Lecture: Accelerator Physics
Winter 2024/25

Free Electron Lasers

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Contents

1 Introduction

Reminder: Quantum Laser

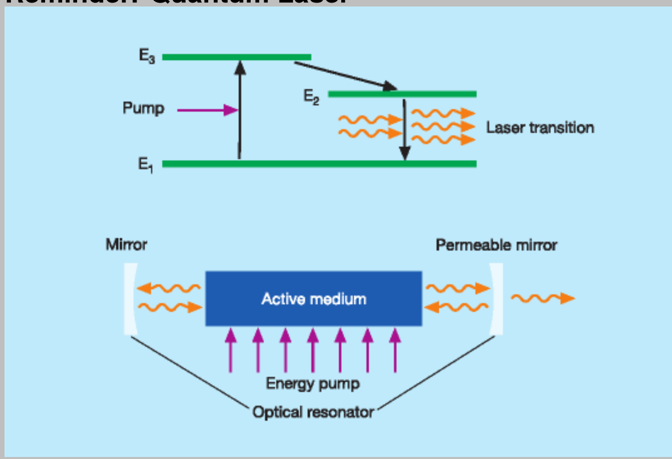


Figure 1: Schematic of the operating principle of a quantum laser

Fig. from Schmüser et al. 2014

Reminder: LASER stands for Light Amplification by Stimulated Emission of Radiation. This stimulated emission is achieved by an *active medium* (solid or gas) with at least three quantum states of the electrons involved, and a pump source that creates a population inversion. In this case, the spontaneously emitted light (partially stored in an optical resonator) is amplified by stimulated emission. The transition probability from the excited state E_2 to the ground state E_1 is proportional to E_0^2 (where E_0 is the amplitude of the electric field of the stimulating wave).

Principle of the free-electron laser

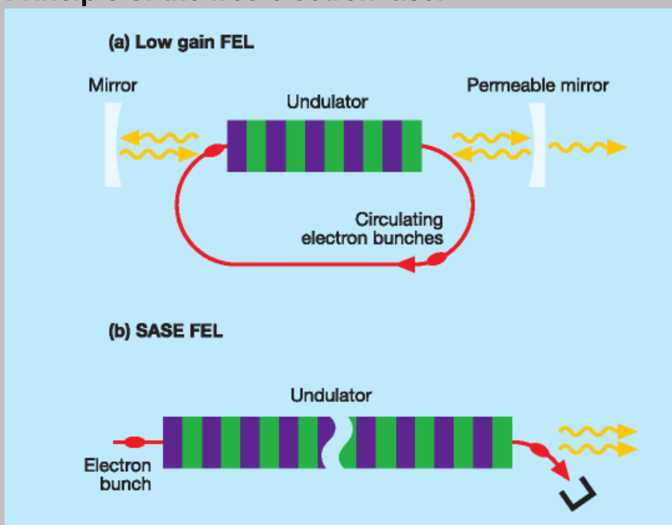


Figure 2: Schematic of the operating principle of low-gain and high-gain FELs (Schmüser et al. 2014)

The FEL is based on the emission of undulator radiation by a relativistic electron beam. The figure shows two types:

1. the *low-gain* FEL for wavelengths from the infrared to the optical range, i.e. for wavelengths at which an optical resonator can be used. Here, the electron beam passes through the undulator many times with a low amplification of the electromagnetic wave per revolution.

2. the *high-gain* FEL for wavelengths from the vacuum ultraviolet to the X-ray range. For these wavelengths, there is no 180°-deflecting mirror. Therefore, the desired gain must be achieved in a single pass through a long undulator.

What is the gain due to? The decisive factor is that the particles in an undulator periodically have a transverse velocity component and therefore an energy transfer from or into the transversely oscillating field can take place.

High Gain FEL: The Basic Process

Energy transfer between particles and the radiation field

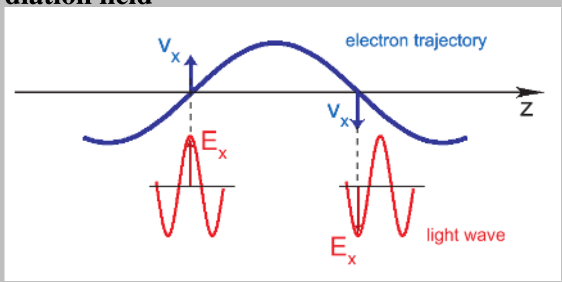


Fig. from Schmüser et al. 2014

- $v_x \parallel \vec{E}_{\text{light}}$
- periodic energy transfer possible

Continuous energy transfer

- fixed phase relation between radiation field and particle motion
- this is fulfilled for:

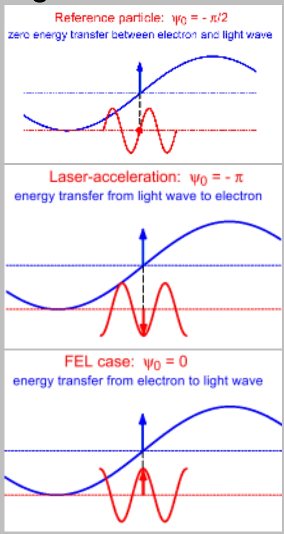
$$\Rightarrow \gamma_r = \sqrt{\frac{\lambda_u}{2\lambda}\left(1 + \frac{K_u^2}{2}\right)}$$

with the resonance energy γ_r

A *continuous* energy transfer, i.e. a net energy transfer in one or the other direction, can only take place if, in the periodic motion of the particles, the maximum velocity component in transverse direction is repeatedly reached in the same phase position to the electromagnetic wave. In the example shown here, for example, at each zero crossing of the trajectory, $v = v_{\max}$ and $E_x = E_{x\,\max}$ are simultaneously fulfilled and the velocity and field vectors point in the same direction. It turns out that this condition is met precisely for the fundamental wavelength of the spontaneous undulator radiation.

In the high-gain FEL, this continuous exchange of energy between wave and particle then leads to microbunching, which is the cause of the FEL amplification:

High Gain FEL: Microbunching



Long bunch, resonance condition fulfilled:

- all relative phase relations are present
- resulting in periodic energy modulation
- periodic modulation of the mean drift velocity
- periodic particle density modulation

$\longrightarrow$ *Microbunching:*

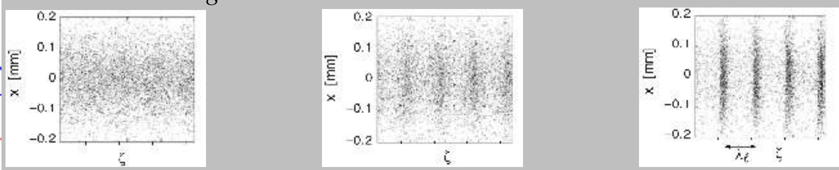


Fig. from Schmüser et al. 2014

- coherent radiation by all particles!

There are characteristic differences and similarities between FELs and quantum lasers:

- The electron energy in the FEL is (virtually) not quantised
- The "pump" energy is provided in the FEL by the high-energy electron beam
- stimulated emission takes place in the FEL as well as in the quantum laser, whereby the stimulating electromagnetic wave can be generated in three ways:
  - as an optical resonator mode (*low gain*)
  - by coupling in an externally generated electromagnetic field (*seeding, high gain harmonic generation (HGHG)*)
  - by spontaneous emission of undulator radiation (*Self Amplification of Spontaneous Emission (SASE)*)

  In all these cases, the coupling between the electromagnetic field and the electrons is proportional to the field amplitude and the amplification is proportional to E_0^2. The properties of FEL radiation and laser radiation (in particular coherence) are the same.

It is therefore fair to speak of an FEL as a laser.

An example: SwissFEL

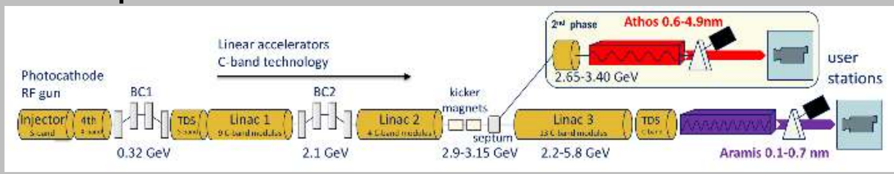


Fig. from Milne et al. 2017

Accelerator	
max. energy	5.8 GeV
Nom. emittance	430 nm rad
Peak current	3 kA
Undulator	
Period	15 mm
K	1.2
Length (active)	48 m
length (total)	60 m

The slide shows the typical setup of a high-gain FEL using the example of the SwissFEL at the Paul Scherrer Institute (PSI) in Villigen, Switzerland. The FEL consists of the injector, a linear accelerator, here divided into three sectors, and the FEL undulators. After the injector and the first Linac sector, there is a magnetic chicane for bunch compression. Background: The effectiveness of the FEL amplification depends on the peak current, i.e. the bunch charge and bunch length. Between sectors two and three, a portion of the bunches is decoupled to the soft X-ray FEL, in the straight-ahead direction further accelerated to the hard X-ray FEL. After passing through the FEL undulators, the electron beams are dumped and the FEL radiation transported further into the beamlines. With 5.8 GeV electron energy and 60 m undulator length, the SwissFEL is a very compact FEL for hard X-rays.

