3 Physics of Synchrotron Radiation

3.1 Geometry and equations of motion

Having dealt so far with the emitted (and in a sense thereby temporally averaged) power densities for the general cases of longitudinally and transversely accelerated charges, we will now look more specifically at the spectrum of radiation emitted during relativistic motion on a circular arc from the point of view of the observer.

Aim: Calculation of the spectral distribution of the power flux for the relativistic particle motion on an arc

Plan:

- Description of the particle motion
- Representation of the radiation field in the frequency domain (Fourier transform, explicit inclusion of the relation between observation and emission time $t(t')$)
- Approximations
- Discussion of the spectral properties of synchrotron radiation

Synchrotron Radiation: Coordinate System

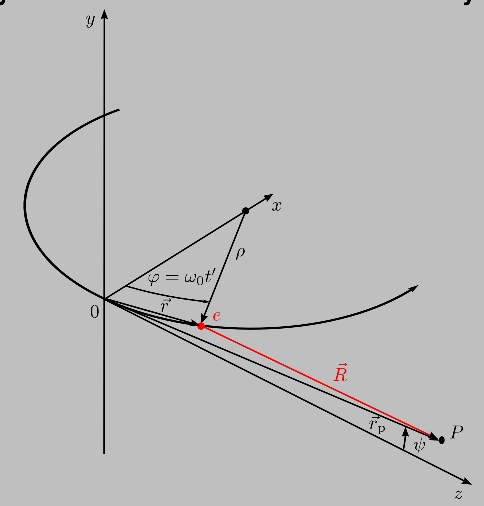


Figure 15: Coordinate system for the further treatment of normal synchrotron radiation

deflection radius $\dfrac{1}{\rho} = \dfrac{eB}{mc\beta\gamma}$

angular frequency $\omega_0 = \beta c/\rho$

Remark. The point of observation P has been placed in the y-z plane without loss of generality, since the problem of the electron's circular motion is cylindrically symmetrical with respect to the axis parallel to y through the centre of the circular orbit.

- The particle with charge $+e$ moves on a circular path with radius ρ in the x-z-plane at the time $t' = 0$ through the origin.
- $\vec{r}_p$ is the (fixed) position vector of the observer

- $\vec{r}(t')$ is that of the particle
- $\vec{R}(t') = \vec{r}_p - \vec{r}(t')$ is the time-varying distance between the observer and the particle.

The angular velocity of the circular motion is

$$\omega_0 = \beta c/\rho. \tag{3.1}$$

with the constant deflection radius

$$\frac{1}{\rho} = \frac{eB}{mc\beta\gamma} \tag{3.2}$$

This means that the equations of motion are

$$\begin{aligned} \vec{r}(t') &= \rho \begin{pmatrix} (1-\cos\omega_0 t') \\ 0 \\ \sin\omega_0 t' \end{pmatrix} \qquad \vec{\beta} = \beta \begin{pmatrix} \sin\omega_0 t' \\ 0 \\ \cos\omega_0 t' \end{pmatrix} \\ \dot{\vec{\beta}} &= \beta\omega_0 \begin{pmatrix} \cos\omega_0 t' \\ 0 \\ -\sin\omega_0 t' \end{pmatrix} \quad \text{with} \quad \beta\omega_0 = \frac{\beta^2 c}{\rho}. \end{aligned} \tag{3.3}$$

With

$$\vec{r}_p = r_p \begin{pmatrix} 0 \\ \sin\psi \\ \cos\psi \end{pmatrix} \tag{3.4}$$

is finally

$$\begin{aligned} \vec{R} &= \begin{pmatrix} -\rho(1-\cos\omega_0 t') \\ r_p \sin\psi \\ r_p\cos\psi - \rho\sin\omega_0 t' \end{pmatrix} =: R\vec{n} \\ R &= r_p \sqrt{1 - 2\frac{\rho}{r_p}\cos\psi\sin\omega_0 t' + 2\left(\frac{\rho}{r_p}\right)^2 (1-\cos\omega_0 t')} \end{aligned} \tag{3.5}$$

3.2 Radiation field in frequency space

To describe the spectral properties of normal synchrotron radiation, the problem is considered in frequency space:

Radiation field:

$$\vec{E}(t) = \frac{e}{4\pi\epsilon_0 c}\left[\frac{\vec{n}\times\left((\vec{n}-\vec{\beta})\times\dot{\vec{\beta}}\right)}{R(1-\vec{n}\cdot\vec{\beta})^3}\right]_{\text{ret}} \tag{3.6}$$

