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Statistical mechanics for storage rings

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In this module, we introduce methods from statistical mechanics far from equilibrium with a view to their application in particular to the particle dynamics in storage rings.

In classical linear (and also non-linear) beam optics and dynamics we are usually concerned with the single-particle equations of motion in an electric and (piecewise constant) magnetic field. Explicitly or implicitly, we consider the whole as a conservative system and assume that the many particles of a bunch do not interact with each other. That is, the particle motion is described approximately by means of the canonical Hamiltonian motion equations, we find conserved quantities such as the emittance and interpret these with the help of Liouville's theorem as something that, in a sense, represents the particle ensemble as a whole.

The success of these descriptions suggests that they are, in some sense, justified, but this is far from obvious when considering the reality. While considering the perhaps 10^{11} particles in the bunch as independent of each other may appear reasonable due to their relativistic velocity, the dynamics of the individual particle is highly non-conservative and, moreover, far from thermodynamic equilibrium: energy is continuously supplied to the particles in the cavities, which they lose again by emitting synchrotron radiation. This process is dissipative and, due to the quantised emission, fluctuating.

The tools for describing such processes are provided by non-equilibrium statistical mechanics. We will see that, despite the fact that a stored particle beam can never reach a thermodynamic equilibrium, the averaging over many (independent) particles leads to distributions in stationary non-equilibrium states, which in some respects show similarities to equilibrium states and even justify the conservative description of particle dynamics in some ways.

In the following, we will focus on the example of electron beam dynamics under emission of synchrotron radiation. However, it should be noted that there are further stochastic phenomena in accelerators that also occur in hadron accelerators, such as the effects of a finite number of particles (Schottky noise), noise in the radio-frequency system or stochastic processes among the particles of a bunch (e.g. intra-beam scattering, IBS).

1 Fundamentals

1.1 The stochastic equations of motion

Our consideration starts from the following, for many dynamic physical systems representative

generalised equation of motion

$$\dot{X}(t) = K(X,t) + Q(X,t)\xi(t) \tag{1.1}$$

with

$$X(t) = (X_1, \cdots, X_N)$$

a tuple of generalised coordinates that microscopically completely represents the system.

Example. In a quasi-hamiltonian system, these would be the canonical coordinates and momenta

$$X(t) = (q_1, \cdots, q_n, p_1, \cdots, p_n), \quad N = 2n$$

In the following, we will think of the "system" initially as a particle with some degrees of freedom, for example a particle in a stored particle beam.

The generalised equation of motion 1.1 has two terms, namely

$$\begin{aligned} K(X,t) &: \text{Representation of the deterministic forces} \\ Q(X,t)\xi(t) &: \text{Representation of the fluctuating forces} \end{aligned}$$

with a normalised noise term $\xi(t)$ and the coupling function $Q(X,t)$. In this general form, it is therefore possible for the coupling to depend on the fluctuation of the coordinates X.

Let us consider these contributions in detail.

1.2 Deterministic systems

In the absence of the fluctuating term, the system's dynamics is completely determined by the coupled ordinary differential equations

$$\dot{X}(t) = K(X,t) \tag{1.2}$$

No further assumption is made here regarding stability under small variations in the initial conditions, i.e. deterministic chaos is in principle also represented in this equation.

If the solution of equation 1.2 is given by $X(t)$, then a completely equivalent description of the system's dynamics is possible using the exact phase space density distribution of this solution, the so-called Klimontovich distribution:

$$\mathcal{F}(X,t) = \delta\left(X - X(t)\right) = \prod_{i=1}^{N} \delta\left(X_i - X_i(t)\right) \tag{1.3}$$

Here X is our general variable and $X(t)$ is a solution of eq. 1.2 for given initial conditions $X(0)$.

The continuity equation for $\mathcal{F}$ can be written by applying the equations 1.2 and 1.3:

$$\begin{aligned} \frac{\mathrm{d}\mathcal{F}}{\mathrm{d}t} &= \partial_t \mathcal{F}(X,t) + \nabla_X \cdot \left(\dot{X}(t)\mathcal{F}(X,t)\right) \\ &= \partial_t \mathcal{F}(X,t) + \nabla_X \cdot \left(K(X,t)\mathcal{F}(X,t)\right) = 0 \end{aligned} \tag{1.4}$$

This is the Liouville equation.

The Liouville equation is not to be confused with the *theorem of* Liouville. This applies to a strictly Hamiltonian system with

$$X = (q_1, \cdots, q_n, p_1, \cdots, p_n) \tag{1.5}$$

and

$$K = \left(\frac{\partial H}{\partial p_1}, \cdots \frac{\partial H}{\partial p_n}, -\frac{\partial H}{\partial q_1}, \cdots, -\frac{\partial H}{\partial q_n}\right), \quad Q = 0 \tag{1.6}$$

with divergence-free flow in phase space

$$\nabla_X \cdot K = 0. \tag{1.7}$$

This leads to the following equation for this special case

$$\partial_t \mathcal{F} + K(q, p, t) \cdot \nabla_X \mathcal{F}(q, p, t) = 0, \tag{1.8}$$

and states: The phase space density of the solutions of the Hamiltonian equations of motion along their trajectories in phase space is constant. In Poisson bracket notation:

$$\partial_t \mathcal{F} = [H, \mathcal{F}] \,. \tag{1.9}$$

Remark. The difference between Liouville's theorem 1.8 and the more general Liouville equation 1.4 is that the latter also describes dissipative systems in which the local phase space density changes. However, the integral over the phase space density $\int \mathcal{F}(X, t)\mathrm{d}^N X$ remains unchanged in every system.

1.3 Fluctuating systems

When describing physical systems, we often reduce the problem to a few macroscopic observables and introduce random forces as a representation of the neglected microscopic degrees of freedom. A prominent example of this is the description of Brownian motion, which disregards the underlying molecular causes and represents them by a random force.

The following description assumes a single source of stochastically acting forces (it will be shown that this is sufficient for the electron beam dynamics with synchrotron radiation. The generalisation to several independent sources of stochastically acting forces, on the other hand, is relatively unproblematic).

For clarification:

As the realisation of $\xi(t)$ we denote one member of the ensemble of possible time courses of the stochastic process. The approach is usually to first develop a mathematical description of the process for one realisation and then to average this over all possible realisations with the corresponding weighting. The final results will therefore always describe expected values.

We make the following assumptions about the random source $\xi(t)$, which we thereby interpret as *Gaussian white noise*:

$$\langle \xi(t) \rangle = 0, \qquad \langle \xi(t)\xi(t') \rangle = \delta(t - t'), \tag{1.10}$$

with the averaging $\langle \cdot \rangle$ over the ensemble of possible temporal sequences. The first restrictive assumption is a relatively weak one, because a net effect of the stochastically acting force can be absorbed in the deterministic term of the motion equation. The second assumption of the non-existence of temporal correlations of the stochastic process is stronger. However, it will be shown that this is also particularly justified for the case of synchrotron radiation.

The delta distribution in the two-time correlation in Eq. **??** means that the underlying random process is treated as a Markov process, i.e. as an instantaneous process. This is an idealisation. Real processes will have a finite correlation time (transition time) τ with

$$\langle \xi_\mathrm{r}(t)\xi_\mathrm{r}(t')\rangle \sim \frac{1}{2\tau}\mathrm{e}^{-|t-t'|/\tau}. \tag{1.11}$$

Here, the subscript r denotes the real process. For the modelling of the stochastic process, we assume that the transition $\tau \to 0$ is allowed. This is justified if τ is shorter than all relevant time scales of the system.

1.4 The phase space density distribution in the Liouville equation

The complete Liouville equation (continuity equation for the Klimontovich distribution) for a given realisation of the stochastic equations of motion 1.1 and their solutions is

$$\partial_t \mathcal{F} + \nabla_X \cdot (K(X,t)\mathcal{F}(X,t)) + \nabla_X \cdot (Q(X,t)\xi(t)\mathcal{F}(X,t)) = 0 \tag{1.12}$$

This is completely equivalent to the set of stochastic motion equations 1.1.