References

Milne, Christopher J. et al. (July 2017). “SwissFEL: The Swiss X-ray Free Electron Laser”. In: *Applied Sciences* 7.7, p. 720. DOI: 10.3390/app7070720.

Schmüser, Peter et al. (2014). *Free-Electron Lasers in the Ultraviolet and X-Ray Regime: Physical Principles, Experimental Results, Technical Realization*. 2nd ed. Springer Tracts in Modern Physics. Springer International Publishing. ISBN: 978-3-319-04080-6. URL: https://www.springer.com/de/book/9783319040806.

2 Low-gain FELs

2.1 Operating principle of low-gain FELs

The *low-gain* FEL

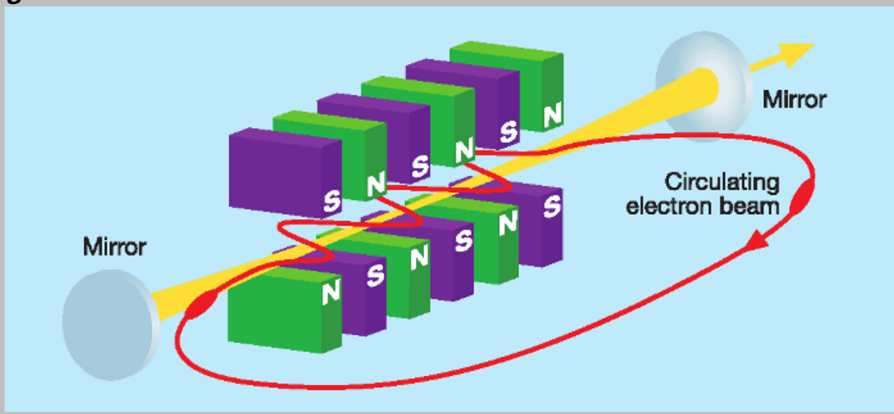


Fig. from Schmüser et al. 2014

Figure 3: Principle design of a low-gain FEL

The amplification in an FEL is based on the feedback of the electromagnetic field to the energy of the electrons, or in other words, on the continuous transfer of energy from the electron beam to the electromagnetic field. In a low-gain FEL, this transfer is only on the order of a few percent of the intensity per round trip. Nevertheless, very high final powers (on the order of GW) are achieved through many round trips:

$$\delta: \quad \text{relative increase in radiant power per round} \Rightarrow P_{\text{out}} = P_{\text{in}}(1+\delta)^N \sim \text{GW} \tag{2.1}$$

What is the origin of this energy transfer? The decisive factor here is the oscillatory motion of the particles in the undulator, or more precisely the fact that the particles periodically exhibit a transverse velocity component and thus a coupling can take place between the *momentary* longitudinal motion of the particle and the *purely transverse* electric field of the free electromagnetic wave.

Let us now consider the interaction between the electron beam and the co-propagating electromagnetic wave

$$E_x(z,t) = E_0 \cos(k_\gamma z - \omega_\gamma t + \psi_0) \quad \text{with} \quad k_\gamma = \omega_\gamma / c = 2\pi/\lambda_\gamma. \tag{2.2}$$

The energy change of an electron in this field is

$$\frac{\mathrm{d}W}{\mathrm{d}t} = \frac{\mathrm{d}\gamma mc^2}{\mathrm{d}t} = -e v_x(t) E_x(t) \tag{2.3}$$

Here, in the harmonic undulator, $v_x(t)$ is also a harmonic function (see the lecture notes *Synchrotron and Undulator Radiation*).

Now, let the phase of the electron motion and the electromagnetic wave at a given time be such that energy transfer can take place, i.e. $v_x(t_0), E_x(t_0) \neq 0$. Then, because of the different propagation velocities of the electron and the wave,

$$\overline{v}_z = \beta^* c = c\left(1 - \frac{1+\frac{K_\mathrm{u}^2}{2}}{2\gamma^2}\right) < c, \tag{2.4}$$

a continuous energy transfer is only possible if the same phase is reached again after passing through half a period of the undulator:

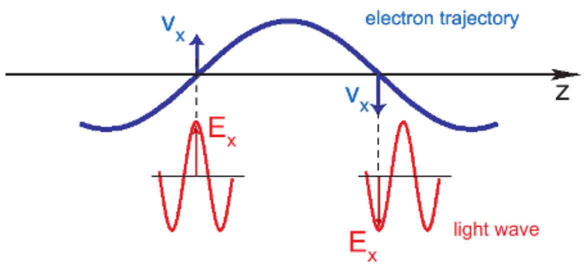


Figure 4: Principle of energy transfer between the electron and the electric field of the emitted light wave (Schmüser et al. 2014)

This means that for the runtime difference of the electron and the wave in a half undulator period, the following must apply:

$$\omega_\gamma(t_e - t_\gamma) = \omega_\gamma\left(\frac{\lambda_\mathrm{u}}{2\overline{v}_z} - \frac{\lambda_\mathrm{u}}{2c}\right) = \pi. \tag{2.5}$$

This is achieved in good approximation precisely when

$$\lambda_\gamma = \frac{\lambda_\mathrm{u}}{2\gamma^2}\left(1 + \frac{K_\mathrm{u}^2}{2}\right) \tag{2.6}$$

This means that the FEL principle works precisely for those wavelengths (λ_γ and their odd harmonics) that are the wavelengths of spontaneous undulator radiation. This is the basis of the SASE principle.

Now we insert the explicit expression for the velocity of the electron in the undulator into equation 2.3:

$$\begin{aligned}\frac{\mathrm{d}W}{\mathrm{d}t} &= -e\underbrace{\frac{cK_\mathrm{u}}{\gamma}\cos(k_\mathrm{u}z)}_{=v_x(z)}\underbrace{E_0\cos(k_\gamma z-\omega_\gamma t+\psi_0)}_{=E_x(z,t)}\\ &= \frac{ecK_\mathrm{u}E_0}{2\gamma}(\cos\psi+\cos\chi) \quad \text{(addition theorem)} \end{aligned} \tag{2.7}$$

mit

$$\begin{aligned}\psi(t) &:= (k_\gamma+k_\mathrm{u})z(t)-\omega_\gamma t+\psi_0\\ \chi(t) &:= (k_\gamma-k_\mathrm{u})z(t)-\omega_\gamma t+\psi_0\end{aligned}$$

ψ is called the *ponderomotive phase*.

Remark. The condition $\psi = const$ or rather $\frac{\mathrm{d}\psi}{\mathrm{d}t} = 0$ is the condition for a continuous energy transfer and leads to equation 2.6.

Remark. The condition $\chi = const$ is not fulfilable (it leads to a backward propagating wave). Because of

$$\chi(z) = \psi(z) - 2k_\mathrm{u}z \tag{2.8}$$

oscillates twice per undulator period, so that this term disappears on average and is therefore neglected in the following.

An electron bunch is usually much longer than the wavelength of the undulator radiation. Therefore, the ponderomotive phase for a single electron contains an arbitrary initial phase ψ_0. For an unmodulated electron bunch, this initial phase is assumed to be uniformly distributed over the ensemble.

In this context, the ponderomotive phase can be interpreted as a longitudinal coordinate of the particle within the bunch.

