Karlsruhe Institute of Technology

Supplementary note

Lecture: Accelerator Physics
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Special Relativity: 4-space-time and electrodynamics

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These supplementary notes repeat material that should basically be known from the BSc lectures and is a prerequisite for the following module. Decide to what extent you need to recapitulate the material for yourself.

Contents

1 Fundamentals of Special Relativity

Synchrotron radiation is electromagnetic radiation that is emitted during the acceleration of relativistic charged particles. The description of the particle kinematics and the physics of the synchrotron radiation must therefore be carried out in the theoretical framework of special relativity.

The aim of this worksheet is to remind the reader of some basics of special relativity.

Newtonian and quantum mechanics can be described in terms of the group of transformations in Hilbert space, which leave the norm of position vectors $\vec{x}$ invariant (namely the Galilean transformations).

This does not apply to Maxwell's electrodynamics: Here, for example, the description of a point charge at rest and of a moving point charge with a co-moving observer lead to different potentials under Galilean transformations, i.e. Maxwell's equations are not Galilean invariant.

The invariance group of electrodynamics results from the postulates of special relativity:

Postulate 1.1. *The laws of physics are independent of the choice of inertial frame. In particular, for any inertial frame, space is homogeneous and isotropic, and time is homogeneous.*

Postulate 1.2. *The speed of light is independent of the motion of its source.*

2 Notes on the mathematical structure of relativistic space-time

The group of transformations which satisfies these postulates and under which the Maxwellian electrodynamics remains invariant, is the *homogeneous Lorentz group*. This group is defined on a non-Euclidean, four-dimensional space, the relativistic space-time.

Definition 2.1 (Space-time)**.** . The transformations of the Lorentz group operate on the vectors of the 4-dimensional space-time

$$(x^0, x^1, x^2, x^3) := (ct, \vec{r}) \tag{2.1}$$

The transformations between two inertial systems K and K' moving against each other, which satisfy the postulates of special relativity, are the transformations of the Lorentz group.

Definition 2.2 (4-norm)**.** On 4-dimensional space-time, the norm of 4-vectors is defined by

$$\begin{aligned} \|X\|^2 &:= (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 \\ &= c^2t^2 - |\vec{x}|^2. \end{aligned} \tag{2.2}$$

The Lorentz group contains all transformations that leave the norm of the 4-vectors invariant. These are the actual Lorentz transformations (see below eq. 3.1) and the normal rotations.

Lorentz invariant kinematics and electrodynamics can be elegantly formulated in the four-dimensional pseudo-Euclidean space. We will take benefit from the four-vector notation in the lecture from time to time. In order to provide a little more background to the statements made in the lecture, we will sketch here some basics of the so-called covariant formulation of relativistic kinematics and electrodynamics. For an exhaustive treatment, please refer to the textbooks.

2.1 The light cone

From the Lorentz invariance of the 4-vector norm it follows: The distance between two space-time points

$$s_{12}^2 = c^2(t_1 - t_2)^2 - |\vec{x}_1 - \vec{x}_2|^2 \tag{2.3}$$

is invariant under Lorentz transformations.

Therefore, it makes sense to use the concept of the light cone defined by

$$\vec{x}^2 = c^2 t^2, \tag{2.4}$$

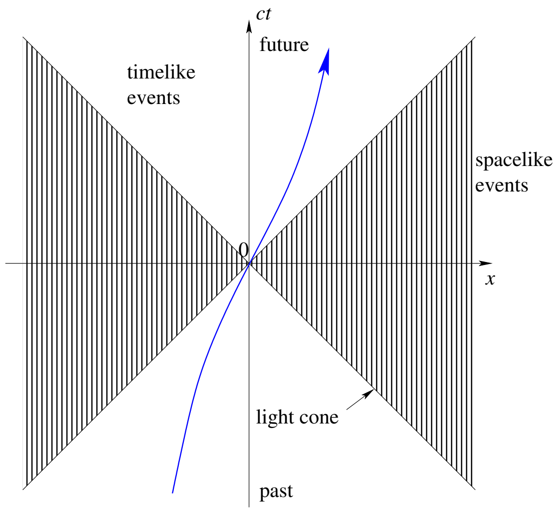

Figure 1: The concept of the light cone in special relativity

and the

Definition 2.3. The spatio-temporal relation of two events $(t_1, \vec{x}_1)$ and $(t_2, \vec{x}_2)$ is called

$$\begin{array}{lcll} \text{timelike} & :\Leftrightarrow & s_{12}^2 = c^2(t_1 - t_2)^2 - |\vec{x}_1 - \vec{x}_2|^2 & > 0 \\ \text{spacelike} & :\Leftrightarrow & s_{12}^2 = c^2(t_1 - t_2)^2 - |\vec{x}_1 - \vec{x}_2|^2 & < 0 \\ \text{lightlike} & :\Leftrightarrow & s_{12}^2 = c^2(t_1 - t_2)^2 - |\vec{x}_1 - \vec{x}_2|^2 & = 0 \end{array} \tag{2.5}$$

Causal relations are only possible between time- or light-like events.