Since $X(t)$ depends on the fluctuating term $\xi(t)$, $\mathcal{F}$ is also a fluctuating quantity. By averaging over the ensemble, $\mathcal{F}$ can be written as the sum of an expectation value and a fluctuating term that disappears on average:

$$\mathcal{F}(X,t) = F(X,t) + \hat{F}(X,t), \qquad F(X,t) := \langle \mathcal{F}(X,t)\rangle. \tag{1.13}$$

The solutions of the Liouville equation are, of course, initially just as difficult to find as those of the stochastic equations of motion. However, there is hope that for the expectation value of the phase space density function $F(X,t)$, which is ultimately what we are actually interested in, the situation will improve somewhat.

1.5 The Fokker-Planck equation

The Fokker-Planck equation describes the time evolution of the expectation value of the phase space density function under the effect of a deterministic drift process K and a diffusion process Q. It can be derived from the Liouville equation by averaging over the ensemble of possible courses of the diffusion process and some approximations.

Averaging the Liouville equation yields

$$\partial_t \langle \mathcal{F}\rangle + \nabla_X \cdot (K\langle \mathcal{F}\rangle) + \nabla_X \cdot (Q\langle \xi\hat{F}\rangle). \tag{1.14}$$

Here, the properties of the Gaussian white noise have already been used to simplify the diffusion term.

This equation contains the unknown correlation function $\langle \xi\hat{F}\rangle$. To eliminate it, we consider the fluctuating part of the un-averaged Liouville equation 1.12:

$$\partial_t \hat{F} + \nabla_X \cdot \left(K\hat{F}\right) + \nabla_X \cdot \left(Q\left(\xi\langle\mathcal{F}\rangle + \xi\hat{F} - \langle\xi\hat{F}\rangle\right)\right) = 0, \tag{1.15}$$

where the identity

$$\widehat{\xi\mathcal{F}} = \xi\mathcal{F} - \langle\xi\mathcal{F}\rangle = \xi\langle\mathcal{F}\rangle + \xi\hat{F} - \underbrace{\langle\xi\rangle}_{=0}\langle\mathcal{F}\rangle - \langle\xi\hat{F}\rangle \tag{1.16}$$

has been applied.

This proves

$$\xi\hat{F} - \langle\xi\hat{F}\rangle = \widehat{\xi\hat{F}} \tag{1.17}$$

to be the fluctuating part of a product of two fluctuating, on average vanishing functions.

Therefore, all fluctuating terms in Eq. 1.15 are neglected except for

$$\partial_t \hat{F} \approx -\nabla_X \left(Q\xi\langle\mathcal{F}\rangle\right). \tag{1.18}$$

This can be read as follows: the time evolution of the fluctuating part of the phase space distribution density on short time scales is determined by the mean phase space distribution density, which changes on longer time scales (determined by K).

Equation 1.18 can now be integrated over time and substituted into equation 1.14

$$\langle\xi\hat{F}\rangle = \int_{-\infty}^{t} \langle\xi\partial_t\hat{F}\rangle \mathrm{d}t' \tag{1.19}$$

$$\partial_t\langle\mathcal{F}\rangle + \nabla_X \cdot \left(K\langle\mathcal{F}\rangle\right) = \nabla_X \cdot \left(Q(X,t) \cdot \nabla_X \cdot \nabla_X \mathrm{d}t'\right) \tag{1.20}$$

and finally leads to

$$\partial_t F(X,t) + \nabla_X \cdot \left(K(X,t)F(X,t)\right) = \frac{1}{2}\nabla_X \cdot \left(Q(X,t)\nabla_X \cdot \left(Q(X,t)F(X,t)\right)\right), \tag{1.21}$$

or explicitly spelled out

$$\boxed{\partial_t F = -\sum_i \frac{\partial}{\partial X_i}\left(\left[K_i + \frac{1}{2}\sum_j \frac{\partial Q_j}{\partial X_j}\right]F\right) + \frac{1}{2}\sum_i\sum_j \frac{\partial}{\partial X_i}\frac{\partial}{\partial X_j}\left(Q_i Q_j F\right)}, \tag{1.22}$$

the FOKKER-PLANCK equation.

Remark. The second term within the square brackets describes the so-called “spurious” drift. It disappears if the diffusion processes, which are described by Q, are independent of amplitude, which is often the case.

Finally, let’s go one step further and write the FOKKER-PLANCK equation for a quasi-Hamiltonian system, i.e. a system whose essential properties can be described by Hamiltonian equations of motion – as is the case for our particle in the storage ring. The stochastic equations of motion can be written as follows:

$$\begin{aligned}\dot{q} &= \frac{\partial H(q,p,t)}{\partial p} + \epsilon\mathcal{K}_q(q,p,t) + \sqrt{\epsilon}Q_q(q,p,t)\xi(t),\\ \dot{p} &= -\frac{\partial H(q,p,t)}{\partial q} + \epsilon\mathcal{K}_p(q,p,t) + \sqrt{\epsilon}Q_p(q,p,t)\xi(t).\end{aligned} \tag{1.23}$$

with the Hamiltonian function H, the non-Hamiltonian deterministic drift $\mathcal{K}$ and a scaling factor ϵ. Here, the scaling by ϵ was chosen such that the dissipative terms cause an averaged effect proportional to $\epsilon\Delta t$ for a short time interval Δt.

The Fokker-Planck equation for the quasi-Hamiltonian system then reads

$$\begin{aligned}\partial_t F =& -\frac{\partial H}{\partial p_i}\frac{\partial F}{\partial q_i} + \frac{\partial H}{\partial q_i}\frac{\partial F}{\partial p_i}\\ &-\epsilon\frac{\partial}{\partial q_i}\left(\left[\mathcal{K}_{q_i} + \frac{1}{2}\frac{\partial Q_{q_i}}{\partial q_j}Q_{q_j} + \frac{1}{2}\frac{\partial Q_{q_i}}{\partial p_j}Q_{p_j}\right]F\right)\\ &-\epsilon\frac{\partial}{\partial p_i}\left(\left[\mathcal{K}_{p_i} + \frac{1}{2}\frac{\partial Q_{p_i}}{\partial q_j}Q_{q_j} + \frac{1}{2}\frac{\partial Q_{p_i}}{\partial p_j}Q_{p_j}\right]F\right)\\ &+\frac{\epsilon}{2}\frac{\partial}{\partial q_i}\frac{\partial}{\partial p_j}(Q_{q_i}Q_{q_j}F)\\ &+\epsilon\frac{\partial}{\partial q_i}\frac{\partial}{\partial p_j}(Q_{q_i}Q_{p_j}F)\\ &+\frac{\epsilon}{2}\frac{\partial}{\partial q_i}\frac{\partial}{\partial p_j}(Q_{p_i}Q_{p_j}F)\end{aligned} \tag{1.24}$$

with summation over identical indices. Here, the terms in the first line describe the incompressible flow in phase space according to the theorem of Liouville, the following two lines describe a possibly irreversible drift (i.e. dissipation, for example a damping), the last three lines describe diffusion processes due to fluctuations.

1.6 Notes on the solution of the Fokker-Planck equation

With the Fokker-Planck equation, we replace N stochastic, possibly coupled and nonlinear ordinary differential equations by a single partial differential equation in $N+1$ variables. The Fokker-Planck equation is always linear in the unknown phase space density function $F(X,t)$. However, it is not necessarily the case that this equation is easier to solve than the system of stochastic motion equations. In fact, closed-form analytical time-dependent solutions cannot usually be found.

More often, it is possible to find analytical *stationary* solutions for the Fokker-Planck equation under the assumption $\partial_t F = 0$.

In the special case of one phase space dimension and time-independent coefficients K and Q, a stationary solution of the form can always be found:

$$F_0(X) = \frac{|Q(X_0)|}{Z(\lambda)|Q(X)|}\exp\left(2\int_{X_0}^{X}\frac{K(X')}{Q(X')^2}dX'\right) \tag{1.25}$$

with X_0 a suitable point in the phase space and parametrisation by λ (induced by K and Q). The normalisation constant Z corresponds conceptually to the partition function as we know it from equilibrium statistical mechanics.