The bunch-internal coordinate Let's define as the *reference electron* an electron with vanishing energy transfer to the electromagnetic wave, i.e. with $\psi_0 = -\pi/2$. Then its position and ponderomotive phase are

$$z_r(t) = \bar{v}_z t, \quad \psi_r(t) = (k_\gamma+k_\mathrm{u})z_r(t)-\omega_\gamma t-\pi/2. \tag{2.9}$$

For an *arbitrary* electron,

$$z(t) = \bar{v}_z t+\zeta(t), \quad \psi(t) = (k_\gamma+k_\mathrm{u})(\bar{v}_z t+\zeta(t))-\omega_\gamma t-\pi/2, \tag{2.10}$$

i.e. the phase and the relative position of the particle are related by

$$\zeta = \frac{\psi+\pi/2}{k_\gamma+k_\mathrm{u}} \approx \frac{\psi+\pi/2}{2\pi}\lambda_\gamma \tag{2.11}$$

Obviously, the phase relationship between the electron motion and the electromagnetic wave determines whether and in which direction energy is transferred between the electron and the wave.:

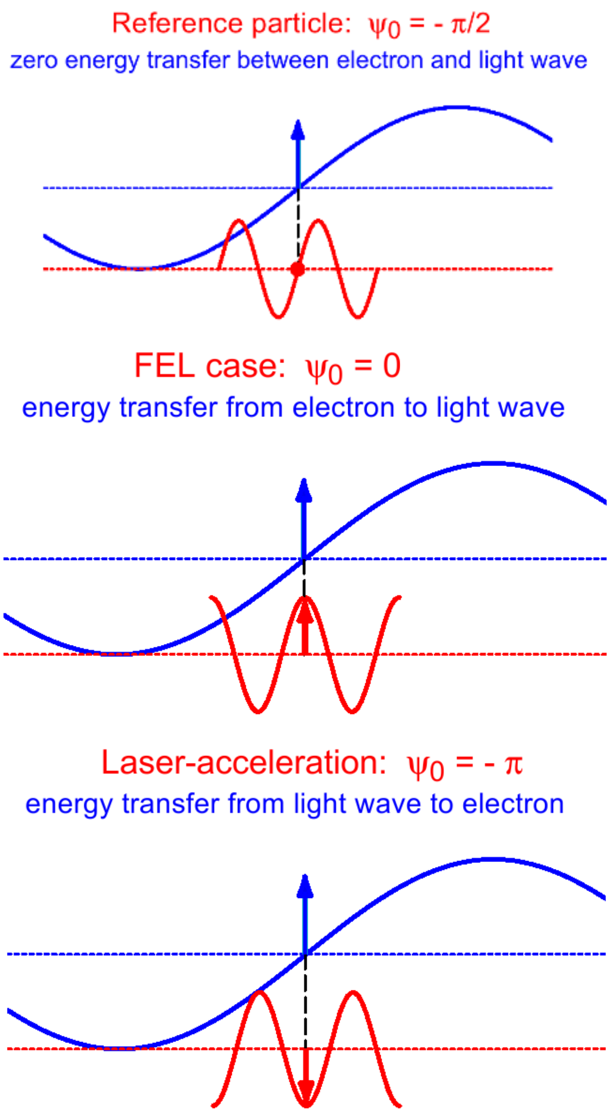


Figure 5: Comparison of energy transfer between light wave and electron (Schmüser et al. 2014)

2.2 The FEL Pendulum Equations

Now for the amplification in the low-gain FEL. We reverse the approach taken in the treatment of the undulator radiation and assume that the laser process is initiated by a plane monochromatic wave with amplitude E_0 and wavelength λ_γ, and define the resonance electron energy $W_r = \gamma_r mc^2$:

$$\boxed{\lambda_\gamma = \frac{\lambda_\mathrm{u}}{2\gamma_r^2}\left(1+\frac{K_\mathrm{u}^2}{2}\right) \quad \Rightarrow \quad \gamma_r = \sqrt{\frac{\lambda_\mathrm{u}}{2\lambda_\gamma}\left(1+\frac{K_\mathrm{u}^2}{2}\right)}} \tag{2.12}$$

Particles with the resonance energy

- emit radiation in the undulator with the “correct” wavelength λ_γ
- have a constant ponderomotive phase when passing through the undulator.

For particles with a (small) relative energy deviation

$$\eta := \frac{W - W_r}{W_r} = \frac{\gamma - \gamma_r}{\gamma_r} \tag{2.13}$$

this is no longer the case, but instead one obtains

$$\frac{\mathrm{d}\psi}{\mathrm{d}t} = \frac{k_\gamma c}{2}\left(1+\frac{K_\mathrm{u}^2}{2}\right)\left(\frac{1}{\gamma_r^2} - \frac{1}{\gamma^2}\right) \tag{2.14}$$

In the low-gain regime, E_0 can be considered constant for a single pass through the undulator can be considered constant, and the coupling between the ponderomotive phase and the relative energy deviation can be described by

$$\boxed{\frac{\mathrm{d}\psi}{\mathrm{d}t} = 2k_\mathrm{u}c\eta, \qquad \frac{\mathrm{d}\eta}{\mathrm{d}t} = -\frac{eE_0K_\mathrm{u}}{2mc\gamma_r^2}\cos\psi,} \tag{2.15}$$

the *FEL pendulum equations*

By ‘transition to the phase variable

$$\varphi = \psi + \pi/2$$

l’a”st this into the equations

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} = 2k_\mathrm{u}c\eta, \qquad \frac{\mathrm{d}\eta}{\mathrm{d}t} = -\frac{eE_0K_\mathrm{u}}{2mc\gamma_r^2}\sin\varphi, \tag{2.16}$$

‘überf”uhren, die analog zu den Bewegungsgleichungen des mathematischen Pendels sind.

As for the longitudinal beam dynamics in a particle accelerator, these equations can be derived from a Hamiltonian

$$H(\eta,\varphi) = k_\mathrm{u}c\eta^2 + \frac{eE_0K_\mathrm{u}}{2mc\gamma_r^2}(1-\cos\varphi) \tag{2.17}$$

and the analogous description of the particle dynamics in the (η, φ) phase space suggests itself. That is, the particles follow trajectories in this phase space with $H = const.$ We remember: There are two classes of trajectories, closed (bound motion) and open. (unstable). The special trajectory that encloses the phase space of bound motions is called the *separatrix*.

Particle dynamics in the FEL bucket

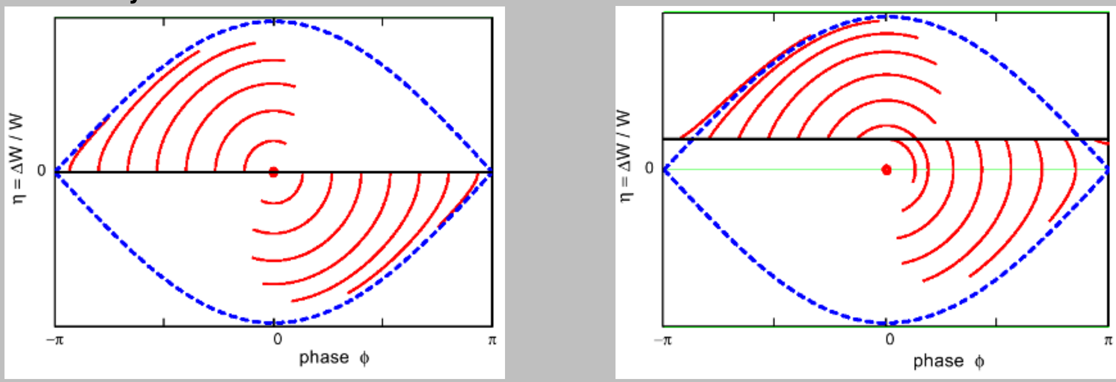


Figure 6: Net energy transfer for an initially uniform phase and $\gamma = \gamma_r$ (left) and $\gamma > \gamma_r$ (right) (Schmüser et al. 2014)

The analogy to longitudinal beam dynamics suggests the designation of the separatrix as a FEL-*bucket*. However, the FEL-*buckets* are much smaller than the radio frequency buckets due to the much shorter wavelength. In fact, a single electron bunch encompasses a multitude of FEL-*buckets*.

The particles in an FEL *bucket* oscillate in terms of their energy and their momentum. If all particles in the ensemble have the same resonance energy and are equally distributed in terms of momentum, then the same number of particles lose and gain energy through interaction with the electromagnetic field, i.e. the net energy transfer between particles and field is zero.

This changes for a monoenergetic particle beam with an energy different from the resonance energy, for example with $\gamma > \gamma_r$. In this case, there are more particles that give energy to the field than those that take energy from it. There is a net energy transfer to the field.

2.3 The Madey theorem

The gain of a *low-gain* FEL, defined as the relative energy increase of the electromagnetic field per pass

$$G = \frac{\Delta W_\gamma}{W_\gamma}$$

is directly related to the derivative of the unulator's spectral function. It is given by the MADEY theorem

$$\boxed{G(\xi) = -\frac{\pi e^2 \hat{K}_\mathrm{u}^2 N_\mathrm{u}^3 \lambda_\mathrm{u}^2 n_e}{4\eta_0 m c^2 \gamma_r^3} \frac{\mathrm{d}}{\mathrm{d}\xi}\left(\frac{\sin^2\xi}{\xi^2}\right)} \tag{2.18}$$

With

$$
\begin{aligned}
n_e &: \quad \text{particle density} \\
\hat{K}_\mathrm{u} := K_\mathrm{u}\left[J_0\left(\frac{K_\mathrm{u}^2}{4+2K_\mathrm{u}^2}\right) - J_1\left(\frac{K_\mathrm{u}^2}{4+2K_\mathrm{u}^2}\right)\right] &: \quad \text{modified undulator parameter} \\
N_\mathrm{u} &: \quad \text{number of undulator periods} \\
\xi &:= \pi N_\mathrm{u}\frac{\omega-\omega_\gamma}{\omega_\gamma}
\end{aligned}
$$

The modified undulator parameter introduced here is not to be confused with the modified undulator parameter K^* introduced in the lecture notes on *Synchrotron and Undulator Radiation*, section 7. This is an abbreviation for the additional intensity factor for the harmonics of the undulator radiation in the case of the strong undulator (see e.g. Eq. 7.26 in the lecture notes *Synchrotron and Undulator Radiation*). The modified undulator parameter used here and in the following results from the first term of the Fourier-Bessel series expansion for the radiation power of the strong undulator, i.e. for the fundamental.