Remark. This classification is also Lorentz invariant.

2.2 Some tensor analysis in four-dimensional non-Euclidean space

Definition 2.4. Let be given a space-time coordinate tuple and a well-defined coordinate transformation with

$$x'^{,\alpha} = x'^{,\alpha}(x^0, x^1, x^2, x^3), \tag{2.6}$$

then is called:

$$
\begin{aligned}
&a \quad \text{4-scalar}\\
&\quad :\Leftrightarrow a \quad \text{tensor of rank 0 with: } a \text{ invariant under the coordinate transformation} \quad x'(x)\\
&A \quad \text{contravariant 4-vector}\\
&\quad :\Leftrightarrow A'^{\,\alpha} = \frac{\partial x'^{\,\alpha}}{\partial x^{\beta}} A^{\beta}\\
&B \quad \text{covariant 4-vector}\\
&\quad :\Leftrightarrow B'_{\alpha} = \frac{\partial x^{\beta}}{\partial x'^{\,\alpha}} B_{\beta}\\
&F \quad \text{contravariant tensor of rank 2}\\
&\quad :\Leftrightarrow F'^{\,\alpha\beta} = \frac{\partial x'^{\,\alpha}}{\partial x^{\gamma}} \frac{\partial x'^{\,\beta}}{\partial x^{\delta}} F^{\gamma\delta}\\
&G \quad \text{covariant tensor of rank 2}\\
&\quad :\Leftrightarrow G'_{\alpha\beta} = \frac{\partial x^{\gamma}}{\partial x'^{\,\alpha}} \frac{\partial x^{\delta}}{\partial x'^{\,\beta}} G_{\gamma\delta}\\
&H \quad \text{mixed tensor of rank 2}\\
&\quad :\Leftrightarrow H'^{\,\alpha}_{\beta} = \frac{\partial x'^{\,\alpha}}{\partial x^{\gamma}} \frac{\partial x^{\delta}}{\partial x'^{\,\beta}} H^{\gamma}_{\delta}
\end{aligned}
\tag{2.7}
$$

Remark. Note that here and in the following the Einstein summation convention for repeated indices is always applied.

Definition 2.5. In these terms the *scalar product of two 4-vectors* is defined as:

$$B \cdot A := B_{\alpha} A^{\alpha}. \tag{2.8}$$

The scalar product is a Lorentz scalar, as can be easily proved by inserting the transformation properties.

2.3 Space-time metric and 4-scalar product

The metric of space-time is given by the invariant norm 2.2:

$$
\begin{aligned}
(\mathrm{d}s)^2 &= (\mathrm{d}x^0)^2 - (\mathrm{d}x^1)^2 - (\mathrm{d}x^2)^2 - (\mathrm{d}x^3)^2\\
&=: g_{\alpha\beta} \mathrm{d}x^{\alpha} \mathrm{d}x^{\beta}
\end{aligned}
\tag{2.9}
$$

with the metric tensor

$$g_{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{2.10}$$

in matrix representation.

Thus, for co- and contravariant 4-vectors, it follows:

$$\begin{aligned} A^\alpha &= (A^0, \vec{A}) \\ A_\alpha &= (A^0, -\vec{A}) \end{aligned} \tag{2.11}$$

Remark. With this metric, the scalar product of two space-time vectors is given by

$$B \cdot A = B^0 A^0 - \vec{B} \cdot \vec{A} \tag{2.12}$$

2.4 4-differential operators

The operators of partial differentiation with respect to a contravariant or a covariant component, respectively, can be formed as:

$$\begin{aligned} \partial^\alpha &:= \frac{\partial}{\partial x_\alpha} \quad = (\frac{\partial}{\partial x_0}, -\operatorname{grad}) \\ \partial_\alpha &:= \frac{\partial}{\partial x^\alpha} \quad = (\frac{\partial}{\partial x^0}, \operatorname{grad}) \end{aligned} \tag{2.13}$$

Remark. The 4-divergence of a 4-vector is the invariant

$$\partial^\alpha A_\alpha = \partial_\alpha A^\alpha = \frac{\partial A^0}{\partial x^0} + \operatorname{div} \vec{A} \tag{2.14}$$

Finally,

Definition 2.6 (D'Alambert operator)**.**

$$\Box := \partial_\alpha \partial^\alpha = \frac{\partial^2}{\partial x^{0\,2}} - \Delta \tag{2.15}$$

is the operator of the wave equation in vacuum.