In fact,

$$\Phi(X;\lambda) = -\log\left(\frac{F_0(X)}{Z(\lambda)}\right) \tag{1.26}$$

can be understood as a generalised thermodynamic potential, and the distribution function formulated with it

$$F_0(X) = \frac{1}{Z(\lambda)}\exp\left(-\Phi(X;\lambda)\right) \tag{1.27}$$

is reminiscent of the canonical ensemble of statistical mechanics at equilibrium.

1.7 Stochastic Calculus

One final addition:

The stochastic equations of motion 1.1 can be formally integrated according to the rule

$$\begin{aligned} X(t) = X(0) &+ \int_0^t K(X,t')\mathrm{d}t' \\ &+ \frac{1}{2}\int_0 tQ(X,t')\cdot\nabla_X Q(X,t')\mathrm{d}t' \\ &+ \int_0^t Q(X,t')\xi(t')\mathrm{d}t'. \end{aligned} \tag{1.28}$$

We do not provide a proof here and refer to the relevant literature (Stratonovich 1967).

References

Jowett, John M. (1987). “Introductory Statistical Mechanics for Electron Storage Rings”. In: PHYSICS OF PARTICLE ACCELERATORS: SLAC Summer School, 1985 and Fermilab Summer School 1984. Stanford, CA and Batavia, IL USA, pp. 864–970. DOI: 10.1063/1.36374.

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2 On the Statistical Physics of Synchrotron Radiation

The emission of synchrotron radiation is a quantum-mechanical phenomenon. The emission time and the energy of each photon emitted are random quantities. With some justification, we have nevertheless classically obtained the mean emitted radiation power and thus the mean emission rate, and we may continue to assume that these results represent an excellent approximation, and we will use them in the following.

Remark. Based on the classical picture, two constitutive basic assumptions can be made for the following:

1. The time interval for the emission of a photon is certainly smaller than the time interval in which an observer can receive radiation from the emitting particle, so

$$\tau_\gamma \lesssim \frac{\rho}{\gamma c}, \tag{2.1}$$

   and this is considerably smaller than the characteristic time scales, which are given by the periods of the particle oscillations around the reference orbit (betatron and synchrotron oscillations). In this sense we may consider the emission to be *instantaneous*

2. The spectral distribution of the synchrotron radiation may be considered to slowly varying with respect to this time scale. In this sense, we may assume the spectral distribution to be locally well-defined .

These assumptions can be rigorously shown; for this, we refer to Sands 1969

2.1 Photon Flux and Emission Probability

To arrive at a description of synchrotron radiation as a stochastic process, we start from what we know about the instantaneous power and spectrum of the radiation emitted by a relativistic particle deflected in a transverse magnetic field $\vec{B}$.

The radiation power

$$P_\mathrm{S} = \frac{2}{3}\frac{r_0 e^2 c^3 B^2 \beta^2 \gamma^2}{mc^2}$$

with

$$r_0 = \frac{e^2}{4\pi\epsilon_0 mc^2}$$

we transform it again by writing it as a function of the magnitude of the 3-linear momentum,

$$P_\mathrm{S} = \frac{2}{3}\frac{r_0 e^2 p^2 B^2}{m^3 c},$$

and we normalise the magnetic field using the beam rigidity to the reference momentum according to

$$\vec{B} =: \frac{p_0}{e}\vec{b}. \tag{2.2}$$

$\vec{b}$, like all the other parameters related to synchrotron radiation, generally depends on the dynamical variables describing the state of the particle (X), as well as on the longitudinal coordinate s, which will play the role of the independent variable. We emphasise this in the following by indexing with X and by making the dependence on s explicit and obtain

$$P_{\mathrm{S}} = \frac{2}{3}\frac{r_0 p_0^2}{m^3 c} p^2 b_X^2(s) \tag{2.3}$$

and, correspondingly, for the critical frequency

$$\omega_c = \frac{3}{2}\frac{c p_0}{(mc)^3} p^2 b_X(s). \tag{2.4}$$

The probability for the emission of a photon of a certain energy is obviously related to the spectral functions of the synchrotron radiation. We had seen:

$$\frac{\mathrm{d}P}{\mathrm{d}\omega} = \frac{P_{\mathrm{S}}}{\omega_c} S_{\mathrm{S}}\left(\frac{\omega}{\omega_c}\right) \tag{2.5}$$

with

$$S_{\mathrm{S}}\left(\frac{\omega}{\omega_c}\right) = \frac{9\sqrt{3}}{8\pi}\frac{\omega}{\omega_c}\int_{\omega/\omega_c}^{\infty} K_{5/3}(z')\mathrm{d}z', \tag{2.6}$$

and ad hoc quantisation to obtain the spectral photon flux

$$\frac{\mathrm{d}\dot{n}_X(s)}{\mathrm{d}W_\gamma/W_\gamma} = \frac{P_{\mathrm{S}}}{W_{\gamma c}} S_{\mathrm{S}}\left(\frac{W_\gamma}{W_{\gamma c}}\right). \tag{2.7}$$

From this, the spectrally integrated photon flux, i.e. the number of photons per unit time, independent of their energy, can be calculated directly:

$$\begin{aligned} \dot{n}_X(s) &:= \int_0^{\infty} \frac{\mathrm{d}\dot{n}_X(s)}{\mathrm{d}W_\gamma/W_\gamma}\mathrm{d}W_\gamma/W_\gamma \\ &= \frac{5\sqrt{3}}{6}\frac{c r_0 p_0}{\hbar} b_X(s), \end{aligned} \tag{2.8}$$

which is independent of the momentum p.

With this quantity and the spectral flux, the relative probability for the emission of a photon of a certain energy can now be given:

$$\begin{aligned} f_X(W_\gamma; s) &= \frac{1}{\dot{n}_X(s)}\frac{\mathrm{d}\dot{n}_X(W_\gamma; s)}{\mathrm{d}W_\gamma/W_\gamma} \\ &= \frac{5\sqrt{3}}{8}\frac{(mc)^3}{\hbar c p_0}\frac{F(W_\gamma/W_{\gamma c})}{p^2 b_X(s)} \end{aligned} \tag{2.9}$$

with

$$F(W_\gamma/W_{\gamma c}) := \frac{S(W_\gamma/W_{\gamma c})}{W_\gamma/W_{\gamma c}} \tag{2.10}$$

the normalised spectral function.

The first two moments of this probability function will be of further interest.

With

$$\langle W_\gamma \rangle_{X;s} = \int_0^\infty W_\gamma f_X(W_\gamma; s) \mathrm{d}W_\gamma = \frac{4}{5\sqrt{3}} \frac{\hbar c p_0}{(mc)^3} p^2 b_X(s) = \frac{8}{15\sqrt{3}} W_{\gamma c} \tag{2.11}$$

the expectation value of the radiation power can be written as

$$\langle P_X(s) \rangle = \dot{n}_X(s) \langle W_\gamma \rangle_{X;s}. \tag{2.12}$$

The second moment

$$\langle W_\gamma^2 \rangle_{X;s} = \frac{11}{12} \frac{(\hbar c p_0)^2}{(mc)^6} p^4 b_X(s)^2 \tag{2.13}$$

will, as we shall see, describe, together with the emission rate, the quantum fluctuation of the particle motion.

2.2 Synchrotron radiation as a stochastic process

With this knowledge about the photon energy distribution of the emitted photons, we can start to formulate the corresponding expressions that we need to derive the stochastic equations of motion and the Klimontovich phase space distribution density.

An emission event in which a photon of energy $W_{\gamma j}$ is emitted at an orbital point s_j is described by the pair of random variables $(W_{\gamma j}, s_j)$. The associated distribution function depends only on the local magnetic field and the particle momentum, both of which are functions of s and time t, respectively.

Remark. To clarify: When we talk about expectation values in the following, this is always to be understood in the sense of an average over all realisations of $(W_{\gamma j}, s_j)$ for a given particle state X at the orbit point s with correct weighting (i.e. in particular not as a temporal average).