Spectral and amplification function of the low-gain FEL

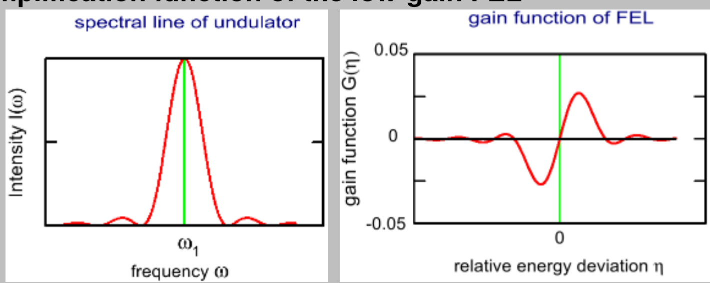
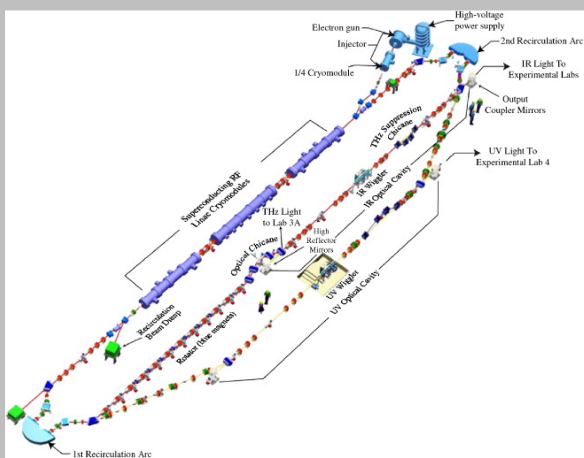


Fig. from Schmüser et al. 2014

Example: JLab infrared FEL



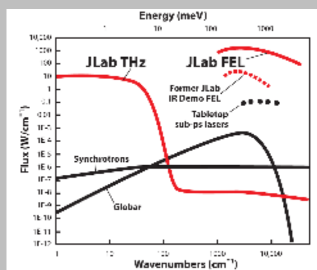


Fig. from Williams 2007

W_e	(80 – 200) MeV
I_e	10 mA
λ_γ	(1.5 – 14) µm
P_out	> 10 kW

References

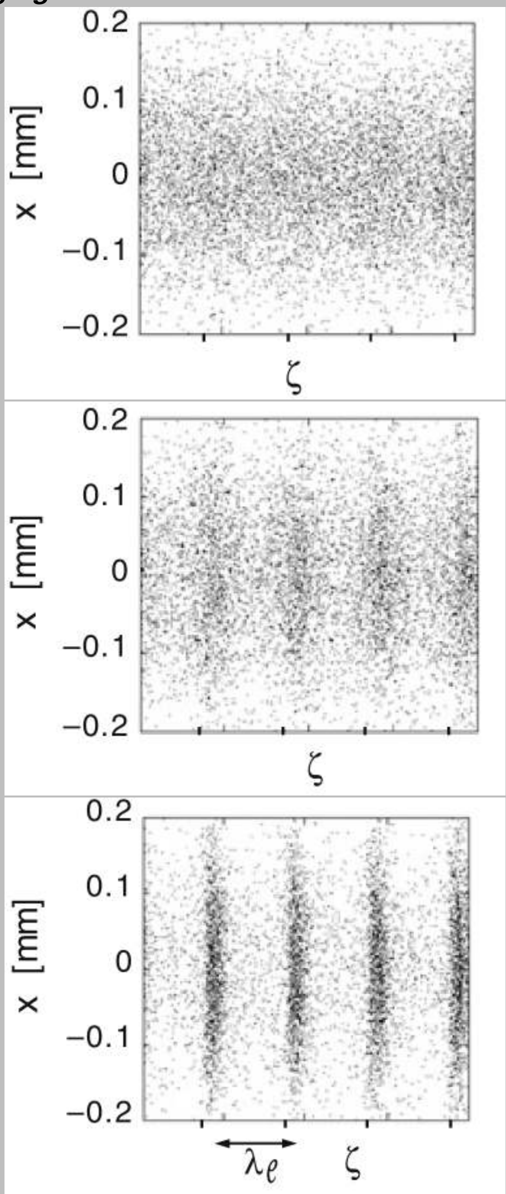
Schmüser, Peter et al. (2014). *Free-Electron Lasers in the Ultraviolet and X-Ray Regime: Physical Principles, Experimental Results, Technical Realization*. 2nd ed. Springer Tracts in Modern Physics. Springer International Publishing. ISBN: 978-3-319-04080-6. URL: https://www.springer.com/de/book/9783319040806.

Williams, Gwyn (2007). *Free-Electron Laser Program*. URL: https://www.jlab.org/FEL/felspecs.html (visited on 07/11/2019).

3 One-dimensional high-gain FEL theory

FELs for vacuum-ultraviolet and X-ray radiation cannot be realised as *low-gain* FELs because no optical resonator can be built for these short wavelengths. The laser amplification must be achieved here in a single pass through a very long undulator. This gives rise to a new amplification phenomenon, known as micro-bunching.

***Micro-bunching* in a *high-gain* FEL**



Micro-bunching is caused by the feedback of the electromagnetic field with the electron bunch: the particles experience an energy loss or gain that depends on their internal longitudinal position, i.e. the bunch is energy-modulated depending on position. This results (due to the stronger deflection of particles with lower energy) in a periodic modulation of the longitudinal velocity and finally in a modulation of the particle density with a period length λ_γ. The bunch thus disintegrates into periodically arranged sub-bunches with a respective longitudinal extent smaller than the wavelength of the electromagnetic wave. The particles of each of these micro-packages now emit *coherently*. In addition, the fixed phase relationship between the micro-packages results in a coherent superposition of the radiation of all micro-packages and thus to the maximum amplification of the radiation with wavelength λ_γ.

The essential physical features of this *high-gain* FEL process can be described in one-dimensional FEL theory, i.e. by neglecting the dependence of the electromagnetic field and the particle density on the transverse coordinates x, y.

So let's write (in complex notation for the sake of simplicity)

$$E_x(z,t) = E_x(z)\mathrm{e}^{\mathrm{i}(k_\gamma z - \omega_\gamma t)} \tag{3.1}$$

with an amplitude $E_x(z)$ that grows slowly in z, and, for the sake of simplicity, we assume that the electron bunch has an infinitely extended charge density that is periodically modulated with a period length of λ_γ

$$\rho(\psi, z) = \rho_0 + \rho_1(z)\mathrm{e}^{\mathrm{i}\psi} \tag{3.2}$$

and, consequently, a correspondingly modulated current density

$$j_z(\psi, z) = j_0 + j_1(z)\mathrm{e}^{\mathrm{i}\psi} \tag{3.3}$$

. Periodicity in z with period length λ_γ is equivalent to periodicity in the Ponderomotive phase with period 2π.

This assumption can be justified by a Fourier expansion of the charge distribution, which shows that for a random initial charge distribution there is always a finite Fourier component of the period λ_γ that is then amplified in the FEL process. Even with a Fourier development in multiples of another period, it can be seen that only components in a narrow band around λ_γ are amplified.