The operator of differentiation by a covariant component behaves like a contravariant vector operator, and that of differentiation by a contravariant component behaves like a covariant vector operator.

3 Lorentz transformations and relativistic kinematics

3.1 Simple Lorentz transformation

Let inertial frame K' be moving against K with velocity v in positive x direction.

$$\Rightarrow \begin{cases} x'^0 = \gamma(x^0 - \beta x^1) \\ x'^1 = \gamma(x^1 - \beta x^0) \\ x'^2 = x^2 \\ x'^3 = x^3 \end{cases} \tag{3.1}$$

$$\begin{aligned} \vec{\beta} &= \frac{\vec{v}}{c}, \quad \beta = |\vec{\beta}| \\ \gamma &= \frac{1}{\sqrt{1-\beta^2}} \end{aligned} \tag{3.2}$$

More generally, for two inertial systems K and K' with parallel coordinate axes with relative velocity $\vec{v}$ of K' against K:

$$\begin{aligned} x^{0\prime} &= \gamma(x^0 - \vec{\beta}\cdot\vec{x}) \\ \vec{x}' &= \vec{x} + \frac{(\gamma-1)}{\beta^2}(\vec{\beta}\cdot\vec{x})\vec{\beta} - \gamma\vec{\beta}x^0 \end{aligned} \tag{3.3}$$

Remember: The Lorentz transformation leads to the phenomena of *length contraction* and *time dilatation*:

If there is a rod of length l at rest in K, an observer in K' will measure the length l.

$$l' = \frac{1}{\gamma}l.$$

If there is a lamp at rest in K which is switched on for a time interval δt, an observer in K' will measure a time interval during which the lamp is switched on,

$$\delta t' = \gamma\delta t.$$

3.2 The concept of proper time

Definition 3.1 (Element of proper time)**.** . The element of time in the instantaneous rest system of a particle (or a clock) is the element of proper time.

$$\mathrm{d}\tau := \frac{1}{\gamma(t)}\mathrm{d}t \tag{3.4}$$

Remark. $\mathrm{d}\tau$ is Lorentz-invariant because $\mathrm{d}\vec{x}^2 = 0$ in the system of rest and thus $c\mathrm{d}\tau = \mathrm{d}s$.

3.3 4-velocity, relativistic momentum and energy

With the element of proper time, the 4-velocity is given by

$$V := \frac{\mathrm{d}s}{\mathrm{d}\tau} = \gamma(c, \vec{v}) \tag{3.5}$$

The 4-momentum P of a particle with rest mass m and velocity V is thus given by

$$P = mV. \tag{3.6}$$

The total energy of such a particle is

$$W = \gamma mc^2. \tag{3.7}$$

Thus the 4-momentum reads

$$\begin{aligned} P &:= (p^0, \vec{p}) \\ \text{with} \quad p^0 &:= W/c, \quad \vec{p} = \gamma m\vec{v} \end{aligned} \tag{3.8}$$

The norm of the 4-momentum,

$$(p^0)^2 - \vec{p} \cdot \vec{p} = (mc)^2, \tag{3.9}$$

is Lorentz invariant. It follows

$$W = \sqrt{c^2p^2 + m^2c^4} \tag{3.10}$$

Proof.

$$\begin{aligned} (p^0)^2 - \vec{p} \cdot \vec{p} &= (\gamma mc)^2 - \gamma^2 m^2 (\vec{v} \cdot \vec{v}) \\ &= \gamma^2 m^2 (c^2 - v^2) \\ &= \gamma^2 m^2 c^2 \left(1 - \frac{v^2}{c^2}\right) \\ &= (mc)^2 \end{aligned}$$

□

4 Lorentz invariant formulation of electrodynamics

Theorem 4.1. Under the (Lorentz-invariant) condition of the *Lorentz gauge*

$$\operatorname{div} \vec{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} = 0, \tag{4.1}$$

the

$$\begin{aligned} \textbf{4-sources:} \qquad & J^\alpha := (c\rho, \vec{j}), \\ \textbf{4-potential:} \qquad & A^\alpha = (\phi/c, \vec{A}) \end{aligned} \tag{4.2}$$

are Lorentz invariant 4-vectors.