Analogous to the Klimontovich distribution, the distribution density of a given realisation (for a given X) in (W_γ, s) space can be written as

$$\Omega_X(W_\gamma, s) = \sum_j \delta(s - s_j)\delta(W_\gamma - W_{\gamma j}) \tag{2.14}$$

with the sum over all events that take place in this realisation.

The expectation value of this distribution is the probability density function for $(W_{\gamma j}, s_j)$:

$$\langle \Omega_X \rangle = \dot{n}_X(s) f_X(W_\gamma; s)/c \tag{2.15}$$

The instantaneous radiated power in this case is

$$\begin{aligned} P_X(s) &= c \int_0^\infty W_\gamma \mathrm{d}W_\gamma \int_{-\infty}^\infty \mathrm{d}s' \Omega_X(W_\gamma, s') \Delta(s - s') \\ &= c \sum_j W_{\gamma j} \Delta(s - s_j). \end{aligned} \tag{2.16}$$

Here, $\Delta(s)$ is a function that describes the shape of the photon emission pulse, with the properties

$$\int_{-\infty}^{\infty} \Delta(s)\mathrm{d}s = 1, \qquad \int_{-\infty}^{\infty} \Delta(s)^2\mathrm{d}s = \frac{1}{c\tau_\gamma} \tag{2.17}$$

Remark (Temporal distribution of the emission probability). Within the framework of this description of synchrotron radiation as a stochastic process, it can be shown that the probability $Pr(n, s)$ for n photons to be emitted in a time interval $[0; t]$, here parameterised by the orbit interval $[0, s(t)]$, follows a Poisson distribution:

$$Pr(n, s) = \frac{1}{n!}\left(\frac{\dot{n}_X s}{c}\right)^n \exp\left(-\frac{\dot{n}_X s}{c}\right) \tag{2.18}$$

2.3 Campbell's theorem

The instantaneous emitted power (Eq. 2.16) can be written as the sum of the average emitted power and the fluctuating component

$$P_X(s) = \langle P_X(s)\rangle + \hat{P}_X(s) \tag{2.19}$$

Theorem 2.1 (Generalised Campbell's Theorem)**.** For the two-time correlation of the fluctuating part of the radiation power, the following applies

$$\langle \hat{P}_X(s)\hat{P}_X(s')\rangle = c\dot{n}_X(s)\langle W_\gamma^2\rangle_{X;s}\Delta(s - s'). \tag{2.20}$$

The proof is possible by substituting and explicitly performing the ensemble averages. The interested listener is referred to Jowett 1987.

To carry out specific calculations, it is useful to consider the limit for $\tau_\gamma \to 0$ in which

$$\begin{aligned}\langle \hat{P}_X(s)\hat{P}_X(s')\rangle &= c\dot{n}_X(s)\langle W_\gamma^2\rangle_{X;s}\delta(s - s') \\ &= \frac{55}{24\sqrt{3}}\frac{r_0\hbar c^4 p_0^3}{(mc)^6}p^4 b_X(s)^3\delta(s - s')\end{aligned} \tag{2.21}$$

.

2.4 The fluctuating radiation power

This allows us to express the result of this chapter, an expression for the fluctuating radiation power as a function of a Gaussian noise source:

$$P_X(s) = p^2c^2\left(c_1 b_X(s)^2 + \sqrt{c_2}|b_X(s)|^{3/2}\xi(s)\right) \tag{2.22}$$

with

$$c_1 := \frac{2r_0 p_0^2}{3(mc)^3}, \qquad c_2 := \frac{55 r_0\hbar p_0^3}{24\sqrt{3}(mc)^6}. \tag{2.23}$$

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Jowett, John M. (1987). "Introductory Statistical Mechanics for Electron Storage Rings". In: PHYSICS OF PARTICLE ACCELERATORS: SLAC Summer School, 1985 and Fermilab Summer School 1984. Stanford, CA and Batavia, IL USA, pp. 864–970. DOI: 10.1063/1.36374.

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3 Beam dynamics with synchrotron radiation

The treatment of beam dynamics under the effect of synchrotron radiation starts with the description of beam dynamics using the tools of Hamiltonian mechanics. This is to be summarised in the following (very briefly) before we add to the Hamiltonian equations of motion the dissipative and fluctuating terms that arise from the physics of synchrotron radiation emission, and which are then incorporated into the associated Fokker-Planck equation.

3.1 Coordinate system and Hamiltonian function

We have already become familiar with the curvilinear coordinate system (x, y, s) for describing beam dynamics, the so-called COURANT-SNYDER coordinate system:

COURANT-SNYDER coordinate system

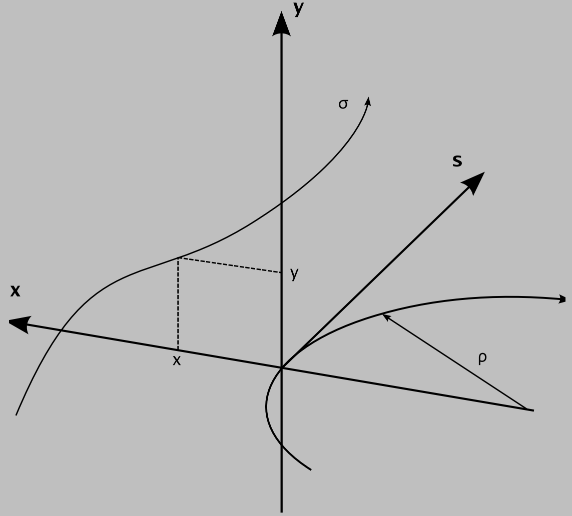

Figure 1: Representation of the moving curvilinear coordinate system for describing particle trajectories with respect to the reference orbit

Let us recall the lecture on Hamiltonian mechanics. The Hamiltonian function for a charged particle in an electromagnetic field in Cartesian coordinates is

$$H = e\Phi + c\sqrt{m^2c^2 + (\vec{p} - e\vec{A})^2} \tag{3.1}$$

Note that the momentum canonically conjugated with the Cartesian coordinates is not the linear momentum but

$$\vec{p} = \vec{p}_L + e\vec{A} \tag{3.2}$$

with the linear momentum $\vec{p}_L$, and the vector potential of the magnetic field $\vec{A}$. Φ is the scalar potential of the electric field.

Let us assume that the magnetic field consists of piecewise constant vertical multipoles along s, in particular, there are no fields deflecting in the vertical direction, and there exists a closed planar reference orbit $\vec{r}_0(s)$ for a hypothetical reference particle with constant reference momentum p_0.

Then the magnetic fields (in Coulomb gauge) can be derived from a scalar function, the *canonical vector potential*

$$\begin{aligned} A_s &= \vec{A} \cdot \vec{e}_s(1 + \kappa_x(s)x) \\ &= \frac{p_0}{e}\left(\kappa_x x + (\kappa_x^2 + k)\frac{x^2}{2} - \frac{ky^2}{2} + O(3)\right) \end{aligned} \tag{3.3}$$

with $\kappa_x(s) = 1/\rho(s)$ the curvature of the orbit. In fact, A_s is the longitudinal component of the vector potential. It follows from the basic assumption of cylinder symmetry of the magnetic field with respect to the reference orbit (from which the above assumption of piecewise constant multipole fields also follows) that only this component of the vector potential is different from zero.

In the second step of the equation, the potential was developed into the multipole potentials and only written explicitly up to the quadrupole term.

For the electric potential, we write

$$\Phi(s) = \sum_k \frac{\hat{V}_k}{\omega_{\mathrm{rf}}}\delta_C(s - s_k)\cos(\omega_{\mathrm{rf}}t + \phi_k). \tag{3.4}$$

Here we assume that the electric field is generated by a set of "thin lens cavities" along s with peak voltages $\hat{V}_k$ and radio frequency ω_{rf}. The delta distribution is periodic with period C, the circumference of the storage ring.