The FEL process now consists of two competing processes:

- a charge density modulation through feedback to the electromagnetic wave
- the repulsive space charge forces acting against it

3.1 Coupling of field and charge density

The interaction between the charge density distribution and the electromagnetic field results from the inhomogeneous wave equation

$$\left[\frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right] E_x(z,t) = \mu_0 \frac{\partial j_x}{\partial t} + \underbrace{\frac{1}{\epsilon_0}\frac{\partial \rho}{\partial x}}_{=0 \text{ in 1D-theory}} \quad . \tag{3.4}$$

It should be recalled that in the case of the *high-gain* FEL, the amplitude of the electromagnetic wave itself is a function of z, so that

$$\left[2\mathrm{i}k_\gamma E_x`(z) + E_x''(z)\right]\mathrm{e}^{\mathrm{i}(k_\gamma z-\omega_\gamma t)} = \mu_0\frac{\partial j_x}{\partial t} \tag{3.5}$$

For further solution, the so-called SVA (*slowly varying amplitude*)-approximation is made, which consists of the following two assumptions:

- Amplitude change within a light wavelength small:

$$\mid E_x'(z) \mid \lambda_\gamma \ll\mid E_x(z) \mid\Rightarrow\mid E_x'(z) \mid\ll k_\gamma \mid E_x(z) \mid \tag{3.6}$$

- change of amplification within a light wavelength negligible:

$$\mid E_x''(z) \mid\ll k_\gamma \mid E_x'(z) \mid\Rightarrow\mid E_x''(z) \mid \text{ negligible} \tag{3.7}$$

With this approximation, we obtain

$$\frac{\mathrm{d}E_x}{\mathrm{d}z} = -\frac{\mathrm{i}\mu_0}{2k_\gamma}\frac{\partial j_x}{\partial t}\mathrm{e}^{-\mathrm{i}(k_\gamma z-\omega_\gamma t)} \tag{3.8}$$

The transversal current density remains to be determined, which results from the relation of longitudinal and transversal velocity of the particles in the undulator:

$$j_x = j_z\cdot\frac{v_x}{v_z} \approx j_z\frac{K_\mathrm{u}}{\gamma}\cos(k_\mathrm{u}z). \tag{3.9}$$

Finally, the following equation holds for the change of the field amplitude

$$\frac{\mathrm{d}E_x}{\mathrm{d}z} = -\frac{\mu_0 c K_\mathrm{u}}{4\gamma}j_1(1+\underbrace{\mathrm{e}^{\mathrm{i}2k_\mathrm{u}z}}_{=0\text{ on average over one period}}) \tag{3.10}$$

3.2 Space charge distribution

The modulation of the charge distribution due to the interaction with the electromagnetic field of the radiation causes, due to the Maxwell equation

$$\frac{\partial E_z(z,t)}{\partial z} = \frac{\rho_1(z)}{\epsilon_0}\mathrm{e}^{\mathrm{i}((k_\gamma+k_\mathrm{u})z-\omega_\gamma t)} \tag{3.11}$$

a modulated longitudinal electric field that counteracts the micro-bunching. Its amplitude is given by

$$\boxed{E_z(z) \approx -\frac{\mathrm{i}}{\epsilon_0 k_\gamma}\rho_1(z) \approx -\frac{\mathrm{i}\mu_0 c^2}{\omega_\gamma}j_1(z)} \tag{3.12}$$

Here again the SVA approximation was used and further approximated $k_u \ll k_\gamma$ and $j_z = \rho v_z \approx \rho c$.

3.3 The coupled FEL differential equations of the first order

The inclusion of the z dependence of the field amplitude leads to a modification of the FEL pendulum equations. To do this, one first switches from the time t to the location $z = v_z t$ as an independent parameter.

If we then consider only a slice of the charge density distribution with a length of λ_γ (or, in other words, an FEL bucket) and represent the current density in such a slice by a distribution N of point-shaped particles with coordinates ψ_n with respect to the ponderomotive phase

$$j = -ec\frac{2\pi}{A_\mathrm{b}\lambda_\gamma}\sum_{n=1}^{N}\delta(\psi - \psi_n) \tag{3.13}$$

with the beam cross-section A_b, then, under the assumption that the particles are either uniformly distributed or their distribution is periodic in ψ with period 2π, we obtain the following

$$\boxed{\begin{aligned}
\frac{\mathrm{d}\psi_n}{\mathrm{d}z} &= 2k_\mathrm{u}\eta_n \qquad \text{for} \quad n = 1...N \\
\frac{\mathrm{d}\eta_n}{\mathrm{d}z} &= -\frac{e}{mc^2\gamma_r}\mathrm{Re}\left[\left(\frac{\hat{K}_u E_x}{2\gamma_r} - \frac{\mathrm{i}\mu_0 c^2}{\omega_\gamma} j_1\right)\mathrm{e}^{\mathrm{i}\psi_n}\right] \\
j_1 &= j_0\frac{2}{N}\sum_{n=1}^{N}\mathrm{e}^{\mathrm{i}\psi_n}, \quad j_0 = -ec\frac{N}{A_b\lambda_\gamma} \\
\frac{\mathrm{d}E_x}{\mathrm{d}z} &= -\frac{\mu_0 c\hat{K}_\mathrm{u}}{4\gamma} j_1.
\end{aligned}} \tag{3.14}$$

This system of equations contains coupled differential equations for each individual particle in the ensemble, and thus directly describes the many-particle problem. However, it cannot be solved analytically. It does, however, provide the basis for numerical simulations that can be extended to any initial distribution of the particles in phase space.

The current density equation (third line) results from the Fourier series expansion of Eq. 3.13.

The last of the equations 3.14 describes the change of the field amplitude as a function of z, i.e. the laser amplification.

3.4 The third-order differential equation for the high-gain FEL

The amplification process in an FEL is obviously a many-particle effect. It is now possible to transform the coupled first-order single-particle equations of motion into a third-order differential equation by applying the concepts of the many-particle description using the particle density distribution and the Vlasov equation that applies to it (see the lecture notes on *Statistical Mechanics for Storage Rings*). A derivation can be found, for example, in Schmüser et al. 2014 in the Appendix.

$$\Psi(\psi, \eta, z) = \Psi_0(\eta) + \mathrm{Re}\left(\Psi_1(\eta, z)e^{\mathrm{i}\psi}\right). \tag{3.15}$$

with

$$|\Psi_1| \ll |\Psi_0| \tag{3.16}$$

and a narrow-band Gaussian energy distribution is assumed for the unshifted term:

$$\Psi_0(\eta) = \frac{1}{\sqrt{2\pi}\sigma_\eta} \exp\left(\frac{-(\eta-\eta_0)^2}{2\sigma_\eta^2}\right) \qquad \text{with} \qquad \eta_0 := \frac{W_0 - W_r}{W_r} \tag{3.17}$$

and W_r is the resonance energy.

With the VLASOV equation

$$\frac{\mathrm{d}\Psi}{\mathrm{d}z} = \frac{\partial\Psi}{\partial z} + \frac{\partial\Psi}{\partial\psi}\frac{\partial\psi}{\partial z} + \frac{\partial\Psi}{\partial\eta}\frac{\partial\eta}{\partial z} = 0 \tag{3.18}$$

The coupled system of differential equations can be approximately converted into an integro-differential-equation for the field amplification, which no longer contains the single-particle dynamics in the FEL*bucket*:

$$\begin{aligned} \frac{\mathrm{d}E_x}{\mathrm{d}z} &= \mathrm{i}k_\mathrm{u}\frac{\mu_0\hat{K}n_e e^2}{2m\gamma_r^2}\int_0^z \left[\frac{\hat{K}}{2\gamma_r}E_x + \mathrm{i}\frac{4\gamma_r c}{\omega_\gamma\hat{K}}\frac{\mathrm{d}E_x}{\mathrm{d}z}\right] h(z-s)\mathrm{d}s \\ \text{with} \quad h(z-s) &= \int_{-1}^{\infty}(z-s)\mathrm{e}^{\mathrm{i}2k_\mathrm{u}\eta(z-s)}\Psi_0(\eta)\mathrm{d}\eta \end{aligned} \tag{3.19}$$

Assuming a monoenergetic particle beam with energy W, the amplification of the electric field in a high-gain FEL can be described by the 3rd-order FEL equation within the framework of 1-dimensional FEL theory:

$$\boxed{\frac{E_x'''}{\Gamma^3} + 2\mathrm{i}\frac{\eta}{\rho_\mathrm{FEL}}\frac{E_x''}{\Gamma^2} + \left(\frac{k_p^2}{\Gamma^2} - \left(\frac{\eta}{\rho_\mathrm{FEL}}\right)^2\right)\frac{E_x'}{\Gamma} - \mathrm{i}E_x = 0.} \tag{3.20}$$

where

$$\begin{aligned} \text{strengthening parameter} \quad & \Gamma = \sqrt[3]{\frac{\mu_0\hat{K}_\mathrm{u}^2 e^2 k_\mathrm{u} n_e}{4\gamma_r^3 m}} \\ \text{Space charge parameter} \quad & k_p = \frac{\omega_p^*}{c}\sqrt{\frac{2\lambda_\gamma}{\lambda_\mathrm{u}}},\ \omega_p^* = \sqrt{\frac{n_e e^2}{\gamma_r\eta_0 m}} \quad \text{(plasma frequency)} \\ \text{FEL parameter (Pierce parameter)} \quad & \rho_\mathrm{FEL} = \frac{\Gamma}{2k_\mathrm{u}}. \end{aligned} \tag{3.21}$$

3.5 Some applications of the FEL equations

within the framework of the coupled one-particle equations of motion, the FEL process can be described at least qualitatively well, also with regard to its initiation.

The amplification process in an FEL can be initiated in two ways: (a) by a modulated charge density distribution or (b) by a seed wave. Fundamental statements about power amplification and particle dynamics can be obtained in the case of (a) by assuming periodic density modulations. The initiation of the amplification process by random density fluctuations, i.e. noise, on which the SASE process is based, can also be described within the framework of the one-dimensional FEL theory.

