Karlsruhe Institute of Technology

IBPT

Institute for Beam Physics and Technology

Particle Accelerator Physics

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III - Beam optics & beam dynamics

- Basics of transverse beam optics
  - Magnetic lenses
  - Equations of motion and transfer matrices
  - Optic functions and emittance
  - Tune, chromaticity and resonances
  - Dispersion and beam size

- Basics of longitudinal beam dynamics
  - Longitudinal oscillations
  - RF buckets and stable phase

- Oscillations and damping
- Many-particle systems

Basics of beam dynamics

- A bunch typically consists of some 10^9 particles.
- Different processes (synchrotron radiation, scattering at residual gas molecules, …) lead to a distribution of beam energy around the nominal energy.
- Without focusing, the vacuum chambers would have to be large requiring extremely strong magnets.
- Electric and magnetic fields are used to steer and focus the particle beams:

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

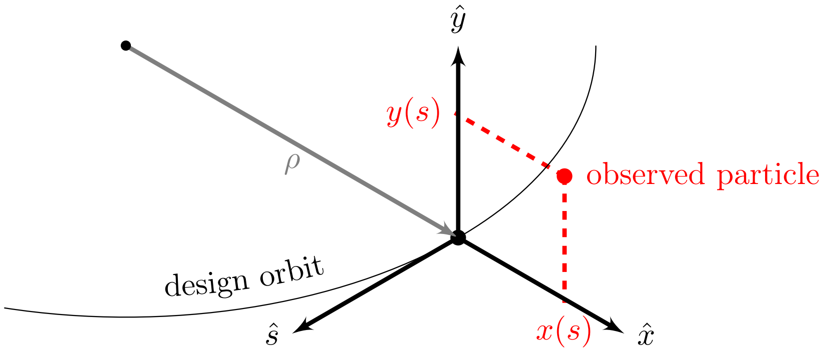
- At relativistic velocities E and B have the same effect if

$$E = cB$$

$$\rightarrow \quad B = 1\,T \ ✓ \quad \Leftrightarrow \quad E = 3 \times 10^8\,\mathrm{V/m} \ ✗$$

- Right-handed orthogonal coordinate system

Simplification: $\vec{v} = (0,0,v_s)$ and $\vec{B} = (B_x, B_y, 0)$



Frenet-Serret coordinate system

Magnetic fields along the beam trajectory

- For highly relativistic particles follows from equality of Lorentz and Centripetal Force

$$\frac{1}{\rho(x,y,s)} = \left|\frac{e}{p}\, B_y(x,y,s)\right| = \left|\frac{e\ c}{\beta E}\, B_y(x,y,s)\right|$$

- Taylor expansion of the magnetic field in the vicinity of the nominal trajectory ($x = 0$):

$$B_y(x) = B_{y0} + \frac{dB_y}{dx}\, x + \frac{1}{2!}\frac{d^2B_y}{dx^2}\, x^2 + \frac{1}{3!}\frac{d^3B_y}{dx^3}\, x^3 + \ldots$$

- This immediately leads to

$$\begin{aligned}\frac{e}{p}\, B_y(x) &= \frac{e}{p}\, B_{y0} && + \frac{e}{p}\frac{dB_y}{dx}\, x && + \frac{1}{2!}\frac{e}{p}\frac{d^2B_y}{dx^2}\, x^2 && + \frac{1}{3!}\frac{e}{p}\frac{d^3B_y}{dx^3}\, x^3 + \ldots \\ &= \frac{1}{\rho} && + k\ x && + \frac{1}{2!}\, m\ x^2 && + \frac{1}{3!}\, o\ x^3 + \ldots \\ &\Rightarrow \text{dipole} && + \text{ quadrupole} && + \text{ sextupole} && + \text{ octupole } + \ldots\end{aligned}$$

- Structures that only consist of **dipole and quadrupole fields** are called **"linear lattices"**. The linear lattice determines trajectory ("orbit") and focusing properties, higher-order field components are used for correction and error compensation.