We obtain the (first) Hamiltonian function for the beam dynamics in the storage ring by a canonical transformation of the Hamiltonian function Eq. 3.1 into our curvilinear coordinate system and the simultaneous transition to s as an independent variable (whereby t becomes a canonical coordinate and $-W_e$ the corresponding canonical momentum). This is done by means of a generating function, which can be read about in the original publication by Courant and Snyder (Courant and Snyder 1958).

The Hamiltonian for a particle with kinematic momentum $\vec{p}$ is

$$H_1(x, y, t, p_x, p_y, -W_e) = -e(A_s + \Phi) - (1 + \kappa_x x)\sqrt{\frac{W_e^2}{c^2} - m^2c^2 - p_x^2 - p_y^2}. \tag{3.5}$$

For exercise the reader should make sure that by insertion of the canonical vector potential, Taylor expansion of the root to the linear term, neglecting the electric potential and differentiating Hamilton's equations of motion for the transversal beam dynamics are obtained!

For our further purposes, we carry out another canonical transformation to avoid complications:

$$(t, -W_e) \mapsto (z_t, p), \tag{3.6}$$

i.e. we go from energy to the absolute value of the total kinematic momentum and to the corresponding canonical coordinate z_t. This transformation is done by means of the generating function

$$F_2(p, t) = -ct\sqrt{p^2 + m^2c^2} \tag{3.7}$$

and leads to

$$H_2(x, y, z_t, p_x, p_y, p) = -e(A_s + \Phi) - (1 + \kappa_x x)\sqrt{p^2 - p_x^2 - p_y^2} \tag{3.8}$$

with the new canonical coordinates expressed as a function of the old ones

$$\begin{aligned} p &= \sqrt{\frac{E^2}{c^2} - m^2c^2} \\ z_t &= -ct\sqrt{1 - \frac{m^2c^4}{W_e^2}} \end{aligned} \tag{3.9}$$

For the Hamiltonian equations of motion, we thus obtain

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}s} &= (1 + \kappa_x x)\frac{p_x}{\sqrt{p^2 - p_x^2 - p_y^2}} \approx (1 + \kappa_x x)\frac{p_x}{p} \\ \frac{\mathrm{d}y}{\mathrm{d}s} &= (1 + \kappa_x x)\frac{p_y}{\sqrt{p^2 - p_x^2 - p_y^2}} \approx (1 + \kappa_x x)\frac{p_y}{p} \\ \frac{\mathrm{d}z_t}{\mathrm{d}s} &= -(1 + \kappa_x x)\frac{p}{\sqrt{p^2 - p_x^2 - p_y^2}} \approx -(1 + \kappa_x x) \\ \frac{\mathrm{d}p_x}{\mathrm{d}s} &= -\kappa_x(p - p_0) - p_0(\kappa_x^2 + k)x + O(2) \\ \frac{\mathrm{d}p_y}{\mathrm{d}s} &= p_0 k y + O(2) \\ \frac{\mathrm{d}p}{\mathrm{d}s} &= -e\sum_k \frac{\hat{V}_k}{c}\delta_C(s - s_k)\sin(\omega_{\mathrm{rf}} z_t/c + \phi_k). \end{aligned} \tag{3.10}$$

3.2 Momentum changes due to radiation

So far, we have described the strictly Hamiltonian (conservative, symplectic) system of a particle orbiting in a storage ring with a given magnetic lattice. We now add non-conservative forces to this description that occur randomly at certain times t_j and/or at certain orbit positions s_j, resulting from the emission of a synchrotron radiation photon of energy $W_{\gamma j}$. We consider the emission time τ_γ to be short compared to all essential time scales of the beam dynamics and can thus treat the effect of the emission as an instantaneous change of the particle momentum. The particle's position remains unchanged during the emission process.

The change in canonical momentum due to the emission of a photon is

$$\begin{aligned} \Delta p &= -W_{\gamma j}/c \\ \Delta p_x &= -\frac{W_{\gamma j}}{c}\frac{p_x}{p} = \frac{W_{\gamma j}}{c}\frac{x'}{\sqrt{1 - x'^2 - y'^2}} = -\frac{W_{\gamma j}}{c^2}\frac{x'}{t'} \\ \Delta p_y &= -\frac{W_{\gamma j}}{c^2}\frac{y'}{t'}. \end{aligned} \tag{3.11}$$

Here and in the following, the prime denotes the derivative with respect to s.

Let us consider a time interval around the emission, short enough to be able to neglect the probability of two photons being emitted in this interval, then

$$W_{\gamma j} = \int_{s_j-\sigma}^{s_j+\sigma} P_X(s)\mathrm{d}s/c \tag{3.12}$$

and consequently

$$\begin{aligned} \mathrm{d}p &= -P_X(s)\mathrm{d}t/c = -P_X(s)t'\mathrm{d}s/c = -P_X(s)z_t'\mathrm{d}s/c^2 + O(\gamma^{-2}) \\ \mathrm{d}p_x &= -P_X(s)x'\mathrm{d}s/c^2 \\ \mathrm{d}p_y &= -P_X(s)y'\mathrm{d}s/c^2 \end{aligned} \tag{3.13}$$

We obtain the complete equations of motion by adding these terms to the canonical Hamiltonian equations of motion:

$$\begin{aligned} x' &= \frac{\partial H_2}{\partial p_x}, & p_x' &= -\frac{\partial H_2}{\partial x} - \frac{P_X(s)}{c^2}\frac{\partial H_2}{\partial p_x}, \\ y' &= \frac{\partial H_2}{\partial p_y}, & p_y' &= -\frac{\partial H_2}{\partial y} - \frac{P_X(s)}{c^2}\frac{\partial H_2}{\partial p_y}, \\ z_t' &= \frac{\partial H_2}{\partial p}, & p' &= -\frac{\partial H_2}{\partial z_t} - \frac{P_X(s)}{c^2}\frac{\partial H_2}{\partial p}. \end{aligned} \tag{3.14}$$

Once again we explicate the equations for the canonical momenta to

$$\begin{aligned} p_x' &= -\frac{\partial H_2}{\partial x} + (1+\kappa_x x)pp_x\left(c_1 b(x,y,s)^2 + \sqrt{c_2}b(x,y,s)^{3/2}\xi(s)\right) \\ &=: -\frac{\partial H_2}{\partial x} + \frac{p_0}{c}\Pi_x, \\ p_y' &= -\frac{\partial H_2}{\partial y} + (1+\kappa_x x)pp_y\left(c_1 b(x,y,s)^2 + \sqrt{c_2}b(x,y,s)^{3/2}\xi(s)\right) \\ &=: -\frac{\partial H_2}{\partial y} + \frac{p_0}{c}\Pi_y, \\ p' &= -\frac{\partial H_2}{\partial z_t} - (1+\kappa_x x)p^2\left(c_1 b(x,y,s)^2 + \sqrt{c_2}b(x,y,s)^{3/2}\xi(s)\right) \\ &=: -\frac{\partial H_2}{\partial z_t} - \frac{p_0}{c}\Pi_t, \end{aligned} \tag{3.15}$$

with the radiative coupling functions Π_x, Π_y, Π_t defined by these equations.

3.3 Robinson's Damping Theorem

The quasi-Hamiltonian stochastic equations of motion 3.14 can be read as equations that describe a flow in phase space. Without the radiation terms, we would have the canonical equations, and the flow would be incompressible (Liouville's theorem), phase space volume and phase space density would be invariant. They would be so regardless of the choice of canonical variables.

With the radiation terms, we introduced a possible temporal variability of these and other invariants of the Hamiltonian system. The rate of change of each of these invariants must be calculable in every set of canonical variables of the underlying Hamiltonian system.

Let us calculate the total dissipation rate, i.e. the rate of change of the phase space volume, or, in other words, the divergence of the phase space flow.