3.5.1 Scaling laws

The analytical solution of the FEL equation with the approach

$$E_x(z) = A\mathrm{e}^{\alpha z} \tag{3.22}$$

leads, for the special case $\eta = 0$ (particle energy = resonance energy) and $k_p = 0$ (small electron density, neglect of space charge), to an exponential amplification of the field amplitude

$$E_x(z) \propto \mathrm{e}^{z/L_{g0}} \tag{3.23}$$

with the gain length

$$L_{g0} := \frac{1}{\sqrt{3}\Gamma}. \tag{3.24}$$

Power amplification in the *high-gain* FEL

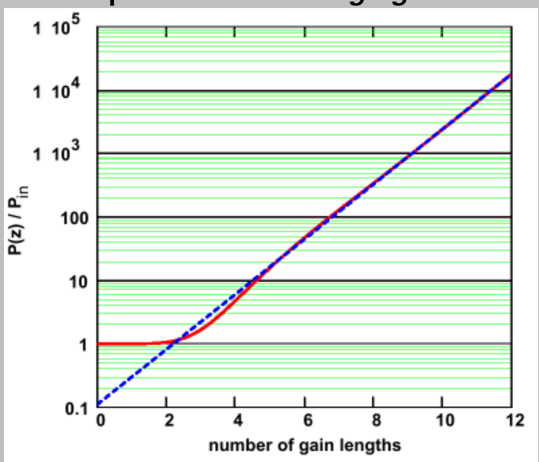


Figure 7: Calculated radiation power gain as a function of z/L_g for a monoenergetic electron beam at resonance energy and negligible space charge. Dashed: Approximation by an exponential function $(P_{\text{in}}/9)\exp(z/L_g)$ (Schmüser et al. 2014)

In the case shown in Fig. 6, *seeding*, i.e. the injection of a stimulating electromagnetic wave with power P_{in}, was assumed.

3.5.2 The gain function

The gain function

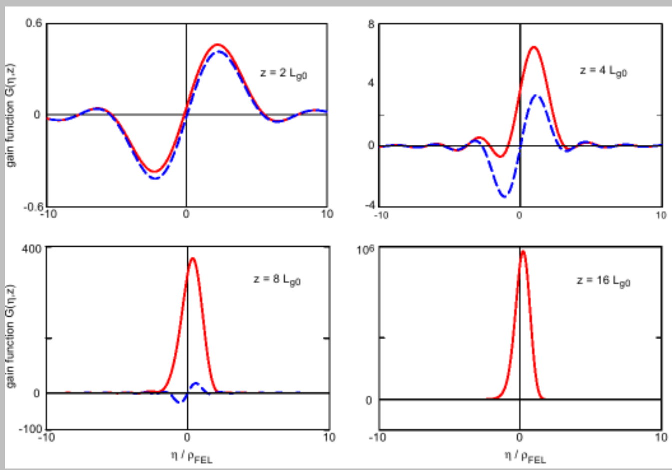


Figure 8: FEL gain function. Blue: low-gain theory, Red: high-gain theory

The solution of the 3rd order FEL equation for short undulators ($z \leq L_g$) leads to similar results for the amplification function as the MADEY theorem of the *low-gain* FEL theory. For long undulators, significant differences arise. Note the extremely different y scales of the plots.

3.5.3 Saturation

The 3rd-order FEL equation is suitable for describing the FEL process in the regime of exponential power amplification. However, the approximations made to derive it do not allow it to correctly describe the saturation regime. It is clear that saturation must occur, because the number of particles in the bunch is finite and consequently also the power amplification, which is based on coherent radiation by as many particles as possible in a bunch.

In order to describe saturation correctly, the coupled system of 1st-order differential equations must be solved (numerically).

The saturation length h depends on the initial conditions of the FEL process (e.g. *seeding power*). The saturation power is independent of the initial conditions and is given by

$$P_{\text{sat}} \approx \frac{4}{3}\rho_{\text{FEL}} P_{\text{beam}}, \tag{3.25}$$

where

$$P_{\text{beam}} = \frac{\gamma_r m c^2 I_0}{e} \tag{3.26}$$

is the power of the electron beam.

The FEL saturation power is typically of the order of 0.1% of the electron beam power.

Saturation

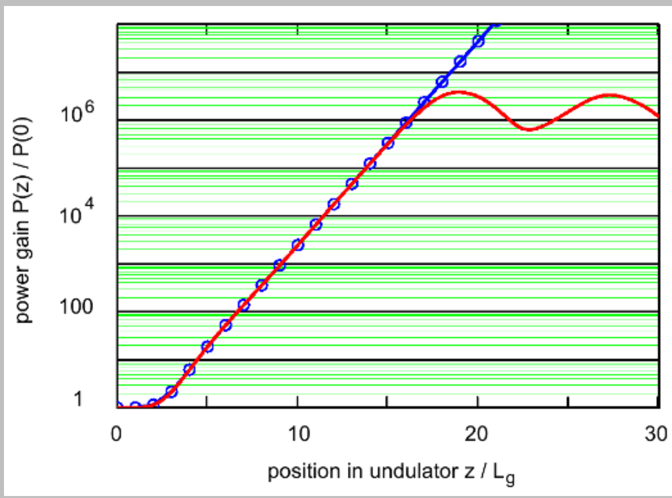


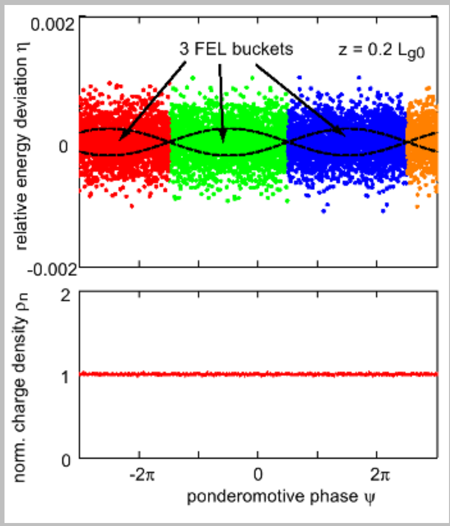
Figure 9: Comparison of the calculated power amplification from the 3rd-order FEL equation (circles) and the coupled 1st-order DGL system (solid line), which also describes the saturation.

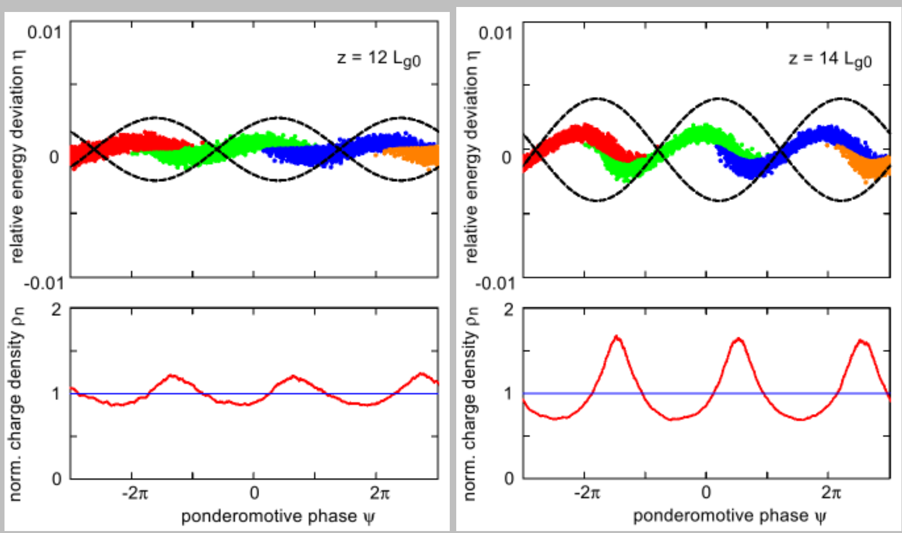
The radiation power oscillation as a function of z in the saturation region is due to the pumping of energy back and forth between the radiation field and the electron beam.

