Theoretical Optics

Exercise 2

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Drop point: 0th floor, Bldg.30.23
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In order to make the return of the corrected exercises easier, please indicate to which tutorial (by the name of the tutor) you normally attend:

Problem 1. Poynting Vector and Normal Mode

Consider a monochromatic plane wave of frequency ω_0, propagating in a homogeneous isotropic weakly lossy dielectric medium of relative permittivity $\varepsilon = \varepsilon' + i\varepsilon''$ (where $\varepsilon', \varepsilon'' > 0$ and $\varepsilon' >> \varepsilon''$). Its electric field has the form $\mathbf{E}_r(\mathbf{r}, t) = E_0\, \mathbf{e}_x\, e^{-\alpha z}\, \cos(\beta z - \omega_0 t + \phi)$, where the subscript r is used for the real valued fields.

(a) Even though the field is given here as a real valued quantity, in some of the following tasks, it might be better to use the complex representation as a mean to simplify your calculations. Therefore, write at first the same field as above but in complex notation. (1 point)

(b) By starting from the dispersion relation of the plane wave in the medium, show that

$$\beta \approx \frac{\omega_0}{c}\sqrt{\epsilon'} \quad \text{and} \quad \alpha \approx \frac{\omega_0}{c}\frac{\varepsilon''}{2\sqrt{\varepsilon'}}$$

(1.5 points)
Useful formula: $\sqrt{1+z} \approx 1 + \frac{1}{2}z, \quad z \in \mathbb{C}, \quad |z| \ll 1.$

(c) Start from Maxwell's equations to find the real valued magnetic field $\mathbf{H}_r(\mathbf{r}, t)$. (2 points)

(d) Continue to use the real valued representation of the field to write down the formula for the instantaneous Poynting vector, $\mathbf{S}_r(\mathbf{r}, t)$. (1 point)

(e) Find the time averaged Poynting vector using the formula $\langle \mathbf{S}_r(\mathbf{r}, t)\rangle = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{+T} \mathbf{S}_r(\mathbf{r}, t)\, dt$. (1.5 points)
Useful formula: $\cos(a)\cos(b) = \frac{1}{2}[\cos(a-b) + \cos(a+b)]$, $\sin(a)\cos(b) = \frac{1}{2}[\sin(a-b) + \sin(a+b)]$.

Problem 2. Maxwell's Stress Tensor

A monochromatic plane wave in vacuum $\mathbf{E}_{\mathrm{in}}(z, t) = \Re\left[E_0 e^{\mathrm{i}k_0 z} e^{-\mathrm{i}\omega_0 t}\right]\mathbf{e}_x$ irradiates an infinitely extending medium interface (at $z = 0$) at normal incidence. The wave is partially reflected at the interface with a complex reflection coefficient r, defined as the ratio of the incident to the reflected electric field amplitude.

(a) Express the total time-averaged Maxwell's stress tensor $\langle \overleftrightarrow{\mathbf{T}}(z, t)\rangle$ in the vacuum as a function of r and I_0, where $I_0 = \epsilon_0 c_0 E_0^2/2$ is the intensity of the radiated light. (4 points)
(*Hint*: The total electromagnetic field in the vacuum can be written as a superposition of two counter-propagating plane waves, *i.e.*, the incident and the reflected waves. Maxwell's stress tensor is defined as $\overleftrightarrow{\mathbf{T}} = \epsilon_0\epsilon_r \mathbf{EE} + \mu_0\mu_r \mathbf{HH} - \frac{1}{2}(\epsilon_0\epsilon_r \mathbf{E}\cdot\mathbf{E} + \mu_0\mu_r \mathbf{H}\cdot\mathbf{H})\overleftrightarrow{\mathbf{I}}$.)

(b) Calculate the time-averaged radiation pressure (*i.e.*, the perpendicular force per surface area) on the medium. (2 points)

(c) Calculate the time-averaged radiation pressure (*i.e.*, the perpendicular force per surface area) on a perfectly absorbing ($|r| \to 0$) and perfectly reflecting medium ($|r| \to 1$). (1 points)

Problem 3. Evanescent waves

As shown in Fig. 1, consider a plane wave propagating from medium 1 to medium 2 that are characterized by the refractive indices $n_1 > n_2$, where the media are assumed to be homogenous, linear, and isotropic. The incoming wave in medium 1 impinges at the interface with an incident angle of θ_i, then yielding the reflected and transmitted waves with angle of θ_r and θ_t, respectively.

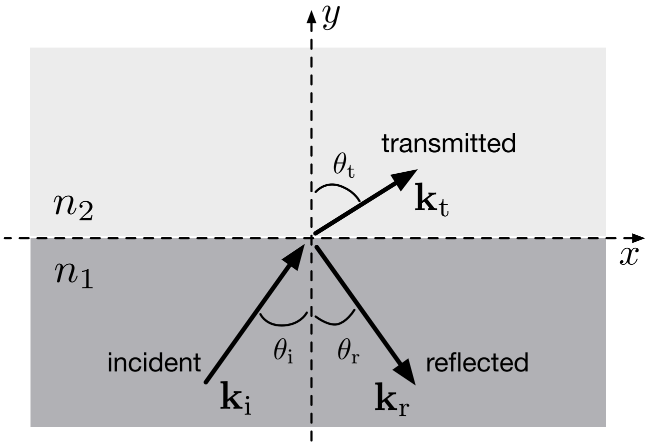

Figure 1: Wave propagation across the interface between two different media with the refractive indices $n_1 > n_2$.

(a) Using the continuity of the tangential component of the wave vector across the interface, derive the critical incident angle θ_c, beyond which (*i.e.*, $\theta_\mathrm{i} \geq \theta_\mathrm{c}$) a total internal reflection occurs. (1 point)

(b) Show that when $\theta_i \geq \theta_c$, the x-component of the wave vector $\mathbf{k}_\mathrm{t}$ is purely real while the y-component is purely imaginary. (2 points)

(c) From the previous result, the electric field of the transmitted (or refracted) wave can be respectively written as

$$\mathbf{E}(x, y, z) = \mathbf{E}_\mathrm{t} e^{ik_x x} e^{-\kappa_y y}, \text{ for } y > 0,$$

where k_x is the (real) x-component of the wave vector $\mathbf{k}_\mathrm{t}$ in medium 2, while κ_y is the (imaginary) y-component. Here, we assume z-component of wave vector is zero, and consider an arbitrarily polarized electric field, *i.e.*, $\mathbf{E}_\mathrm{t} = E_x \hat{\mathbf{e}}_x + E_y \hat{\mathbf{e}}_y + E_z \hat{\mathbf{e}}_z$. This is called *evanescent wave*. Show that the time-averaged Poynting vector in medium 2 is zero along y-axis, but non-zero along x-axis. (3 points)