Dipole

- The magnetic field is calculated by

$$\oint \vec{H} d\vec{s} = hH_0 + lH_E = nI.$$

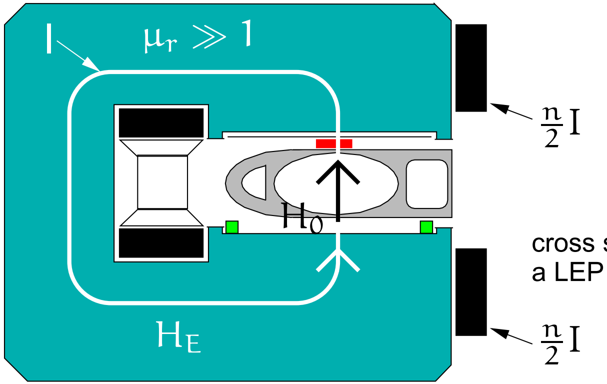
$$H_E = \frac{1}{\mu_r} H_0$$

- For $\mu_r \gg 1$ the field of a dipole magnet is

$$B_0 = \mu_0 H_0 = \frac{\mu_0 n I}{h} \quad \text{with gap } h.$$

- With radius ρ the dipole strength is

$$\frac{1}{\rho}[\mathrm{m}^{-1}] = \frac{e}{p} B_0 = 0.2998 \, \frac{B_0[\mathrm{T}]}{p[\mathrm{GeV/c}]}.$$



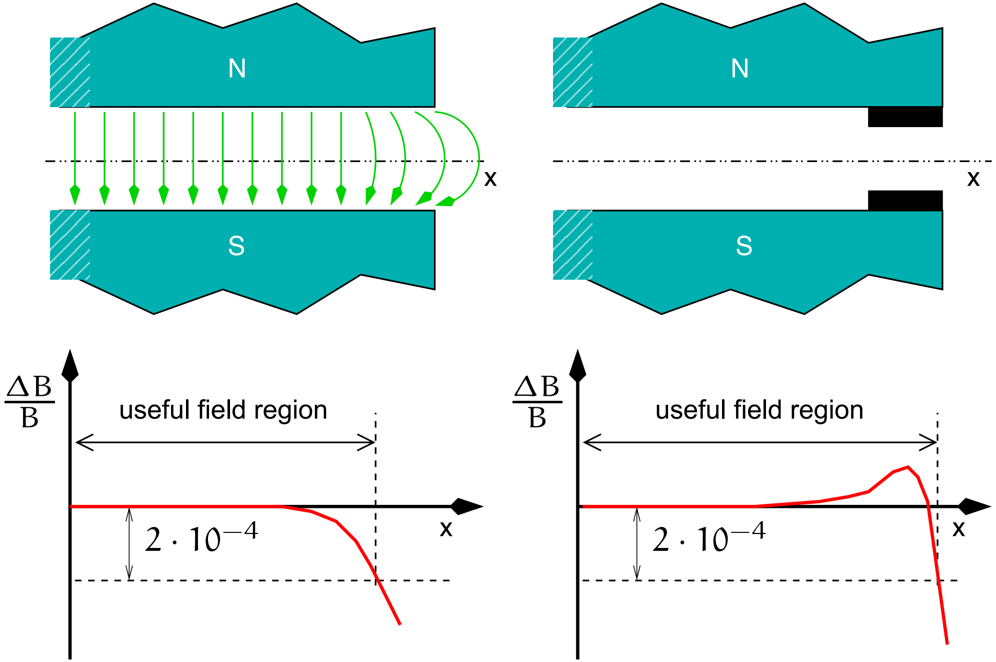
cross section of a LEP dipole


KARA dipole

Dipol — edge field

- The magnetic field decreases at the edge.
- The useful field region can be increased by partial compensation of the edge field.

$\Rightarrow$ Fit so-called “shims” to the edge of the poles.