3.5.4 Simulation of particle dynamics

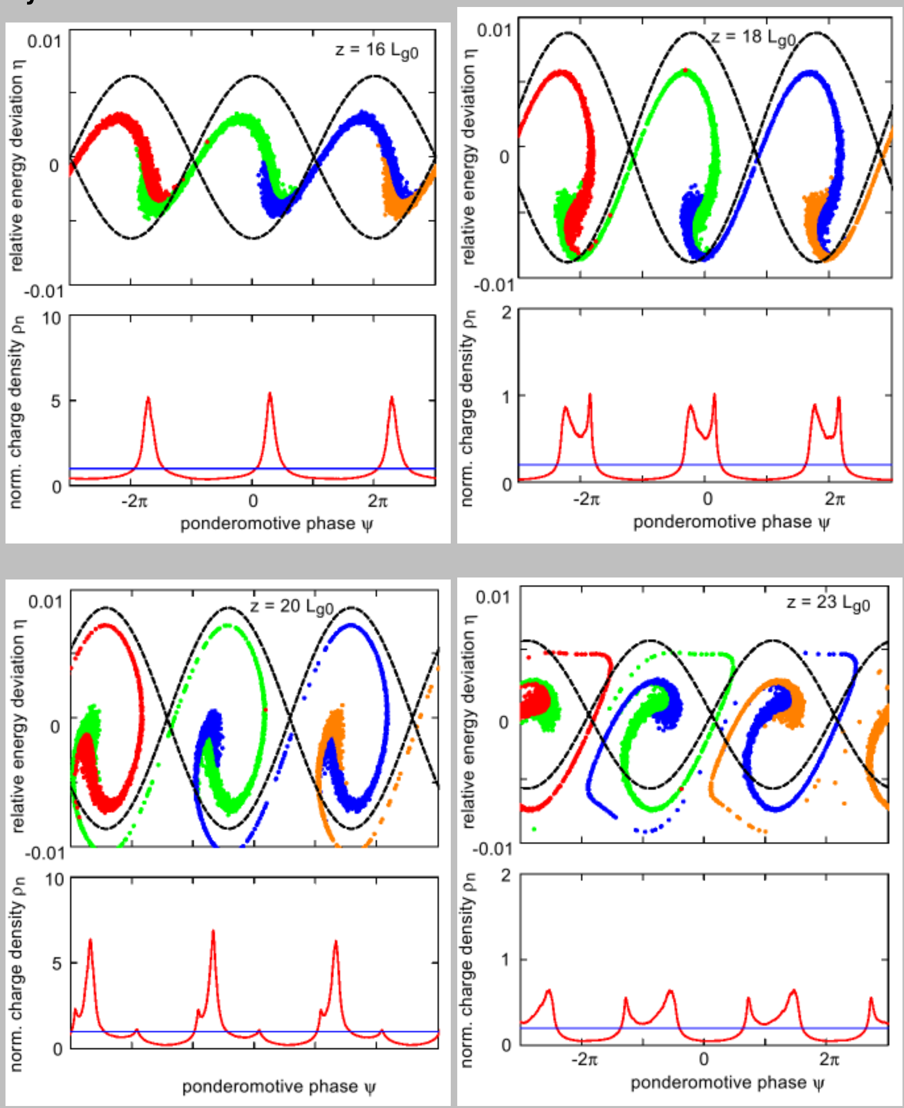
The phenomenon of microbunching can also be investigated within the framework of the one-dimensional FEL theory by numerical integration of the system of first-order differential equations. The following figures show, starting from an uniform distribution of the particles in phase space, the development of the particle distribution in the FEL buckets and the particle density distribution projected onto the ponderomotive phase as a function of the displacement z/L_g:

Particle dynamics





Particle Dynamics



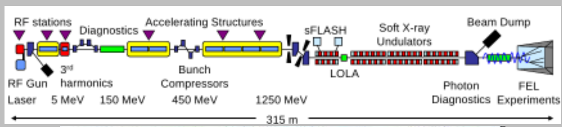
Figures from Schmüser et al. 2014

Conclusions:

- with increasing z, the FEL buckets are shifted towards smaller phases and the amplitude of the separatrix increases
- up to $14L_g$, the energy modulation of the bunch remains approximately harmonic and becomes increasingly distorted for $z > L_g$ the particles diffuse from one *bucket* to the next the particle density forms (narrow) maxima at values of the ponderomotive phase at which energy is transferred from the electrons to the electromagnetic field.

3.5.5 Examples of high-gain FELs

Example: FLASH, Hamburg



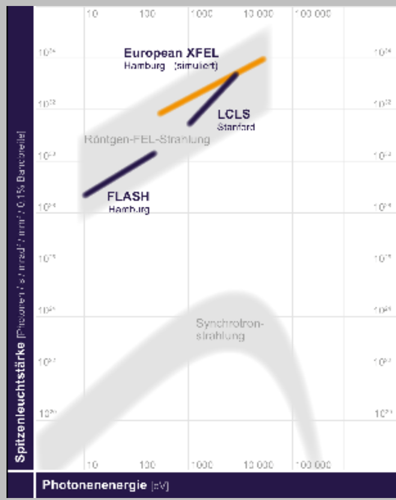
W_e	$1.25\,\mathrm{GeV}$
λ_γ	$(4.1 - 45)\,\mathrm{nm}$
P_{out}	$400\,\mathrm{mW}$
$\hat{P}_{\mathrm{out}}$	$(1 - 3)\,\mathrm{GW}$

Example: LCLS, Stanford

W_e	$(6.7 - 14.7)\,\mathrm{GeV}$
λ_γ	$(6.2 - 1.3)\,\mathrm{\mathring{A}}$

Example: European XFEL, Hamburg

W_e	17.5 GeV
λ_γ	0.5 Å



References

Schmüser, Peter et al. (2014). *Free-Electron Lasers in the Ultraviolet and X-Ray Regime: Physical Principles, Experimental Results, Technical Realization*. 2nd ed. Springer Tracts in Modern Physics. Springer International Publishing. ISBN: 978-3-319-04080-6. URL: https://www.springer.com/de/book/9783319040806.

4 Energy distribution, space charge, 3D effects

A truly (also quantitatively) realistic description of free-electron lasers can only be given in a three-dimensional theory. Our assumptions, which we have more or less explicitly stated above, namely a transversely and longitudinally infinitely extended electron beam and light wave, a monoenergetic electron beam, negligible space charge — are not fulfilled in a realistic FEL.

For 3D-FEL physics, there are a number of very good simulation programs, but no closed analytical theory. Nevertheless, in this section we want to give an overview of the most important effects that lead to deviations from the 1D FEL theory, and give an example of modified semi-empirical scaling laws that can be profitably applied for realistic estimates can be profitably applied.

4.1 Finite energy distribution width, space charge

Let us return to the solution of the third-order FEL equation, which we treated a little carelessly above.

We have searched for a solution of the FEL equation with the approach

$$E_x(z) = A\mathrm{e}^{\alpha z} \quad \text{for} \quad W = W_r \Leftrightarrow \eta = 0, \sigma_\eta = 0 \tag{4.1}$$

. This leads to

$$\alpha^3 = \mathrm{i}\Gamma^3 \tag{4.2}$$

with the three eigenvalues

$$\alpha_1 = (\mathrm{i} + \sqrt{3})\Gamma/2, \quad \alpha_2 = (\mathrm{i} - \sqrt{3})\Gamma/2, \quad \alpha_2 = -\mathrm{i}\Gamma/2, \tag{4.3}$$

of which only the first leads to an exponential amplification, and which we have identified with the *gain* length:

$$\exp(2\Re\{\alpha_1\}z) =: \exp\left(z/L_{g0}\right) \tag{4.4}$$

With $W \neq W_\mathrm{r} \Leftrightarrow \eta \neq 0$, but still $\sigma_\eta = 0$ and $k_p = 0$, the eigenvalues α_i become functions of the energy deviation, in particular $\alpha_1 = \alpha_1(\eta)$, and thus the gain length

$$L_g = \frac{1}{\max[2\Re\{\alpha_2(\eta)\}]} \tag{4.5}$$

or expressed using the dimensionless growth rate

$$f_\mathrm{gr}(\eta) := 2\Re\{\alpha_1(\eta)\}L_{g0} \tag{4.6}$$

$$\max[f_\mathrm{gr}(\eta)] = \max[\Re\{\alpha_1(\eta)\}L_{g0}]. \tag{4.7}$$

Detuning: FEL growth rate

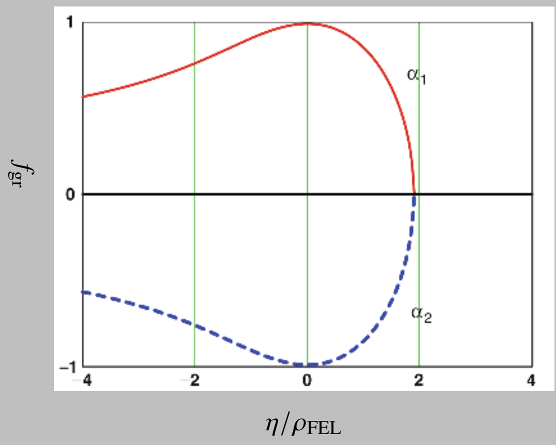


Figure 10: Growth rate as a function of the relative energy deviation. Figure from Schmüser et al. 2014

The graph shows the growth rate for the first and second eigenvalue of the FEL equation as a function of the relative energy deviation from the resonance energy. The amplification stops at $\eta = 1.88\rho_{\mathrm{FEL}}$.