To calculate the spectrum 'transition to the Fourier transform

$$\hat{\vec{E}}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \vec{E}(t)\mathrm{e}^{-\mathrm{i}\omega t}\mathrm{d}t. \tag{3.7}$$

Remark. Integration is performed here over the *observation time* t. However, the emission time t' is given in the retarded field. Therefore:

Transformation of the integration variable $t \to t'$:

$$t = t' + \frac{R(t')}{c}, \qquad \mathrm{d}t = (1 - \vec{n} \cdot \vec{\beta})\mathrm{d}t' \tag{3.8}$$

$$\hat{\vec{E}}(\omega) = \frac{e}{4\pi\epsilon_0 c} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\frac{\vec{n} \times \left((\vec{n} - \vec{\beta}) \times \dot{\vec{\beta}} \right)}{R(1 - \vec{n} \cdot \vec{\beta})^2} \right) \mathrm{e}^{-\mathrm{i}\omega(t' + R(t')/c)} \mathrm{d}t' \tag{3.9}$$

Partial integration, neglecting terms for $t' = \pm\infty$ and approximating $\vec{n}$ =const. and $R \approx r_p$ in the large bracket (not in the exponent!) leads to

$$\boxed{\hat{\vec{E}}(\omega) = \frac{\mathrm{i}\omega e}{\sqrt{2\pi} 4\pi\epsilon_0 c r_p} \int_{-\infty}^{\infty} \left(\vec{n} \times (\vec{n} \times \vec{\beta}) \right) \mathrm{e}^{-\mathrm{i}\omega(t' + R(t')/c)} \mathrm{d}t'} \tag{3.10}$$

The approximations made are justified by the fact that the arc of the circle on which radiation is emitted, which arrives at the observer, is very short compared to the distance of the observation point due to the radiation pattern during relativistic motion.

3.3 Approximations

$$\hat{\vec{E}}(\omega) = \frac{\mathrm{i}\omega e}{\sqrt{2\pi} 4\pi\epsilon_0 c r_p} \int_{-\infty}^{\infty} \underbrace{\left(\vec{n} \times (\vec{n} \times \vec{\beta}) \right)}_{I} \mathrm{e}^{-\mathrm{i}\omega \underbrace{(t' + R(t')/c)}_{II+III}} \mathrm{d}t'$$

To evaluate this Fourier integral, we make the following legitimate approximations for the special case of interest to us:

1. Dipole approximation:

   Expansion of $R(t')$ in a Taylor series and truncation after the linear term

2. Limiting to „relevant motion“

   $\vec{n} = const. \forall t'$

3. Ultra-relativistic approximation

$$\beta \approx 1, \qquad \gamma \gg 1, \qquad \psi \lesssim \frac{1}{\gamma} \ll 1 \tag{3.11}$$

1. Dipole approximation: Taylor series expansion for R:

$$R = r_p \left(1 - \frac{\rho}{r_p} \cos\psi \underbrace{\sin\omega_0 t'}_{\lesssim \frac{1}{\gamma}} + \frac{1}{2}\left(\frac{\rho}{r_p}\right)^2 (\underbrace{2 - 2\cos\omega_0 t'}_{\lesssim \frac{1}{\gamma^2}} - \cos^2\psi \underbrace{\sin^2\omega_0 t'}_{\lesssim \frac{1}{\gamma^2}}) + ... \right) \tag{3.12}$$

Approximations for the trigonometric functions in the ultra-relativistic case (see below) are anticipated here in order to obtain the condition on ρ/r_p for the Taylor series to terminate.

The cancellation of this development after the linear term (dipole approximation) is justified if

$$\frac{\rho}{r_p \gamma} \ll 1. \tag{3.13}$$

This makes the exponent

$$II + III \approx t' + \frac{r_p}{c} - \frac{\rho\cos\psi\sin\omega_0 t'}{c} \tag{3.14}$$

The constant term $\frac{r_p}{c}$ is essentially due to the choice of origin and of little interest. It is therefore introduced:

Definition 3.1 (Reduced observer time)**.**

$$t_p := t - \frac{r_p}{c} \approx t' - \frac{\rho\cos\psi\sin\omega_0 t'}{c} \tag{3.15}$$

For the double cross product, we now further assume the constancy of the direction vector $\vec{n}$, again with the justification that the arc of the circle on which the particle emits visible radiation for the observer is short compared to the distance between the observer and the emitting particle:

2. Limiting to the “relevant motion”

$$\vec{n} \approx \begin{pmatrix} 0 \\ \sin\psi \\ \cos\psi \end{pmatrix} \forall t' \tag{3.16}$$

$$I \approx \beta \begin{pmatrix} -\sin\omega_0 t' \\ \sin\psi\cos\psi\cos\omega_0 t' \\ -\sin^2\psi\cos\omega_0 t' \end{pmatrix} \tag{3.17}$$

So far, only approximations have been carried out under the condition $\frac{\rho}{r_p\gamma} \ll 1$. Now, finally, the ultrarelativistic approximation for the trigonometric functions is made:

3. Ultrarelativistic approximation:

$$\begin{aligned} &\beta \approx 1, \qquad \gamma \gg 1 \\ &\Rightarrow \psi \lesssim \frac{1}{\gamma} \ll 1 \\ &\Rightarrow \sin\psi \approx \psi, \quad \cos\psi \approx 1 - \frac{\psi^2}{2} \end{aligned}$$

Circular motion and relation of time scales:

$$\begin{aligned} &\omega_0 t' = \frac{\beta c}{\rho} t' \lesssim \frac{1}{\gamma} \\ &\Rightarrow \sin\omega_0 t' \approx \omega_0 t' - \frac{(\omega_0 t')^3}{6}, \ \cos\omega_0 t' \approx 1 - \frac{(\omega_0 t')^2}{2} \end{aligned}$$

$$I \approx \beta \begin{pmatrix} -\omega_0 t' \\ \psi \\ 0 \end{pmatrix} \tag{3.18}$$

$$\begin{aligned} II + III &\approx t' \left(1 - \beta + \beta \frac{\psi^2}{2}\right) + t'^3 \frac{c^2 \beta^3}{6\rho^2} \\ &\approx t' \frac{1 + \gamma^2 \psi^2}{2\gamma^2} + t'^3 \frac{c^2}{6\rho^2} \end{aligned} \tag{3.19}$$

$\sin\omega_0 t'$ is expanded to third order here, because in the ultrarelativistic approximation the linear term is afflicted with a factor $\frac{1}{\gamma^2}$ and thus comes into the same order of magnitude as the third-order term.

3.4 Spectrum of the emitted radiation when moving on an arc with constant angular velocity

Equipped with the approximations discussed above, we now set about calculating the Fourier-transformed electric field, i.e. the spectrum of the synchrotron radiation.

To do this, we also switch to the reduced observer time introduced above under the Fourier integral:

$$\Rightarrow \vec{E}(\omega) = \frac{\mathrm{i}\omega e}{4\pi\epsilon_0\sqrt{2\pi}cr_p}\int_{-\infty}^{\infty}\begin{pmatrix}\omega_0 t'\\ \psi\\ 0\end{pmatrix}\exp\left(-\mathrm{i}\omega\left(\frac{t'(1+\gamma^2\psi^2)}{2\gamma^2}+\frac{c^2t'^3}{6\rho^2}\right)\right)\mathrm{d}t' \tag{3.20}$$

This is the frequency spectrum of the electric radiation field as measured by the observer. The emission time t' now only plays the role of an integration variable.

To solve the integral (which is not easy), we transform it in the following way: The complex exponential function is represented by its trigonometric components. We can take advantage of the fact that the x component of the triple vector product is odd and the y component is even in t', so that:

Trigonometric representation:

$$\Rightarrow \begin{cases} E_x(\omega) = \dfrac{-e\omega}{4\pi\epsilon_0\sqrt{2\pi}cr_p}\displaystyle\int_{-\infty}^{\infty}\omega_0 t'\sin\left(\omega t'\frac{1+\gamma^2\psi^2}{2\gamma^2}+\frac{\omega c^2t'^3}{6\rho^2}\right)dt' \\ E_y(\omega) = \dfrac{\mathrm{i}e\omega}{4\pi\epsilon_0\sqrt{2\pi}cr_p}\displaystyle\int_{-\infty}^{\infty}\psi\cos\left(\omega t'\frac{1+\gamma^2\psi^2}{2\gamma^2}+\frac{\omega c^2t'^3}{6\rho^2}\right)dt' \end{cases} \tag{3.21}$$

