points)

(b) Assume the input signal $u(t)$ is given by a function $F(t)$, i.e. $u(t) = F(t)$, with

$$F(t) = \frac{A}{\sqrt{\pi}T} e^{-\frac{t^2}{T^2}} e^{-i(\omega_0 t + \phi)}.$$

(i) Calculate the intensity of the output signal $I_D(h)$. (1.5 points)

(ii) Calculate the fringe visibility $\nu(h) = \left|g\left(\frac{2h}{c}\right)\right|$. (1.5 points)

(iii) Calculate the value of the displacement h that gives a fringe visibility of $\nu(h) = e^{-\frac{\pi}{2}}$. (1.5 points)