FEL growth rate, effect of space charge

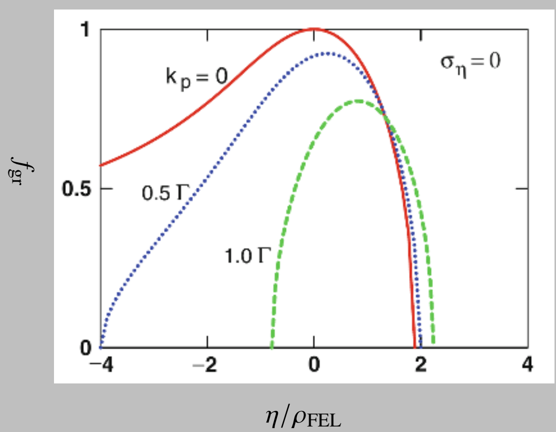


Figure 11: Growth rate as a function of the relative energy deviation for different values of the space-charge parameters. Figure taken from Schmüser et al. 2014

The influence of space charge on the amplification is relatively small. At FEL FLASH, for example, the space charge parameter has a value of about 0.2Γ, resulting in a change of the gain length of less than 1 %.

FEL growth rate, effect of energy spread

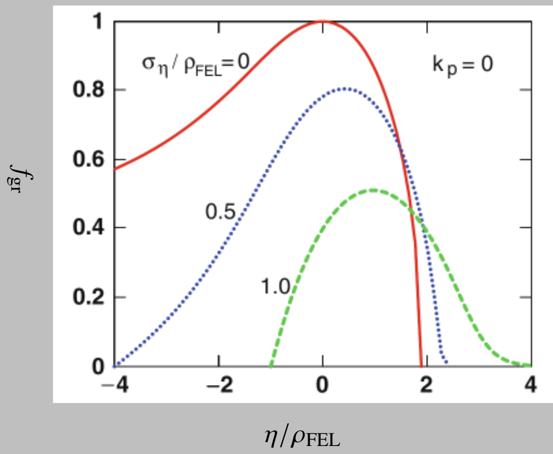


Figure 12: Growth rate as a function of the relative energy deviation for different energy spreads. Fig. from Schmüser et al. 2014

The influence of the energy spread is stronger. For $\sigma_\eta = 0.5\rho_{\mathrm{FEL}}$ the gain length increases by 25 %, for $\sigma_\eta = \rho_{\mathrm{FEL}}$ it increases by a factor of 2. $\sigma_\eta = 0.5\rho_{\mathrm{FEL}}$ is regarded as a reasonable upper limit for the energy distribution width in the FEL.

4.2 3D Effects

It is possible to extend the derivation of the 1D FEL equations to three dimensions to the extent that the dependence of the eigenvalues of the FEL equation on the transversal extent of the electron beam (assuming a cylindrical beam) can be described.

It turns out that the influence of the transverse beam expansion r_b scales with the parameter

$$w_m := \sqrt{L_{g0}\lambda_\gamma}. \tag{4.8}$$

The description within the 1D-FEL theory proves to be adequate if

$$r_b \gg w_m. \tag{4.9}$$

FEL growth rate, effect of transverse beam size

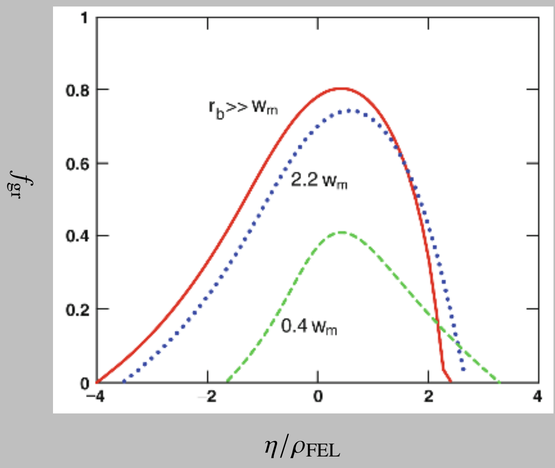


Figure 13: Growth rate as a function of the relative energy deviation for different transverse beam sizes. Figure taken from Schmüser et al. 2014

4.3 Overlap between electron beam and photon beam, emittance

An essential requirement for effective FEL amplification is a good overlap between the photon and electron beams.

This overlap is subject to various influences:

- mean particle beam size
- spatial deviation of the entire electron beam (e.g. due to field errors in the undulator)
- photon beam size (Gaussian optics), influenced e.g. by diffraction effects

As we know, the electron beam size and divergence are determined by the emittance and the (mean) beta function. However, the betatron oscillations of the electrons have yet another effect: they reduce the mean longitudinal drift velocity of the particles. This has the effect of FEL process like an additional effective energy spread.

An estimate of this effect leads to a practical upper limit for the acceptable emittance

$$\epsilon_{x,y} < \frac{\bar{\beta}_{x,y}}{2\sqrt{2}\gamma_r^2}\rho_{\mathrm{FEL}} \tag{4.10}$$

A large average betafunction reduces the beam divergence and thus also this effect. On the other hand, it leads to a transversely large beam and thus a reduced particle density. In fact, an optimum for the beta function can be found with regard to these conflicting effects.

4.4 Parameterisation of the gain length for X-ray FELs

We have seen that various 3D effects lead to an increase in the gain length compared to the 1D theory. In general, calculation of the gain length taking these effects into account, a suitable 3D simulation program must be used. Now, on the basis of such simulations, various parameterisations or scaling laws have been proposed that allow a reasonably correct estimation of the 3D effects. We present here the parameterisation by Ming Xie (Xie 2000).

Ming Xie's parameterisation is based on a round electron beam with

$$\beta_x = \beta_y = \bar{\beta}, \epsilon_x = \epsilon_y = \tilde{\epsilon} \quad \text{and} \quad \sigma_r = \sqrt{\bar{\beta}\tilde{\epsilon}} \tag{4.11}$$

and uses the following parameters:

$$\begin{aligned} X_\gamma &= \frac{L_{g0} 4\pi\sigma_\eta}{\lambda_\mathrm{u}} && \text{energy spread parameter} \\ X_d &= \frac{L_{g0}\lambda_\gamma}{4\pi\sigma_r^2} && \text{diffraction parameter} \\ X_{\tilde{\epsilon}} &= \frac{L_{g0}4\pi\tilde{\epsilon}}{\bar{\beta}\lambda_\gamma} && \text{Angular distribution parameters} \end{aligned} \tag{4.12}$$

The meaning of these parameters follows from the previous considerations regarding upper limits for energy distribution width and emittance. We have not explicitly treated the diffraction of the FEL radiation in the beam pipe. Our considerations lead to a condition for the ratio of the Rayleigh length of the FEL radiation to the gain length, which is included in the diffraction parameter. The conditions for effective FEL amplification mean that the three parameters must be lower than 1 in order to achieve FEL amplification.

The scaling law of Ming Xie now has the form

$$L_g = L_{g0}(1 + \Lambda) \tag{4.13}$$

with the parameterisation of the correction factor

$$\begin{aligned} \Lambda = a_1 X_d^{a_2} + a_3 X_{\tilde{\epsilon}}^{a_4} + a_5 X_\gamma^{a_6} + a_7 X_{\tilde{\epsilon}}^{a_8} X_\gamma^{a_9} + a_{10} X_d^{a_{11}} X_\gamma^{a_{12}} \\ + a_{13} X_d^{a_{14}} X_{\tilde{\epsilon}}^{a_{15}} + a_{16} X_d^{a_{17}} X_{\tilde{\epsilon}}^{a_{18}} X_\gamma^{a_{19}}. \end{aligned} \tag{4.14}$$

The parameterisation was determined numerically and yields

$$\begin{array}{llll} a_1 = 0.45, & a_2 = 0.57, & a_3 = 0.55, & a_4 = 1.6, \\ a_5 = 3.0, & a_6 = 2.0, & a_7 = 0.35, & a_8 = 2.9, \\ a_9 = 2.4, & a_{10} = 51, & a_{11} = 0.95, & a_{12} = 3.0, \\ a_{13} = 5.4, & a_{14} = 0.7, & a_{15} = 1.9, & a_{16} = 1140, \\ a_{17} = 2.2, & a_{18} = 2.9, & a_{19} = 3.2. & \end{array} \tag{4.15}$$

Example. FLASH For example, the 3D gain length for the beam parameters of FLASH is 32 % larger than the 1D gain length L_{g0}.

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Schmüser, Peter et al. (2014). *Free-Electron Lasers in the Ultraviolet and X-Ray Regime: Physical Principles, Experimental Results, Technical Realization*. 2nd ed. Springer Tracts in Modern Physics. Springer International Publishing. ISBN: 978-3-319-04080-6. URL: https://www.springer.com/de/book/9783319040806.

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