Quadrupole

- The quadrupole field increases linearly with transverse distance from the magnet center:

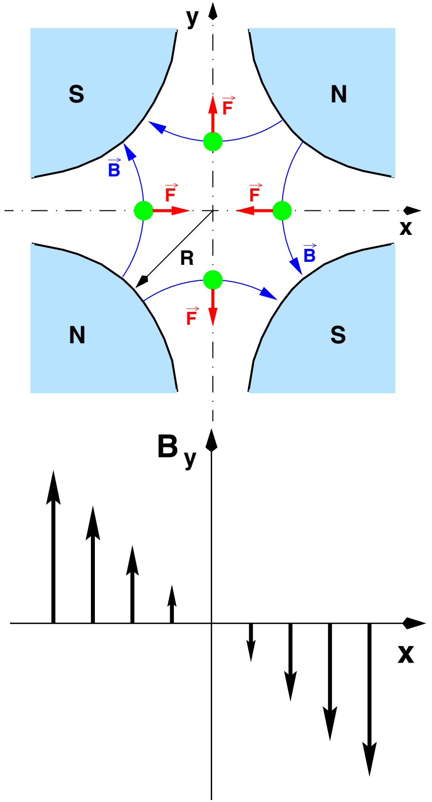
$$B_x(y) = -\ g \cdot y$$
$$B_y(x) = -\ g \cdot x$$

- Is I the current in n windings of the coils and R the distance of the poles to the magnet center, a perfectly formed pole has the gradient

$$g\ =\ \frac{\partial B_y}{\partial x}\ =\ \frac{\partial B_x}{\partial y}\ =\ g = \frac{2\mu_0 n I}{R^2}.$$

- Definition of a normalized quadrupole strength in analogy of the deflection strength $1/\rho$ of a dipole magnet:

$$k = \frac{e}{p}\ g$$



Quadrupole gradient

- Like the dipole strength, the field gradient is determined from the relationship of field integral and current in the coil windings:

$$\left| \oint \vec{H}\, \vec{ds} \right| = nI$$

- The integral from I via II to III may be broken up as follows:

$$\oint \vec{H}\, \vec{ds} = \int_{I}^{II} \vec{H_0}\, \vec{ds} + \int_{II}^{III} \vec{H_E}\, \vec{ds} + \int_{III}^{I} \vec{H}\, \vec{ds}$$

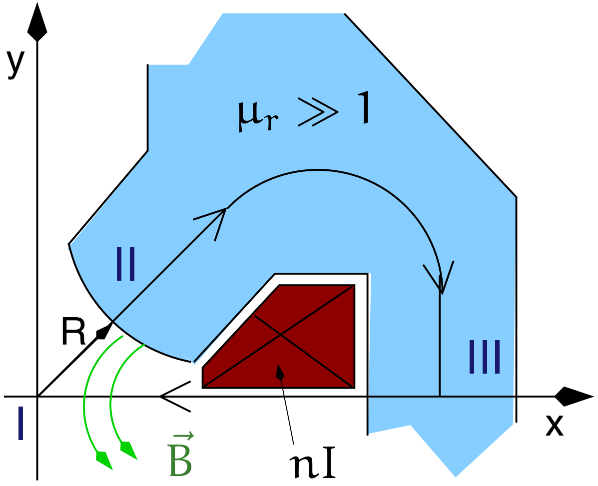
- The second and third term vanish:

$$\int_{II}^{III} \vec{H_E}\, \vec{ds} = 0 \quad \text{because} \quad \mu_r \gg 1 \quad \text{and} \quad \int_{III}^{I} \vec{H}\, \vec{ds} = 0 \quad \text{because} \quad \vec{H} \perp \vec{ds}$$

- The field between the poles is determined by $B_x = -g\,y$ and $B_y = -g\,x$. From this follows that

$$H = B/\mu_0 = -(g/\mu_0)\sqrt{x^2+y^2} = -(g/\mu_0)\, r$$

- Integration up to the pole radius R: $nI = \left| \int_{I}^{II} H ds \right| = \left| -\frac{g}{\mu_0} \int_{0}^{R} r ds \right| = \frac{g}{\mu_0} \frac{R^2}{2} \quad \rightarrow \quad g = \frac{2\mu_0 n I}{R^2}$



Quadrupole “focal length”