With a variable substitution, the integrals occurring here can be transformed into the integral representation of the so-called Airy functions:

$$\begin{aligned} \mathrm{Ai}(v) &:= \frac{1}{2\pi}\int_{-\infty}^{\infty}\cos\left(vt+\frac{t^3}{3}\right)\mathrm{d}t \\ \mathrm{Ai}'(v) &= \frac{\mathrm{d\,Ai}}{\mathrm{d}v} = -\frac{1}{2\pi}\int_{-\infty}^{\infty}t\sin\left(vt+\frac{t^3}{3}\right)\mathrm{d}t \end{aligned} \tag{3.22}$$

$$\boxed{\Rightarrow \begin{cases} E_x(\omega) = \dfrac{e\gamma}{\sqrt{2\pi}\epsilon_0 cr_p}\left(\dfrac{3\mid\omega\mid}{4\omega_c}\right)^{1/3}\mathrm{Ai}'\left(\left(\dfrac{3\omega}{4\omega_c}\right)^{2/3}(1+\gamma^2\psi^2)\right) \\ E_y(\omega) = \dfrac{\mathrm{i}e\gamma(\omega/\mid\omega\mid)}{\sqrt{2\pi}\epsilon_0 cr_p}\left(\dfrac{3\mid\omega\mid}{4\omega_c}\right)^{2/3}\gamma\psi\,\mathrm{Ai}\left(\left(\dfrac{3\omega}{4\omega_c}\right)^{2/3}(1+\gamma^2\psi^2)\right) \end{cases}} \tag{3.23}$$

with the *critical frequency*

$$\boxed{\omega_c := \frac{3c\gamma^3}{2\rho} = \frac{3\omega_0\gamma^3}{2}.} \tag{3.24}$$

The Airy functions initially appear to be just another name for the integral expressions. In fact, they are solutions of the differential equation

$$\mathrm{Ai}''(x) - x\,\mathrm{Ai}(x) = 0.$$

They and their properties, in particular integrals involving the Airy functions, are well documented in the mathematical literature.

Remark. In principle, the polarisation properties are already contained in the equations 3.23. The fact that $E_y(\omega)$ is imaginary means that the y component of the oscillating field is phase-shifted by $\pi/2$ with respect to the x component. In general, synchrotron radiation is elliptically polarised. The amplitudes of both components are now essentially determined by the Airy functions. The angular distribution of the two polarisation components will be discussed in more detail below. However, a more detailed examination of the polarisation and the transformation into Stokes parameters will be omitted here.

Remark. The representation of the synchrotron radiation spectrum with modified Bessel functions is widely used in the literature, which is why we also present it here.

The Airy functions are related to the modified Bessel functions of the orders 1/3 and 2/3 by

$$\begin{aligned}\mathrm{Ai}(x) &= \frac{1}{\pi}\sqrt{\frac{x}{3}}K_{1/3}\left(\frac{2x^{3/2}}{3}\right)\\ \mathrm{Ai}'(x) &= -\frac{1}{\pi}\frac{x}{\sqrt{3}}K_{2/3}\left(\frac{2x^{3/2}}{3}\right),\end{aligned} \tag{3.25}$$

so that it can also be written as

$$\begin{aligned}E_x(\omega) &= \frac{-\sqrt{3}e\gamma}{(2\pi)^{3/2}\epsilon_0 c r_p}\left(\frac{|\,\omega\,|}{2\omega_c}\right)(1+\gamma^2\psi^2)K_{2/3}\left(\frac{\omega}{2\omega_c}(1+\gamma^2\psi^2)^{3/2}\right)\\ E_y(\omega) &= \frac{\mathrm{i}\sqrt{3}e\gamma}{(2\pi)^{3/2}\epsilon_0 c r_p}\left(\frac{\omega}{2\omega_c}\right)\gamma\psi\sqrt{1+\gamma^2\psi^2}K_{1/3}\left(\frac{\omega}{2\omega_c}(1+\gamma^2\psi^2)^{3/2}\right)\end{aligned} \tag{3.26}$$

Now we have one step left to do: the user is usually not so much interested in the electric field, but in the spectral power density at the observation point or the spectral photon flux, from which, for example, the brilliance can be calculated.