Let us interpret the equations of motion 3.14 as a (hydrodynamic) equation for the local flow velocity in phase space in the form

$$X' = V(X). \tag{3.16}$$

For the phase space divergence we get

$$\nabla_X \cdot V = \frac{P_X(s)}{c^2}\left(\frac{\partial^2 H}{\partial p_x^2} + \frac{\partial^2 H}{\partial p_y^2} + \frac{\partial^2 H}{\partial p^2}\right) + \frac{2P_X(s)}{c^2 p}\frac{\partial H}{\partial p}. \tag{3.17}$$

Here we have exploited the fact that, due to LIOUVILLE's theorem, all contributions not containing P_X add up to zero, and the fact that P_X is proportional to p^2 and independent of p_x, p_y.

For the complete derivation, the Hamiltonian function 3.8 is to be inserted and differentiated. In doing so, the terms that contain the vector potential disappear under the assumption that there are no places on the orbit for which the Hamiltonian is explicitly time-dependent (as in the cavities, for example) and at the same time b_X does not disappear.

The derivative then leads to

$$\nabla_X V = -(1 + \kappa_x(s)x)\frac{4P_X(s)}{c^2\sqrt{p^2 - p_x^2 - p_y^2}}. \tag{3.18}$$

With

$$t' = (1 + \kappa_x x)\frac{W_e/c^2}{\sqrt{p^2 - p_x^2 - p_y^2}} \tag{3.19}$$

from Eq. 3.5 and the usual definition of the total damping increment as $-1/2\times$ the rate of change of the phase space volume, we obtain

$$\boxed{\alpha_{\text{tot}} = \frac{2P_X(s)}{W_e},} \tag{3.20}$$

the (generalised) ROBINSON Theorem.

Remark. Due to the symplectic structure of the underlying phase space, the total damping increment calculated here is invariant under canonical transformations into new canonical phase space coordinates and rescaling that preserve the Hamiltonian form of the equations of motion.

In particular, if transformed into a set of canonical variables that describe the natural oscillations of the particles around an equilibrium orbit, then ROBINSON's theorem must apply to the sum of the damping decrements of these natural oscillations.

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4 Closed orbits and eigenmodes

4.1 Dispersion function and closed orbit

We know the eigenmodes of vibration mentioned at the end of the last section. These are the transverse betatron and longitudinal synchrotron oscillations around — ideally — the reference orbit. In the real case, the betatron and synchrotron oscillations are about the dispersion orbit to the mean momentum p_s, i.e. to the momentum of the (fictitious) synchronous particle, which, as we have learned, results from the real deflection fields and the real frequency of the accelerating cavities.

In fact, this can be calculated within the framework of the Hamiltonian description of beam dynamics given here. We will refrain from doing so here. We assume that we know it. From the lecture on the dispersion function we know that there is a uniquely defined closed orbit for a particle of given deviation from the reference momentum p_0, which is described by the dispersion function.

The dispersion function itself, however, is parametrically dependent on the reference momentum.

We therefore understand the dispersion function in the following

$$\eta = \eta(s) = \eta(\delta_s, s) \quad \text{with} \quad \delta_s = \frac{p_s - p_0}{p_0}. \tag{4.1}$$

The corresponding dispersion orbit is

$$x_\delta(s) = \eta(s)\delta_s. \tag{4.2}$$

If the particle also has an individual relative momentum deviation from the mean momentum $\epsilon = \frac{p-p_s}{p_0}$, then the total displacement is

$$x(s) = \eta(\delta_s, s) \cdot (\delta_s + \epsilon) + x_\beta(s), \tag{4.3}$$

where $x_\beta(s)$ is the deviation due to the betatron oscillation of the particle around its own dispersion orbit, i.e. the orbit associated with the relative momentum deviation $(\delta_s + \epsilon)$.

What is important for longitudinal beam dynamics (as we have seen from the example of phase focusing) is the fact that the circumference C of the dispersion orbit depends on the momentum deviation. This dependence is described by the *momentum compaction factor* α_c, which in turn is parametrically dependent on the reference momentum:

$$\frac{1}{C}\frac{\mathrm{d}C}{\mathrm{d}\delta_s} =: \alpha_c(\delta_s) \tag{4.4}$$

4.2 Hamiltonian for synchrotron and betatron oscillations

Further procedure: To evaluate the effect of synchrotron radiation on the eigenmodes of the particle motion, the dynamical variables of the equations of motion 3.14 and the underlying hamilton function are canonically transformed

1. by normalisation of H and the momentum coordinates to the mean momentum

$$H, (p_x, p_y, p) \mapsto H/p_s, (P_x = p_x/p_s, P_y = p_y/p_s, P = p/p_s) \tag{4.5}$$

   (rendering all momentum quantities dimensionless)

2. to the individual dispersion orbit

$$(x, y, z_t, P_x, P_y, P) \mapsto (x_\beta, y, z, p_\beta, p_y, \epsilon) \tag{4.6}$$

3. in action-angle variables (transversal coordinates)

$$(x_\beta, y, p_\beta, p_y) \mapsto (\psi_x, \psi_y, I_x, I_y) \tag{4.7}$$

Developing the root term in Eq. 3.8 to the linear term and applying the so-called *smooth approximation* for the electric potential

$$\Phi = \frac{e\hat{V}}{2\pi p_s c h} \cos\left(\frac{h(z - z_s)}{R_\delta}\right) \tag{4.8}$$

with R_δ the mean radius of the dispersion orbit to the mean momentum, hz_s/R_δ the (constant) phase of the (fictitious) synchronous particle and $\hat{V} = \sum_k \hat{V}_k$, results in the first step

$$\boxed{\begin{aligned} H_4(x_\beta, y, z, p_\beta, p_y, \epsilon; s) = &- \frac{\alpha_c \epsilon^2}{2} - \frac{e\hat{V}}{2\pi p_s c h} \cos\left(\frac{h(z - z_s)}{R_\delta}\right) \\ &+ \frac{p_\beta^2 + p_y^2}{2} + \frac{1}{2}(K_x x_\beta^2 + K_y y^2) \\ &+ O(3). \end{aligned}} \tag{4.9}$$

with $K_x := \frac{(\kappa_x^2 + k)}{1+\delta_s}$ and $K_y := -\frac{k}{1+\delta_s}$. *Smooth approximation* means effectively that the discrete cavities of any number and placement but with the appropriate phase relation among each other, with which we started, are replaced by h periods of a standing wave along the equilibrium dispersion orbit. h here stands for the harmonic number, i.e. the integer ratio of the radio frequency and the orbital frequency.

When setting up the stochastic equations of motion, it must be noted that the radiation coupling functions must also be transformed. Since the transformation to the dispersion orbit is linear, this is relatively simple, but leads to an important consequence for the equations of motion in (x_β, p_β):

$$x'_\beta = \frac{\partial H_4}{\partial p_\beta} + \frac{p_0}{p_s c}\eta \Pi_t, \qquad p'_\beta = -\frac{\partial H_4}{\partial x_\beta} + \frac{p_0}{p_s c}(\Pi_x + \Pi_t \eta') \tag{4.10}$$

with η' the canonically conjugated function to η. The emission of synchrotron radiation in these coordinates leads to an instantaneous change of the position variable, because the change of the momentum changes the dispersion orbit and the same position (in Cartesian coordinates) corresponds to a different position x_β of the reference orbit.

If the periodic beta function $\beta(s)$ (and with it the optical functions $\alpha(s)$, $\gamma(s)$, $\psi(s)$) are known, then the transversal variables and the transversal part of the Hamiltonian function can be further canonically transformed into action-angle variables using the generating function

$$\begin{aligned} S(\psi_x, x_\beta, \psi_y, y) = &- \frac{x_\beta^2}{2\beta_x(s)}(\alpha_x(s) + \tan\phi_x(\psi_x, s)) \\ &- \frac{y^2}{2\beta_y(s)}(\alpha_y(s) + \tan\phi_y(\psi_y, s)) \end{aligned} \tag{4.11}$$

with the phase functions

$$\phi_{x,y}(\psi_{x,y}, s) := \psi_{x,y} + \int_0^s \left(\frac{1}{\beta_{x,y}(s')} - \frac{\nu_{x,y}}{R}\right) \mathrm{d}s' \tag{4.12}$$

with the mean radius of the reference orbit R and the betatron tunes $\nu_{x,y} = \frac{1}{2\pi}\oint_C \frac{\mathrm{d}s}{\beta_{x,y}(s)}$.

