THEORETICAL OPTICS

EXERCISE 3

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Drop point: 0th floor, Bldg.30.23
Due Date: June 3 2019, 16:00

Please indicate which tutorial you normally attend by the name of the tutor!

Problem 1. Electrostatic stress tensor

Consider two spherical shells with a radius of R are charged with a surface charge density of σ, and are separated by a center-to-center distance $d > 2R$ in free space (see Figure below).

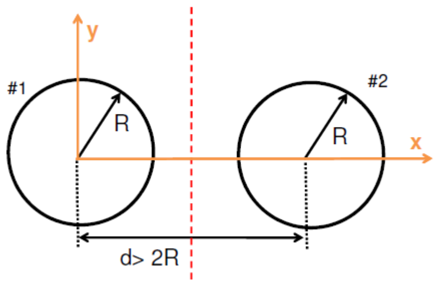

Figure 1: Two identical spherical shells in free space with a surface charge density of σ are depicted in x-y plane.

(a) Show that the total electric field caused by two spherical shells at the planar surface sketched with a red dashed-line ($x = \frac{d}{2}$) in Figure 1 is written as (4 points)

$$\mathbf{E} = \frac{2\sigma R^2}{\epsilon_0}\left(\frac{1}{d^2/4 + y^2 + z^2}\right)^{3/2}(y\hat{y} + z\hat{z}).$$

(b) Calculate Maxwell's stress tensor $\overleftrightarrow{\mathbf{T}}$ at $x = d/2$. (2 points)

(c) By using the stress tensor $\overleftrightarrow{\mathbf{T}}$ obtained above, show that the force that the shell #1 exerts on the shell #2 is written as (3 points)

$$\mathbf{F} = \frac{4\pi\sigma^2 R^4}{\epsilon_0 d^2}\hat{\mathbf{x}}.$$

(*Hint*: A proper surface for the surface integration is y-z plane at $x = d/2$.)

(d) Verify that the calculated force above is identical to the force between two charged *point* particles when the charge of each particle is $Q = 4\pi R^2\sigma$. (1 points)

Problem 2. Scalar diffraction theory

We consider two infinitely narrow slits, located in the plane $z = 0$, infinitely extended in y-direction, and placed at $x = a$ and $x = -a$, respectively. These slits are illuminated by a linearly polarized plane wave with wavelength λ and amplitude A_0. In this question, we consider the scalar approximation.

(a) Write down the field $u_0(x, z = 0_+)$ after the slits. (0.5 points)

(b) Calculate the spatial frequency spectrum $U_0(\alpha, z = 0_+)$ with α being the spatial frequency. (0.5 points)

(c) Write down the exact transfer function in free space. Also, derive the transfer function in the paraxial (Fresnel) approximation. (*Hint*: Consider a reduced dimensionality, i.e., $\beta = k_y = 0$, for simplicity.) (0.5 points)

(d) Calculate the spatial frequency spectrum $U(\alpha, z)$ using the paraxial (Fresnel) approximation. (0.5 points)

(e) Calculate the field $u(x, z)$ at a distance z where the paraxial (Fresnel) approximation holds. (*Useful formula*: $\int_{-\infty}^{\infty} e^{-(A\eta^2+B\eta)}d\eta = \sqrt{\frac{\pi}{A}}e^{\frac{B^2}{4A}}$ for $\Re[A] > 0$.) (1 point)

(f) By referring to the previous steps, calculate the field distribution $u(x, z)$ for $2N$-slits where the distance between the adjacent slits are same as in the above questions, *i.e.*, the slits are located at $z = -(2N-1)a, -(2N-3)a, \cdots, -3a, -a, a, 3a, \cdots, (2N-3)a, (2N-1)a$. (1 point)

Problem 3. Scalar diffraction theory

Consider a one-dimensional grating as illustrated in Fig. 2, modeled as a transmitting structure with amplitude transmittance

$$t(x', y') = A\left[1 + 2\cos\left(\frac{2\pi}{G}x'\right)\right],$$

where A and G are some non-negative constants. In scalar approximation, assume that this object is illuminated by a normally incident, unit-amplitude plane wave propagating along the z direction, and the paraxial approximation holds.

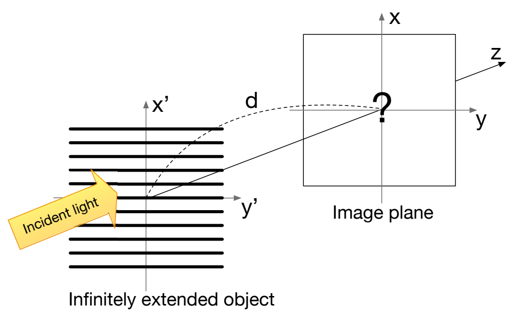

Figure 2: A grating in xy plane.

(a) Find the expression for the field distribution in the image plane at $z = d$. (4 points)
(*Hint*: Use the formula, $\frac{1}{2\pi}\int e^{-i\alpha x}dx = \delta(\alpha)$, where $\delta(x)$ is the Dirac delta distribution.)

(b) From the solution to the above question, at what distances z behind the object we will find a field distribution that is of the same form as that of the object, up to possible complex constants (*i.e.*, ignore the overall phase factor)? (2 points)