Spectral Power Flux Reminder:

$$\text{Energy flux:}\quad \frac{\mathrm{d}W}{\mathrm{d}\Omega} = \int_{-\infty}^{\infty}\frac{\mathrm{d}P}{\mathrm{d}\Omega}\mathrm{d}t' \quad = \int_{-\infty}^{\infty}\frac{\mathrm{d}P_p}{\mathrm{d}\Omega}\mathrm{d}t = \frac{R^2}{\mu_0 c}\int_{-\infty}^{\infty}|\,\vec{E}(t)\,|^2\,\mathrm{d}t$$

For the Fourier transform, (1) applies because of Plancherel's theorem (Parseval's equation) and (2) because $|\vec{E}(t)|^2$ is real

$$\begin{aligned}\frac{\mathrm{d}W}{\mathrm{d}\Omega} &= \frac{R^2}{\mu_0 c}\int_{-\infty}^{\infty}|\,\vec{E}(\omega)\,|^2\,\mathrm{d}\omega\\ &= \frac{2R^2}{\mu_0 c}\int_{0}^{\infty}|\,\vec{E}(\omega)\,|^2\,\mathrm{d}\omega\end{aligned} \tag{3.27}$$

and finally, because of the fundamental theorem of the differential and integral calculus, for the spectral and angular distribution of the radiated energy

$$\frac{\mathrm{d}^2 W}{\mathrm{d}\omega \mathrm{d}\Omega} = \frac{2R^2 \mid \vec{E}(\omega) \mid^2}{\mu_0 c} \tag{3.28}$$

If the particle is orbiting with the angular frequency ω_0, i.e. regularly recurring after an orbital period of $\omega_0/2\pi$ (as is the case with the storage ring), it makes sense to specify a time-averaged observed spectral power flux

$$\frac{\mathrm{d}^2 P}{\mathrm{d}\omega \mathrm{d}\Omega} = \frac{\omega_0}{2\pi} \frac{\mathrm{d}^2 W}{\mathrm{d}\omega \mathrm{d}\Omega} \approx \frac{2R^2 \mid \vec{E}(\omega) \mid^2}{2\pi \mu_0 \rho}. \tag{3.29}$$

Here, $\omega_0 = \frac{\beta c}{\rho}$ and $\beta \approx 1$ was used.

Remark. **Caution:** Without further comment, we are here making the transition from a single flyby of the particle to a periodic motion. This should then lead to a line spectrum with the fundamental frequency ω_0 and the associated harmonics, instead of a continuous spectrum. The reason why such a line spectrum is not observed in a storage ring is that the particles are decelerated by the emission of synchrotron radiation and their revolution frequency is therefore not constant. Due to this uncertainty, the synchrotron radiation spectrum at the bending magnet of a storage ring actually transitions into the continuous spectrum discussed here.

$\vec{E}$ has two components corresponding to two polarisation states. These are referred to in the literature as σ- and π-polarisation. We therefore write the spectral power density as the sum of two polarisation contributions

$$\boxed{\begin{aligned}
\frac{\mathrm{d}^2 P}{\mathrm{d}\Omega \mathrm{d}\omega} &= \frac{\mathrm{d}^2 P_\sigma}{\mathrm{d}\Omega \mathrm{d}\omega} + \frac{\mathrm{d}^2 P_\pi}{\mathrm{d}\Omega \mathrm{d}\omega} \\
&= \frac{P_\mathrm{S}\gamma}{\omega_c} \left(F_{\mathrm{S}\sigma}(\omega,\psi) + F_{\mathrm{S}\pi}(\omega,\psi)\right) \\
&\text{with the dimensionless spectral functions} \\
F_{\mathrm{S}\sigma}(\omega,\psi) &= \frac{9}{2\pi} \left(\frac{3\omega}{4\omega_c}\right)^{2/3} \mathrm{Ai}'^2 \left(\left(\frac{3\omega}{4\omega_c}\right)^{2/3} (1+\gamma^2\psi^2)\right) \\
F_{\mathrm{S}\pi}(\omega,\psi) &= \frac{9}{2\pi} \left(\frac{3\omega}{4\omega_c}\right)^{2/3} \gamma^2\psi^2\, \mathrm{Ai}^2 \left(\left(\frac{3\omega}{4\omega_c}\right)^{2/3} (1+\gamma^2\psi^2)\right) \\
\text{and} \quad P_\mathrm{S} &= \frac{2 r_0 c m c^2 \gamma^4}{3\rho^2}
\end{aligned}} \tag{3.30}$$

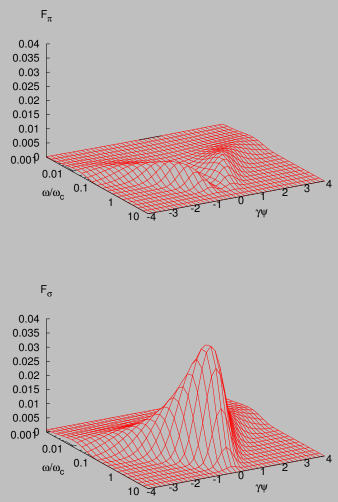
Here, the subscript S was introduced for the power of the synchrotron radiation.

