Theoretical Optics

Exercise 5

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Drop point: 0th floor, Bldg.30.23
Due Date: July 4 2019, 16:00

Please indicate which tutorial you normally attend by the name of the tutor!

Problem 1. Fraunhofer diffraction

Consider two apertures, a circular aperture A_1 and an elliptical aperture A_2 that are located at the origin of x-y plane. As illustrated in Fig. 1, the circular aperture A_1 has a radius of r, but the elliptical aperture A_2 is stretched in y-axis by a factor μ, compared to A_1. Suppose that the apertures shall be illuminated with a plane wave at normal incidence.

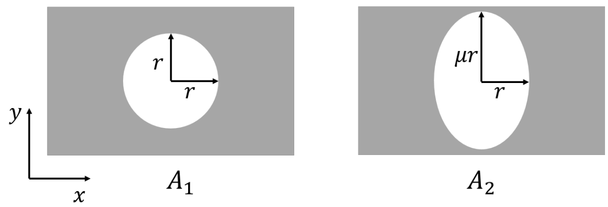

Figure 1: Two apertures A_1 and A_2, where aperture A_2 is stretched in one direction.

(a) Show that in a distance d where the far-field approximation holds, the Fourier spectrum of the field diffracted by the aperture A_2 can be written as

$$U_2\left(k\frac{x}{z}, k\frac{y}{z}; z = d\right) = \mu U_1\left(k\frac{x}{z}, \mu k\frac{y}{z}; z = d\right),$$

where $U_1\left(k\frac{x}{z}, \mu k\frac{y}{z}; z = d\right)$ denotes the Fourier spectrum of the field diffracted by the aperture A_2. (2 points)

(b) Given the diffraction patterns in the far-field for a circular and square apertures as shown in Fig. 2, sketch the diffraction patterns by an elliptical and rectangular apertures that are stretched in one direction but keep the same geometric width in the other direction. Also discuss why you think so. (2 points)

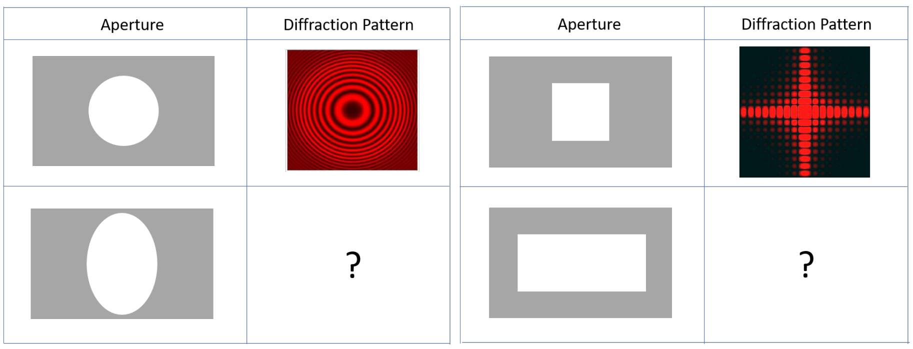

Figure 2: Diffraction patterns.

Problem 2. Principal axis of anisotropic crystals

Consider an anisotropic crystal characterized in the laboratory coordinate system by the following relative permittivity tensor

$$\hat{\varepsilon} = \begin{pmatrix} a & 0 & 0 \\ 0 & 1.25a & \alpha a \\ 0 & \alpha a & 1.75a \end{pmatrix},$$

where a is some non-zero number that eventually defines the material properties.

(a) Find all possible solutions for α such that the crystal is uniaxial. (2 points)

(b) For α's obtained from the previous question, rewrite the crystal permittivity tensor in the respective principal axes. (1 point)

(c) For the real positive solution of α from the previous question, write the principal axes of the crystal in terms of the laboratory coordinate basis. (2 points)

(d) How much does the crystal need to be rotated with respect to laboratory coordinate such that we have a diagonalized permittivity tensor in the new coordinate system? (1 point)

Problem 3. Anisotropic material

Let us consider the interface between an isotropic medium [characterized by $\varepsilon_i(\omega)$] and a uniaxial crystal [$\varepsilon_{or}(\omega)$, $\varepsilon_e(\omega)$] and light impinging at an angle ϕ_i to the surface normal with a wavevector $\mathbf{k}_i$ in the y-z plane. The optic axis (z-axis) and the surface normal forms the angle α (see Fig. 3). We know from the theory of anisotropic media that there will be two refracted waves into the uniaxial material, namely the *extraordinary* wave and the *ordinary* wave, if the polarization of the electric field of the incident wave is not parallel to any of the principal axes of the anisotropic medium.