The canonical transformation leads to

$$\begin{aligned} u &= \sqrt{2\beta_u I_u}\cos\phi_u \\ p_u &= -\sqrt{\frac{2I_u}{\beta_u}}(\alpha_u\cos\phi_u + \sin\phi_u) \end{aligned} \tag{4.13}$$

for $u = x_\beta, y$, and

$$H_{\mathrm{tr}}(\psi_x, \psi_y, I_x, I_y) = \nu_x I_x/R + \nu_y I_y/R. \tag{4.14}$$

Remark. The Hamiltonian in action-angle variables does not depend on the angle variables, i.e. these are cyclic and the actions (in the conservative system) are therefore constants of motion. In fact, these are the Courant-Snyder invariants!

When transforming into action-angle variables, the non-canonical terms must also be transformed in order to set up the stochastic equations of motion. The rules for doing this can be found in Jowett 1987 and will not be explained here.

The overall result for the stochastic equations of motion is

$$
\begin{aligned}
\psi_x' &= \frac{\nu_x}{\rho} - \frac{1}{\sqrt{2\beta_x I_x}} \left((\beta_x \eta' + \alpha_x \eta)\cos\phi_x + \eta \sin\phi_x\right) \frac{p_0 \Pi_t}{p_s c} - \sqrt{\frac{\beta_x}{2I_x}} \cos\phi_x \frac{p_0 \Pi_x}{p_s c}, \\
I_x' &= -\sqrt{\frac{2I_x}{\beta_x}} \left((\beta_x \eta' + \alpha_x \eta)\sin\phi_x + \eta \cos\phi_x\right) \frac{p_0 \Pi_t}{p_s c} - \sqrt{2\beta_x I_x} \sin\phi_x \frac{p_0 \Pi_x}{p_s c}, \\
\psi_y' &= \frac{\nu_y}{\rho} - \sqrt{\frac{\beta_y}{2I_y}} \cos\phi_y \frac{p_0 \Pi_y}{p_s c}, \\
I_y' &= -\sqrt{2\beta_y I_y} \sin\phi_x \frac{p_0 \Pi_y}{p_s c}, \\
\epsilon' &= \frac{e\hat{V}}{2\pi p_s c R_\delta} \cos\left(\frac{h(z - z_s)}{R_\delta}\right) - \frac{p_0 \Pi_t}{p_s c} \\
z' &= -\alpha_c \epsilon - \eta \frac{p_0 \Pi_x}{p_s c} - \eta\eta' \frac{p_0 \Pi_t}{p_s c}.
\end{aligned}
\tag{4.15}
$$

References

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5 Radiation damping

The equations of motion 4.15 show us immediately: damping and quantum fluctuations due to the emission of synchrotron radiation occur in all three eigenmodes of motion. In addition, the effects couple the longitudinal phase space variables to the phase space of the horizontal betatron oscillations via the dispersion.

In this section, we look at the radiation damping, which is a consequence of the *mean* effect of the radiation on the particle dynamics, i.e. the deterministic dissipative part of the stochastic equations of motion.

When dealing with the stochastic equations of motion and also the Fokker-Planck equations, a series of averaging integrals arise, for which the name *synchrotron radiation integral* and the following notational convention has been adopted (there are others; we mention the ones most important in our context):

definition	influences
$I_1 = \oint_C \kappa\eta \mathrm{d}s$	Momentum compaction factor
$I_2 = \oint_C \kappa^2 \mathrm{d}s$	energy loss, energy spread, damping time, emittance, damping partition numbers
$I_3 = \oint_C \lvert\kappa^3\rvert \mathrm{d}s$	energy spread, polarisation time, polarisation level
$I_{3a} = \oint_C \kappa^3 \mathrm{d}s$	polarisation level
$I_4 = \oint_C (\kappa^2 + 2k)\kappa\eta \mathrm{d}s$	energy spread, emittance, damping partition numbers
$I_5 = \oint_C \lvert\kappa\rvert^3 \mathcal{H} \mathrm{d}s$	emittance

with the dispersion invariant

$$\mathcal{H} := \left(\gamma_x \eta^2 + 2\alpha_x \eta\eta' + \beta_x \eta'^2\right). \tag{5.1}$$

If we now look at the equations of motion for the longitudinal and the two transverse modes of particle motion in detail and assume assume that the linearisation of all oscillations is permissible (which is approximately the case here), then the damping decrement can be determined directly from the equations of motion for each mode.

They are

$$\begin{aligned}
\alpha_x &= \frac{1}{2}\frac{\langle\langle P_X\rangle_X\rangle_C}{W_s}\left(1 - \frac{I_4}{I_2}\right) &&=: \frac{1}{2}\frac{\langle\langle P_X\rangle_X\rangle_C}{W_s} J_x \\
\alpha_y &= \frac{1}{2}\frac{\langle\langle P_X\rangle_X\rangle_C}{W_s} &&=: \frac{1}{2}\frac{\langle\langle P_X\rangle_X\rangle_C}{W_s} J_y \\
\alpha_\epsilon &= \frac{1}{2}\frac{\langle\langle P_X\rangle_X\rangle_C}{W_s}\left(2 + \frac{I_4}{I_2}\right) &&=: \frac{1}{2}\frac{\langle\langle P_X\rangle_X\rangle_C}{W_s} J_\epsilon
\end{aligned} \tag{5.2}$$

with the damping partition numbers

$$\begin{aligned} J_x(\delta_s) &= 1 - \frac{I_4}{I_2} \\ J_y &= 1 \\ J_\epsilon(\delta_s) &= 2 + \frac{I_4}{I_2}. \end{aligned} \tag{5.3}$$

With the *damping partition numbers*, Robinson's damping theorem writes itself

$$\sum_i J_i = 4. \tag{5.4}$$

The damping partition numbers depend on the mean momentum p_s. This allows the damping of the longitudinal and radial vibration components to be redistributed within certain limits, for example by changing the radio frequency .

It can be seen that for $I_4/I_2 > 1$ the damping partition number and thus the damping decrement for the radial betatron oscillation can become negative, i.e. the oscillation is anti-damped. In storage rings with separated multipole magnets, this does not normally occur. In storage rings with combined function magnets, however, it is quite possible.

References

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6 Quantum fluctuations and Fokker-Planck equations

We now take the fluctuating part of the radiation power into the consideration. This fluctuation leads to the excitation of synchrotron and betatron oscillations due to the instantaneous momentum changes, and is therefore fundamentally destabilising. The damping effects discussed in the last section counteract this destabilisation. It should be noted that excitation and damping have the same physical origin and that one cannot be had without the other.

6.1 Longitudinal phase space

The stochastic equations of motion (ignoring terms of higher order in δ_s) are

$$\begin{aligned} z' &= K_z + Q_z\xi(s) &&= -\alpha_c\epsilon \\ \epsilon' &= K_\epsilon + Q_\epsilon\xi(s) &&= \left[\left(\frac{\Omega_s}{c}\right)^2 z - \frac{J_\epsilon\alpha_\epsilon}{c}\epsilon\right] - p_0\sqrt{c_2\kappa_x^3}\xi(s) \end{aligned} \tag{6.1}$$

with the frequency of the (damped) synchrotron oscillation Ω_s

This can be used to construct the associated Fokker-Planck equation

$$\frac{\partial F}{\partial s} = \alpha_c\epsilon\frac{\partial F}{\partial z} - \left(\frac{\Omega_s}{c}\right)^2 z\frac{\partial F}{\partial\epsilon} + \frac{J_\epsilon\alpha_\epsilon}{c}\frac{\partial}{\partial\epsilon}(\epsilon F) + \frac{c_2p_0^2I_3}{4\pi R}\frac{\partial^2 F}{\partial\epsilon^2} \tag{6.2}$$

This equation can be solved completely with the help of GREEN's function or by expansion in eigenfunctions.

