Theoretical Optics

Exercise 1

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Problem 1. Maxwell's equation

(a) Write down Maxwell's equations in time domain in the presence of media.

(b) From Maxwell's equations, derive the continuity equation written as

$$\nabla \cdot \mathbf{J}(\mathbf{r},t) + \frac{\partial \rho(\mathbf{r},t)}{\partial t} = 0,$$

where $\mathbf{J}(\mathbf{r},t)$ and $\rho(\mathbf{r},t)$ denote the current density and the charge density, respectively. Furthermore, explain in one sentence the meaning of the continuity equation.

(c) The dynamics of an electron bound to an ionic core in the presence of an external electric field can be described by the Lorentz model, where the equation of motion for an individual electron is written as

$$\frac{d^2\mathbf{r}(t)}{dt^2} + \gamma\frac{d\mathbf{r}(t)}{dt} + \omega_0^2\mathbf{r}(t) = -\frac{e}{m_e}\mathbf{E}(t),$$

where $\mathbf{r}(t)$ is the displacement vector from the equilibrium point of the electron, m_e is the effective mass, γ is a phenomenological damping factor, ω_0 is the resonance frequency, e is the charge of the electron, and $\mathbf{E}(t)$ is the external electric field. With the help of the above equation, explain the physical difference between the Lorentz model and the Drude model.

(d) The solution of the latter equation in frequency domain can be written as

$$\bar{\mathbf{r}}(\omega) = \frac{e}{m_e}\frac{\bar{\mathbf{E}}(\omega)}{\omega^2 - \omega_0^2 + i\gamma\omega}.$$

Use this to obtain the expressions of the susceptibility $\chi(\omega)$ and relative permittivity $\varepsilon_\mathrm{r}(\omega)$.

Problem 2. Wave propagation in dispersive media

Let us consider a linear, homogeneous, isotropic, non-magnetic, dispersive material that can be modeled by a simple Lorentz oscillator with a single undamped resonance. The relative dielectric function (or electric permittivity) is written as

$$\varepsilon_\mathrm{r}(\omega) = 1 + \frac{Ne^2}{\varepsilon_0 m_0}\frac{1}{(\omega_0^2 - \omega^2)},$$

where ω_0 is the resonance frequency, N is the electron density in the material, e is the electron charge, and m_0 is the electron mass.

(a) At which frequencies an electromagnetic wave can propagate in the medium?

(b) Show that the group velocity v_g of an electromagnetic wave in a dispersive medium considered above can be written as

$$\frac{1}{v_\mathrm{g}} = \frac{1}{c}\left(n + \omega\frac{dn}{d\omega}\right),$$

where $n = \sqrt{\epsilon_\mathrm{r}(\omega)}$ is the refractive index of the medium, and c is the group velocity in vacuum.

(c) Show that the group velocity is different from the phase velocity in a dispersive medium, and compare it to the case of a non-dispersive medium.

(d) At frequencies for which light can propagate, show that the group velocity is always smaller than the group velocity in vacuum.

Problem 3. Kramers-Kronig relation

(a) In the lecture we derived the Kramers Kronig relations in the following form:

$$\Re\left[\chi(\omega)\right] = \frac{1}{\pi}\mathrm{PV}\int_{-\infty}^{\infty}\frac{\Im\left[\chi(\overline{\omega})\right]}{\overline{\omega}-\omega}d\overline{\omega}, \qquad \Im\left[\chi(\omega)\right] = -\frac{1}{\pi}\mathrm{PV}\int_{-\infty}^{\infty}\frac{\Re\left[\chi(\overline{\omega})\right]}{\overline{\omega}-\omega}d\overline{\omega},$$

where PV denotes the Cauchy's principal value, and $\chi(\omega)$ is the suceptability. Express the above equations in terms of the integration over positive frequencies.

(b) Use the Kramers-Kronig relations to calculate the real part of the relative permittivity $\varepsilon_{\mathrm{r}}(\omega)$, given its imaginary part is

$$\Im\left[\varepsilon_{\mathrm{r}}(\omega)\right] = \frac{A\gamma\omega}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2},$$

where A is a constant.

Problem 4. Lorentz model

In good approximation a dielectric medium can be modeled by an ensemble of damped harmonic oscillators. The response function of the medium will then be of the form

$$R(t) = \Theta(t)g(t), \quad \text{with} \quad g(t) = \frac{f}{\Omega}\,e^{-\gamma t}\sin(\Omega t),$$

where $\Theta(t)$ is the Heaviside step function, $\Omega = \sqrt{\omega_0^2-\gamma^2}$, ω_0 is the characteristic frequency of the oscillator, $\gamma > 0$ is the damping constant, $\gamma < \omega_0$, and $f > 0$ is the oscillator strength.

Show that the electric susceptibility is

$$\chi(\omega) = \frac{f}{\omega_0^2-\omega^2-i2\gamma\omega}.$$

Useful formula: $\chi(\omega) = \int_{-\infty}^{\infty} R(t)e^{i\omega t}dt$