Theoretical Optics

Exercise 4

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Drop point: 0th floor, Bldg.30.23
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Please indicate which tutorial you normally attend by the name of the tutor!

Problem 1. Scalar diffraction theory

Consider the circularly symmetric object shown in Fig. 1. It is infinite in extent in the x-y plane. Its amplitude transmission function is given by

$$t_{\mathrm{A}}(r) = 2\pi J_0(ar) + 4\pi J_0(2ar),$$

where $J_0(x)$ is the zeroth-order Bessel function of the first kind, a is some positive real number signifying a spatial frequency (*i.e.*, it has the units of m^{-1}), and $r = \sqrt{x^2 + y^2}$ is the radial coordinate in the two-dimensional plane. In scalar approximation, this object is illuminated by a normally incident, unit-amplitude plane wave propagating along z direction, and the paraxial condition is assumed to hold.

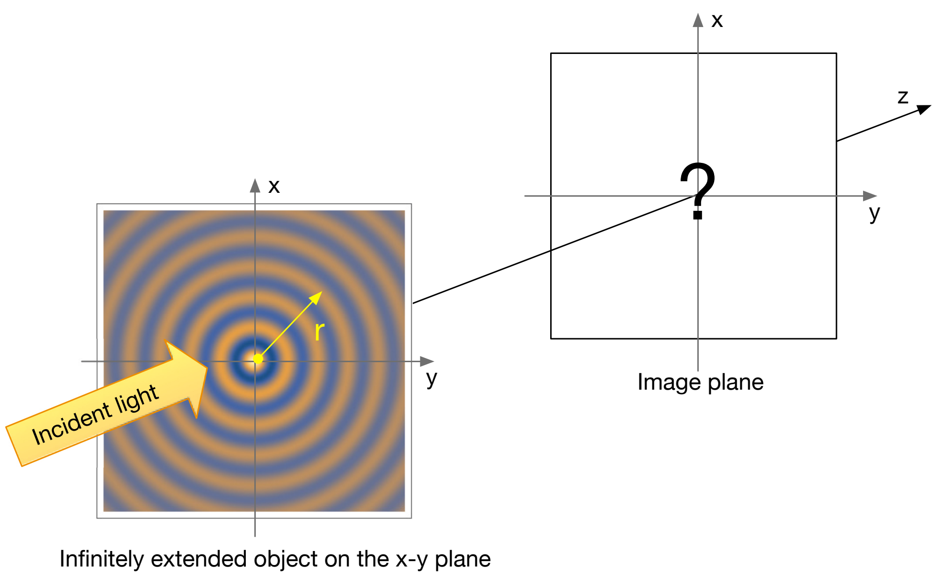

Figure 1: A circularly symmetric object in $x - y$ plane.

(a) Find the expression of the field distribution in the image plane at z. (3 points)

($Hint$: Use the formula for the Fourier transformation ($\mathcal{FT}$) of the Bessel function: $\frac{1}{(2\pi)^2} \iint 2\pi\gamma J_0(\gamma\sqrt{x^2+y^2}) e^{-i(\alpha x + \beta y)} dx dy = \delta(\sqrt{\alpha^2+\beta^2} - \gamma)$, where γ is a some positive real constant. Also use the property of the Dirac delta function: $\mathcal{FT}[\delta(x - x_0) f(x)] = \mathcal{FT}[\delta(x - x_0) f(x_0)]$ where $f(x)$ has no singularity in the whole space.)

(b) At what distances z behind the object, will we find a field distribution that is of the same form as that of the object, up to possible complex constants (*i.e.*, ignore the overall phase factor)? (2 points)

Problem 2. Fresnel diffraction

Fresnel approximation plays an important role in diffraction theory. We want to understand its physical and geometrical meaning. Start with the expression for a spherical wave in real space (which is an *exact* solution of Helmholtz's equation), written as

$$u(\mathbf{r}) = \frac{A}{r} e^{i\mathbf{k}\cdot\mathbf{r}} = \frac{A}{\sqrt{z^2+\rho^2}} e^{ik\sqrt{z^2+\rho^2}}.$$

where A is the amplitude of the spherical wave, ρ is the radial distance in the x-y plane and $\mathbf{k}$ is the wavevector.

(a) Derive the Fresnel approximation $u_F(\mathbf{r})$ of this spherical wave by expanding $u(\mathbf{r})$ into a Taylor series for points around the optical axis i.e. $\rho \ll z$. (3 points)

(*Hint*: Carefully think about where to truncate the expansion in the amplitude and phase term.)

(b) In practical situations, one might prefer to use the angle θ, under which a beam propagates with respect to the optical axis, to determine if the Fresnel approximation is valid. Using above expansion, show that the condition of θ for the Fresnel approximation therefore is

$$N_F \theta^2 \ll 4,$$

where $N_F = \frac{\rho^2}{\lambda z}$ is the so-called *Fresnel number*. (2 points)

(*Hint*: At points nearby the optical axis, $\tan\theta \approx \theta$.)

Problem 3. Diffraction free beam

Diffraction free beam is a beam that preserves its amplitude distribution upon propagation and only experiences a trivial phase accumulation, i.e.:

$$u(x, y, z) = u(x, y, 0)e^{i\phi(x,y,z)}.$$

(a) Identify the necessary condition for the Fourier spectrum of a beam $U(\alpha, \beta)$ in order to be free of diffraction. (2 points)

(b) Give one example for the Fourier spectrum of a diffraction free beam that is localized in both transverse dimension. (2 points)

(c) Give one example for the Fourier spectrum of a diffraction free beam that is localized in one transverse dimension and invariant in the other dimension. (2 points)

(d) Give one example for the Fourier spectrum of a diffraction free beam that is invariant in both transverse dimension (please be not surprised by the triviality of the last solution). (1 points)

(e) Show that the following diffraction free beam is indeed a solution to the scalar wave equation: $u(x, y, z, t) = u_0 J(k_\perp \rho) e^{i(kz-\omega t)} \quad \rho^2 = x^2 + y^2; \quad k_\perp = \sqrt{\frac{\omega^2}{c^2} - k^2}$

J is a Bessel function. (3 points)

(Hint: Solve the wave equation in cylindrical coordinates.)