However, we only want to consider the stationary solution here, i.e. (in a not strictly understood sense) the equilibrium energy distribution.

We obtain the stationary solution with the assumption that

$$\langle z\rangle = \int \mathrm{d}z \int \mathrm{d}\epsilon z F(z,\epsilon,s) = 0, \tag{6.3}$$

the so-called *phase mixing assumption*.

In this case, we can derive the stationary Fokker-Planck equation for the reduced distribution function

$$\tilde{F}(\epsilon,s) = \int_{-\infty}^{\infty} F(z,\epsilon,s)\mathrm{d}z : \tag{6.4}$$

$$\frac{\partial\tilde{F}_0}{\partial s} = 0 = +\frac{J_\epsilon\alpha_\epsilon}{c}\frac{\partial}{\partial\epsilon}(\epsilon\tilde{F}_0) + \frac{c_2p_0^2I_3}{4\pi R}\frac{\partial^2\tilde{F}_0}{\partial\epsilon^2} \tag{6.5}$$

Twofold integration leads to the stationary solution

$$F_0(\epsilon) = \frac{1}{\sqrt{2\pi\sigma_\epsilon^2}}\exp\left(-\frac{\epsilon^2}{2\sigma_\epsilon^2}\right) \tag{6.6}$$

with the *RMS Energy Spread*

$$\sigma_\epsilon^2 = \frac{55}{32\sqrt{3}}\frac{\hbar}{mc}\left(\frac{p_0}{mc}\right)^2\frac{I_3}{J_\epsilon I_2} \tag{6.7}$$

6.2 Radial phase space

The Fokker-Planck equation for the radial phase space can be formulated in the same way:

After neglecting some higher-order terms, we obtain

$$\begin{aligned}\tau_x \frac{\partial F}{\partial t} = &-\underbrace{\frac{\partial}{\partial I_x}(J_x(-I_x+\epsilon_x)F)}_{\text{emittance damping}} + \underbrace{\omega_{\mathrm{rf}}\nu_x\tau_x\frac{\partial F}{\partial \psi_x}}_{\text{phase advance}} \\ &+ \underbrace{2J_x\epsilon_x\frac{\partial^2}{\partial I_x^2}(I_xF)}_{\text{Emittanzdiffusion}} + \underbrace{\frac{J_x\epsilon_x}{2I_x}\frac{\partial^2 F}{\partial \psi_x^2}}_{\text{Phasendiffusion}}\end{aligned} \tag{6.8}$$

with the stationary solution

$$F_0(I_x) = \frac{1}{\epsilon_x}\exp\left(-\frac{I_x}{\epsilon_x}\right) = \frac{1}{\epsilon_x}\exp\left(-\frac{\gamma_x x_\beta^2 - \beta_x' x_\beta p_\beta + \beta_x p_\beta^2}{\epsilon_x}\right). \tag{6.9}$$

Here, we additionally introduced the damping time $\tau_x = 1/\alpha_x$ and the *equilibrium emittance*

$$\epsilon_x = \langle I_x\rangle = \frac{55}{32\sqrt{3}}\frac{\hbar}{mc}\left(\frac{p_0}{mc}\right)^2\frac{I_5}{J_xI_2}. \tag{6.10}$$

The stationary solution of the Fokker-Planck equation for the radial phase space is a two-dimensional Gaussian distribution in the phase space (x_β, p_β) with the emittance as the width of the distribution and the Courant-Snyder invariant as the $2-\sigma$ contour. This justifies to a certain extent our conservative single-particle description.

In beam dynamics for the (y, p_y) phase space, the excitation term due to synchrotron radiation is missing. The emittance in y is essentially determined by the coupling with the radial phase space due to rotated dipole and quadrupole components as well as higher-order multipoles.

References

Jowett, John M. (1987). "Introductory Statistical Mechanics for Electron Storage Rings". In: PHYSICS OF PARTICLE ACCELERATORS: SLAC Summer School, 1985 and Fermilab Summer School 1984. Stanford, CA and Batavia, IL USA, pp. 864–970. DOI: 10.1063/1.36374.

7 The Vlasov equation

The previous discussion of statistical mechanics was (strictly speaking) still a consideration of *single-particle dynamics* under the influence of random variables.

In the following, the application of the concepts we have become familiar with to many-particle descriptions will be briefly discussed.

7.1 Many-particle systems

Within the description of particle dynamics with the stochastic equations of motion and the corresponding Klimontovich phase space density distribution, the extension to many-body systems is possible without any constraints:

$$\dot{X}_i(t) = K(X_i,t) + \sum_{i\neq j} L(X_i,X_j,t) + Q(X_i,t)\xi(t), \quad i = 1,\cdots,N, \tag{7.1}$$

this time with N the number of particles and X_i the tuple of e.g. canonical coordinates of the ith particle.

Formally, this differs from the single-particle equations of motion by the term $L(X_i,X_j,t)$, which describes the pairwise interaction of the particles with each other (e.g. Coulomb interaction).

Analogous to single-particle dynamics, the Klimontovich distribution can also be formally formed for the solutions of these coupled equations:

$$\mathcal{F}(X,t) = \frac{1}{N}\sum_{i=1}^{N}\delta(X-X_i) \tag{7.2}$$

It should be noted that this is a distribution in 6-dimensional, not in $6N$-dimensional phase space. That is, Eq. 7.2 describes a distribution of N points that move dynamically through the phase space under the influence of mutual interaction.

7.2 The Vlasov equation

As in the single-particle case, the continuity equation can be formulated for the Klimontovich distribution and an averaging can be carried out. This averaging no longer occurs over all realisations of the random process, but over the so-called

Liouville distribution = ensemble of systems with identical macroscopic means.

For the sake of clarity, let us consider a deterministic system, i.e. we leave out the fluctuating part of the equations of motion $Q\xi$.

By averaging the continuity equation over the Liouville distribution and defining the

- single-particle distribution function

$$\Psi(X,t) := \langle \mathcal{F}(X,t) \rangle \tag{7.3}$$

- Two-particle distribution function

$$\Psi_2(X,X',t) := \frac{1}{N^2} \sum_{i \neq j} \langle \delta(X - X_i(t)) \delta(X' - X_j(t)) \rangle \tag{7.4}$$

- Two-particle correlation

$$g_2(X,X',t) = \Psi_2(X,X',t) - \Psi(X,t)\Psi(X',t) \tag{7.5}$$

we arrive at

$$\begin{aligned} \partial_t \Psi(X,t) + \nabla_X \left(K(X,t) \Psi(X,t) \right) \\ = -\nabla_X \int \mathrm{d}X' L(X,X',t) \Psi(X,t) \Psi(X',t) \\ - \nabla_X \int \mathrm{d}X' L(X,X',t) g_2(X,X',t). \end{aligned} \tag{7.6}$$

Neglecting the two-particle correlation finally leads to the VLASOV equation

$$\boxed{\partial_t \Psi(X,t) + \nabla_X \left(K(X,t) \Psi(X,t) \right) = -\nabla_X \int \mathrm{d}X' L(X,X',t) \Psi(X,t) \Psi(X',t).} \tag{7.7}$$

This now describes a many-particle system in 6-D phase space with mutual interaction of the particles, which is expressed in a self-consistent field term on the right side of the equation.

The Vlasov equation neglects possible correlations between the phase space coordinates of two particles. It should be noted that such correlations do occur in reality, for example when direct collision processes take place between two particles.

In many cases in accelerator physics, the particles can be considered independent. If, in addition, one considers non-dissipative systems, the Vlasov equation takes the form

$$\partial_t \Psi + K \nabla_X \Psi = 0. \tag{7.8}$$

In this form, it is often cited as a statistical counterpart to the LIOUVILLE theorem.

References

Jowett, John M. (1987). “Introductory Statistical Mechanics for Electron Storage Rings”. In: PHYSICS OF PARTICLE ACCELERATORS: SLAC Summer School, 1985 and Fermilab Summer School 1984. Stanford, CA and Batavia, IL USA, pp. 864–970. DOI: 10.1063/1.36374.