Remark. The spectral functions fulfil the normalisation condition

$$\int_0^{2\pi} \mathrm{d}\varphi \int_{-\infty}^{\infty} \mathrm{d}(\gamma\psi) \int_0^{\infty} \left(F_{\mathrm{S}\sigma}(\omega,\psi) + F_{\mathrm{S}\pi}(\omega,\psi)\right) \mathrm{d}(\omega/\omega_c) = 1 \tag{3.31}$$

3.5 Discussion and Summary

Synchrotron radiation: spectral and angle distribution



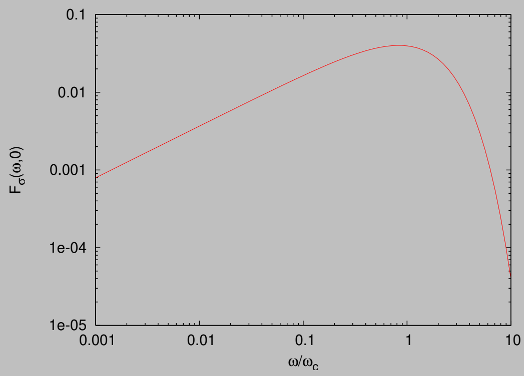


Figure 16: Normalised spectral and angular distribution of the normal synchrotron radiation for σ- and π-polarisation

Fig. 16 shows the normalised spectral functions, plotted over frequency and observation angle. Particularly noteworthy is the node line of the spectral function for π-polarisation for $\gamma\psi = 0$. I.e. in the deflection plane, the radiation is σ-polarised, otherwise elliptical. ω_c turns out to be a "typical frequency" of the spectrum in the sense mentioned in the introduction. More about this below.

Spectrum In many cases, only the spectrum is of interest, i.e. the angular-integrated spectral distribution of the synchrotron radiation, which in turn can be written with dimensionless spectral functions:

$$\boxed{\frac{\mathrm{d}P}{\mathrm{d}\omega} = \int \frac{\mathrm{d}^2 P}{\mathrm{d}\Omega \mathrm{d}\omega}\mathrm{d}\Omega = \frac{P_\mathrm{S}}{\omega_c}\left(S_{\mathrm{S}\sigma}\left(\frac{\omega}{\omega_c}\right) + S_{\mathrm{S}\pi}\left(\frac{\omega}{\omega_c}\right)\right) = \frac{P_\mathrm{S}}{\omega_c} S_\mathrm{S}\left(\frac{\omega}{\omega_c}\right)} \tag{3.32}$$

Here are

$$\begin{aligned} S_{\mathrm{S}\sigma}\left(\frac{\omega}{\omega_c}\right) &= \frac{27}{16}\frac{\omega}{\omega_c}\left(-3\frac{\mathrm{Ai}'(z)}{z} - \frac{1}{3} + \int_0^z \mathrm{Ai}'(z')\mathrm{d}z'\right) \\ S_{\mathrm{S}\pi}\left(\frac{\omega}{\omega_c}\right) &= \frac{27}{16}\frac{\omega}{\omega_c}\left(-\frac{\mathrm{Ai}'(z)}{z} - \frac{1}{3} + \int_0^z \mathrm{Ai}'(z')\mathrm{d}z'\right) \\ S_\mathrm{S}\left(\frac{\omega}{\omega_c}\right) &= \frac{54}{16}\frac{\omega}{\omega_c}\left(-2\frac{\mathrm{Ai}'(z)}{z} - \frac{1}{3} + \int_0^z \mathrm{Ai}'(z')\mathrm{d}z'\right) \end{aligned} \tag{3.33}$$