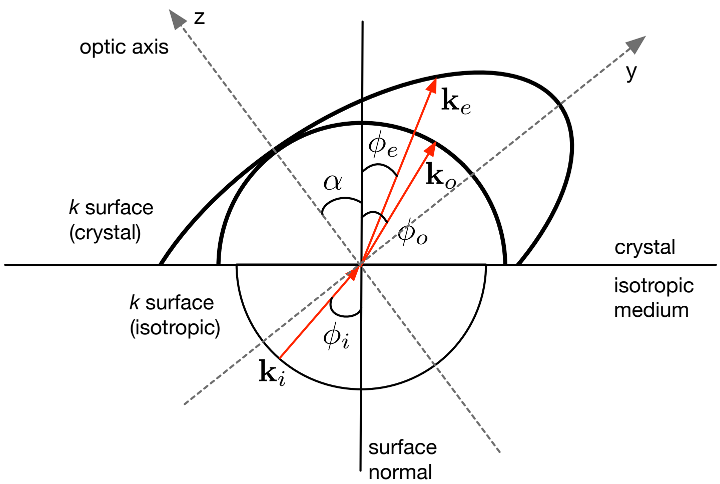

Figure 3: Intersection of the k surfaces with the plane of incidence

(a) The expression of the refractive index for the extraordinary wave $n_b(\omega, \theta)$ is given by

$$n_b(\omega, \theta) = \frac{n_e(\omega) n_{or}(\omega)}{\sqrt{n_{or}^2(\omega)\sin^2(\theta) + n_e^2(\omega)\cos^2(\theta)}},$$

where θ is the angle between the direction of the wavevector in a medium and the optic axis (z-axis), $n_{or}(\omega) = \sqrt{\varepsilon_{or}(\omega)}$, and $n_e(\omega) = \sqrt{\varepsilon_e(\omega)}$. Explain what represents the value of $n_b(\omega, \theta)$ in the geometrical construction in Fig. 3. (*Hint*: You may explain it in a text or draw it if you like.) (2 points)

(b) Write the expression of Snell's law for both the ordinary and extraordinary rays. Furthermore, explain what Snell's law means in the geometrical construction in Fig. 3. (*Hint*: You may explain it in a text or draw it if you like.) (2 points)

(c) Find all the possible values for ϕ_i and α, from the coordinate above, for which the energy flow for ordinary and extraordinary waves is along the same direction. (2 points)

(d) Assume that the above anisotropic material ($\varepsilon_1 = \varepsilon_2 = \varepsilon_{or}, \varepsilon_3 = \varepsilon_e$) with thickness d is cut and oriented such that the surface normal is oriented in the direction of $\mathbf{e}_y = \{0, 1, 0\}$, (*i.e.*, $\alpha = \pi/2$) and one crystal axis is oriented along $\mathbf{e}_\mathrm{c} = \{\sin(\beta), 0, \cos(\beta)\}$ (*i.e.*, the rotated z-axis by an angle of $-\beta$ around y-axis). Consider an x-polarized plane wave, with a vacuum wavevector $\mathbf{k} = \frac{2\pi}{\lambda_0}\{0, 1, 0\}$ that hits this slab at normal incidence (*i.e.*, $\phi_i = 0$), propagates through it and leaves the slab. We assume lossless propagation, and further assume the Fresnel reflection at both interfaces to be negligible. Decompose the incident electric field $\mathbf{E} = E_0 \mathbf{e}_x e^{i(ky-\omega t)}$ in the crystal basis given above and show that it reads as

$$\mathbf{E}(y) = E_0 \left[\cos(\beta)(\mathbf{e}_y \times \mathbf{e}_c) + \sin(\beta)\mathbf{e}_c\right] e^{i(ky-\omega t)}.$$

(2 points)

(e) When $\beta = \pi/4$, show that the output electric field $\mathbf{E}(y)$ at $y = d$ when it comes out of the crystal can be written in the laboratory axis as

$$\mathbf{E}\left(y = d\right) = E_0 \frac{1}{2}\left[(e^{i\sqrt{\varepsilon_e}\frac{\omega}{c}d} + e^{i\sqrt{\varepsilon_{or}}\frac{\omega}{c}d})\mathbf{e}_x + (e^{i\sqrt{\varepsilon_e}\frac{\omega}{c}d} - e^{i\sqrt{\varepsilon_{or}}\frac{\omega}{c}d})\mathbf{e}_z\right] e^{-i\omega t}.$$

(2 points)