The wave equation, the continuity equation and the Lorentz condition thus obtain the Lorentz-invariant form

$$\begin{aligned}\Box A^{\alpha} &= \mu_0 J^{\alpha} \\ \partial_{\alpha} J^{\alpha} &= 0 \\ \partial_{\alpha} A^{\alpha} &= 0\end{aligned} \tag{4.3}$$

With the electric field tensor

$$F^{\alpha\beta} := \partial^{\alpha} A^{\beta} - \partial^{\beta} A^{\alpha}, \tag{4.4}$$

or, respectively, in the matrix representation

$$F^{\alpha\beta} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix}, \tag{4.5}$$

and the *dual electric field tensor*

$$\mathcal{F}^{\alpha\beta} := \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta} F_{\gamma\delta} \tag{4.6}$$

with

$$F_{\alpha\beta} = g_{\alpha\gamma} F^{\gamma\delta} g_{\delta\beta} \tag{4.7}$$

and with the 4th-rank totally antisymmetric unit tensor

$$\epsilon^{\alpha\beta\gamma\delta} := \begin{cases} +1 & \text{for even permutations of indices} \\ -1 & \text{for odd permutations of the indices} \\ 0 & \text{if two indices are equal} \end{cases} \tag{4.8}$$

the Maxwell equations in vacuum can also be formulated Lorentz-invariantly under the Lorentz gauge condition:

$$\begin{aligned}\partial_{\alpha} F^{\alpha\beta} &= \mu_0 J^{\beta} \\ \partial_{\alpha} \mathcal{F}^{\alpha\beta} &= 0\end{aligned} \tag{4.9}$$

5 Transformation of the electromagnetic fields

The electric field tensor transforms under the transition

$$K \mapsto K'$$

according to

$$F'^{\alpha\beta} = \frac{\partial x'^{\alpha}}{\partial x^{\gamma}} \frac{\partial x'^{\beta}}{\partial x^{\delta}} F^{\gamma\delta} \tag{5.1}$$

The Lorentz transformation of the electric field tensor in the case of relative motion of equally oriented coordinate systems K and K' along the common z-axis leads to

$$\begin{aligned} E'_x &= \gamma(E_x + \beta_z B_y) \qquad & B'_x &= \gamma(B_x - \beta_z E_y) \\ E'_y &= \gamma(E_y - \beta_z B_x) \qquad & B'_y &= \gamma(B_y + \beta_z E_x) \\ E'_z &= E_z & B'_z &= B_z \end{aligned}$$

Under Lorentz transformation between inertial frames with relativistic relative velocities (e.g. the rest frame of an accelerated particle and the laboratory frame), pure electric or magnetic fields always become a combination of electric *and* magnetic fields.

6 Acceleration of charged particles: The Lorentz force

The acceleration of charged particles, as we already know, is described by the *Lorentz force*

$$\vec{F} = \frac{\mathrm{d}\vec{p}}{\mathrm{d}t} = q\left(\vec{E} + (\vec{v} \times \vec{B})\right) \tag{6.1}$$

which can also be expressed in covariant notation. (note: the Lorentz force is Lorentz invariant):

$$\boxed{\frac{\mathrm{d}p^\alpha}{\mathrm{d}\tau} = qF^{\alpha\beta}V_\beta} \tag{6.2}$$

Finally, a remark on the stability of particle beams. The fact that beams of charged particles can be stored in accelerators at all does not seem self-evident in view of their mutual Coulomb repulsion. The suppression of this repulsion is a consequence of the relativistic motion of the particles:

Let be given a longitudinally infinitely extended continuous particle beam of velocity v along an axis s with a constant charge density ρ_0. The electromagnetic fields in s-direction compensate each other, i.e. due to the cylindrical symmetry of the problem $\vec{E}$ has only a radial, $\vec{B}$ only an azimuthal component. *In the interior of the particle beam* these are

$$\begin{aligned} E_r &= \frac{1}{2\epsilon_0}\rho_0 r \\ B_\phi &= \frac{\mu_0}{2}\rho_0 v r. \end{aligned} \tag{6.3}$$

This gives the radial component of the Lorentz force on a particle at a distance r from the axis.

$$F_r = e\left(E_r - vB_\phi\right) = \frac{e}{2\epsilon_0}\left(1 - \frac{v^2}{c^2}\right)\rho_0 r = \frac{e}{2\epsilon_0}\frac{\rho_0}{\gamma^2}r \tag{6.4}$$

The repulsive force therefore vanishes with γ^2.

Remark. The same result is obtained (of course) in the rest system of the particle. In this system, the charge density decreases with $1/\gamma$. Another factor $1/\gamma$ is added by the time dilatation.