$$\text{mit} \quad z := \left(\frac{3\omega}{2\omega_c}\right)^{2/3}$$

or expressed with the modified Bessel functions

$$
\begin{aligned}
S_{S\sigma}\left(\frac{\omega}{\omega_c}\right) &= \frac{9\sqrt{3}}{16\pi}\frac{\omega}{\omega_c}\left(\int_{\omega/\omega_c}^{\infty} K_{5/3}(z')\mathrm{d}z' + K_{2/3}\left(\frac{\omega}{\omega_c}\right)\right) \\
S_{S\pi}\left(\frac{\omega}{\omega_c}\right) &= \frac{9\sqrt{3}}{16\pi}\frac{\omega}{\omega_c}\left(\int_{\omega/\omega_c}^{\infty} K_{5/3}(z')\mathrm{d}z' - K_{2/3}\left(\frac{\omega}{\omega_c}\right)\right) \\
S_{S}\left(\frac{\omega}{\omega_c}\right) &= \frac{9\sqrt{3}}{8\pi}\frac{\omega}{\omega_c}\int_{\omega/\omega_c}^{\infty} K_{5/3}(z')\mathrm{d}z'
\end{aligned}
\tag{3.34}
$$

Integral spectrum of synchrotron radiation

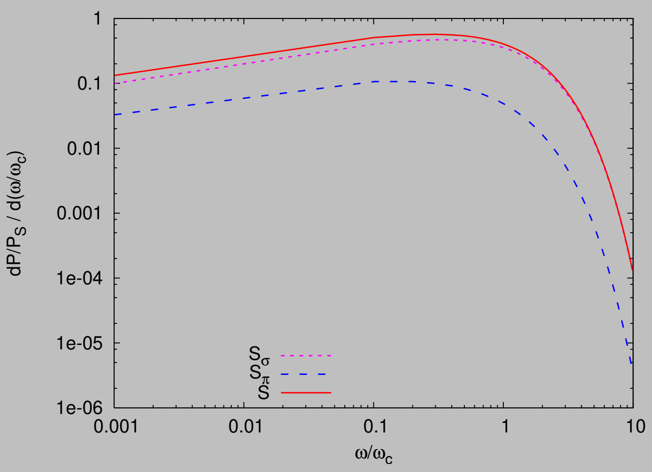


Figure 17: Representation of the integral spectrum of normal synchrotron radiation.

Remark. The integral of the spectral functions over all frequencies yields

$$
\int_0^{\infty} \left(S_{S\sigma}(\omega/\omega_c) + S_{S\pi}(\omega/\omega_c)\right) \mathrm{d}(\omega/\omega_c) = \frac{7}{8} + \frac{1}{8} = 1 \tag{3.35}
$$

Remark. The critical frequency divides the spectrum into two parts of equal integral power:

$$
\int_0^{\omega_c} S_S \mathrm{d}(\omega/\omega_c) = \frac{1}{2} \tag{3.36}
$$

The spectral distribution of synchrotron radiation is fully characterised by the critical frequency and the total radiation power. As a function of the particle energy and the deflecting magnetic field, it is given by:

$$
\boxed{\omega_c = \frac{3c\gamma^3}{2\rho} = \frac{3}{2}\frac{ec^2BW_e^2}{(mc^2)^3}} \tag{3.37}
$$

This is equivalent to the customised equation of magnitude

$$\epsilon_c(\text{keV}) = 2.218\frac{W_e^3(\text{GeV}^3)}{\rho(\text{m})} = 0.665 W_e^2(\text{GeV}^2)B(\text{T}).$$

e.g. KARA:

$$\begin{aligned} W_e &= 2.5\,\text{GeV} \\ B &= 1.5\,\text{T} \\ \hbar\omega_c &= 6\,\text{keV} \end{aligned}$$

Let us finally discuss the radiated power again:

$$P_\text{S} = \frac{2}{3}\frac{r_0 cmc^2\beta^4\gamma^4}{\rho^2} \tag{3.38}$$

For a given particle mass and orbit radius, the power radiated at any one moment scales with γ^4. Since the circumference of a circular accelerator scales approximately with ρ, it can be seen from this equation that the circumference of circular accelerators must increase approximately with γ^4 if the total power radiated is to remain the same. This fact defines the technical limit for high-energy circular accelerators.

Equation 3.38 can be transformed:

$$P_\text{S} = \frac{2}{3}\frac{r_0 e^2c^3B^2\beta^2\gamma^2}{mc^2} \tag{3.39}$$

$$P_\text{S} = \frac{2}{3}\frac{e^4c^3B^2\beta^2W_e^2}{4\pi\epsilon_0(mc^2)^4} \tag{3.40}$$

Equation 3.39 shows that, for a given particle energy and mass, the power radiated scales with B^2. If high radiation power is desired, it is therefore advisable to use bending magnets with high magnetic fields. In some cases, a high radiation power at the end of a bending magnet is undesirable, for example if a superconducting insertion device is located in the subsequent straight section, which must not be heated by radiation. At some storage ring sources, so-called *soft bends* are therefore used at the transition from a bending section to a straight section (example MAX III in Lund, Sweden).

From equation 3.40 it can be seen that for a given particle energy and a given bending field, the radiated power is inversely to the fourth power of the rest energy (mass) of the accelerated particle; synchrotron radiation is therefore a negligible phenomenon in most cases when protons and heavy ions are accelerated (incidentally, this no longer applies to the LHC).

The equations discussed above describe the instantaneous radiation power, corresponding to the *local* energy loss of the particle. Technically more illustrative is the total energy loss along the particle's trajectory during one revolution around the ring, which is obtained by integration:

$$\begin{aligned} W_\text{S} &= \oint \frac{\mathrm{d}W}{\mathrm{d}t'}\frac{\mathrm{d}t'}{\mathrm{d}s}\mathrm{d}s = \oint \frac{P_\text{S}}{c}\mathrm{d}s = \frac{2r_0mc^2\gamma^4}{3}I_\text{S2} \\ \text{with} \quad I_\text{S2} &= \oint \frac{1}{\rho^2}\mathrm{d}s \quad : \text{2nd synchrotron radiation integral} \end{aligned} \tag{3.41}$$

Note: In a storage ring with straight sections, ρ is a function of s. If ρ is the same in all non-field-free sections (isomagnetic ring), then

$$I_{S2} = \frac{2\pi}{\rho}$$

The total radiation power in an isomagnetic storage ring with a stored electron current

$$I = \frac{eN_e}{T_{rev}}$$

is given by

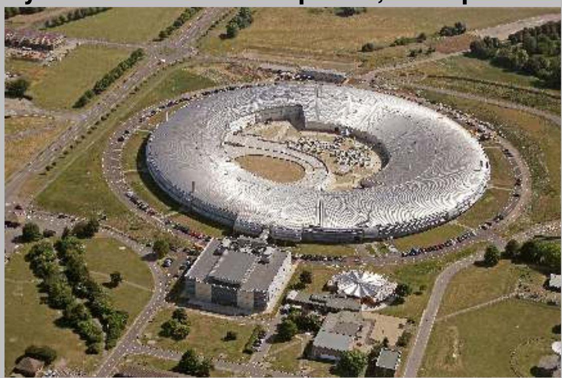
$$P_I = \frac{N_e W_S}{T_{rev}} = \frac{4\pi r_0 mc^2 \gamma^4 I}{3e\rho} \tag{3.42}$$

Synchrotron radiation power, example KARA

$$\begin{aligned} W_e &= 2.5\,\mathrm{GeV} \\ I &= 200\,\mathrm{mA} \\ B &= 1.5\,\mathrm{T} \\ \rho &= 5.7\,\mathrm{m} \\ C &= 110.4\,\mathrm{m} \\ P_I &= 130\,\mathrm{kW} \end{aligned}$$

Figure 18: KARA-Speicherring

Synchrotron radiation power, example Diamond



$$\begin{aligned} W_e &= 3\,\mathrm{GeV} \\ I &= 300\,\mathrm{mA} \\ B &= 1.4\,\mathrm{T} \\ \rho &= 7.16\,\mathrm{m} \\ C &= 561.6\,\mathrm{m} \\ P_I &= 300\,\mathrm{kW} \end{aligned}$$

Figure 19: Diamond Light Source, Didcot

Synchrotron Radiation Power, Example ESRF

Figure 20: European Synchrotron Radiation Facility (ESRF), Grenoble

$$W_e = 6.03\,\mathrm{GeV}$$
$$I = 200\,\mathrm{mA}$$
$$B = 0.8\,\mathrm{T}$$
$$\rho = 24.86\,\mathrm{m}$$
$$C = 844.4\,\mathrm{m}$$
$$P_\mathrm{I} = 910\,\mathrm{kW}$$

Note again: The radiation intensity goes with γ^4 and with $1/\rho$. This is the reason why the ring circumference has to increase disproportionately with increasing particle energy.

Remarks:

- Lost power must be supplied.
- Radiated power must be dissipated.
- Regarding accelerators for particle physics collision experiments: Since γ is the ratio of energy to rest energy, it is considerably easier from the point of view of synchrotron radiation loss to accelerate heavy particles (hadrons) to high energies in circular accelerators than light particles (leptons). However, the scattering process then becomes more complex due to the larger number of elementary particles involved.

References

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