- In analogy to the dipole magnet, the normalized quadrupole gradient can be written as

$$k[\mathrm{m}^{-2}] \;=\; 0.2998\,\frac{g[\mathrm{T/m}]}{p[\mathrm{GeV/c}]}$$

- The deflection (“kick”) of a particle passing through a quadrupole of length l with offset x is

$$\Delta x' \;=\; \Delta\left(\frac{dx}{ds}\right) \;=\; klx.$$

- The focal length f of a quadrupole of length l is given by

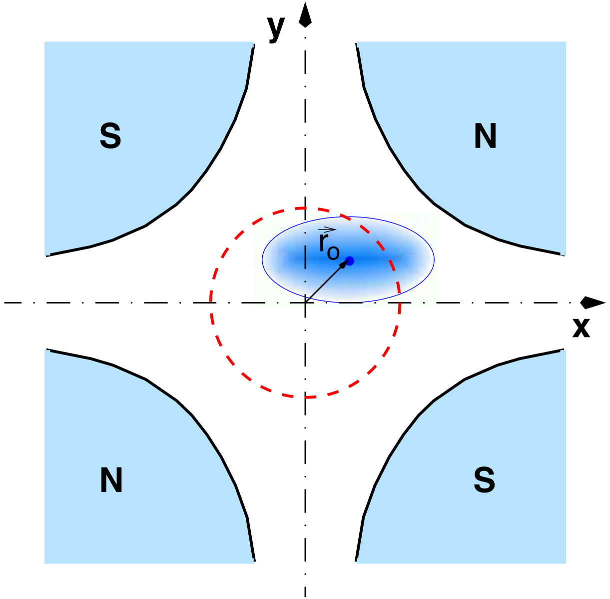
$$\frac{1}{f} = -k\ l.$$

- In the case of $f \gg l$, we speak of **thin lenses**, no matter how large l actually is.

KARA quadrupole

Synchrotron radiation in quadrupoles

- A particle undergoes a deflection in the quadrupole: again synchrotron radiation occurs!
- Since the quadrupole field increases linearly with the particle offset, the energy loss is also position dependent.
- The ensemble of particles has an average energy loss, which can be calculated from beam size and position.



Synchrotron radiation in quadrupoles II

- The transverse radiation power is $P_\gamma = \frac{C_\gamma}{2\pi}\, c\, \frac{E^4}{\rho^2}$.
- The quadrupole field is $B_x = \frac{k\,p}{e}\, y$ and $B_y = \frac{k\,p}{e}\, x$, respectively.
- Thus the bending radius inside the quadrupole is $1/\rho = k\ u$ and $P_\gamma = \frac{c\,C_\gamma}{2\pi}\, E^4\, k^2\, u^2$ for $u = (x, y)$.
- Integration over the time of flight in the quadrupole field provides the energy loss:

$$\Delta E_i = \int_{\Delta t_i} \mathrm{d}t P_\gamma \ = \frac{c\ C_\gamma}{2\pi}\, E^4\, k^2\, u^2\, \frac{l_i}{c}$$

- For a bunch with Gaussian-distributed charge density is $\Delta E_i = \frac{C_\gamma\, l_i}{2\pi}\, k^2\, E^4 \int \mathrm{d}u\, \frac{1}{\sqrt{2\pi}\,\sigma_u}\, u^2\, e^{\boxed{-\frac{1}{2}\frac{(u-u_0)^2}{\sigma_u^2}}}$.
- Substitutions: $\xi^2 = (u-u_0)^2/(2\sigma_u^2)$, $\xi = (u-u_0)/\left(\sqrt{2}\sigma_u\right)$, $\mathrm{d}\xi/\mathrm{d}u = 1/\left(\sqrt{2}\sigma_u\right)$, $\mathrm{d}u = \sqrt{2}\sigma_u \mathrm{d}\xi$ und $u = \sqrt{2}\sigma_u \xi + u_0$
- Finally, the energy loss is $\Delta E_i = \frac{C_\gamma\, l_i}{2\pi}\, k^2\, E^4 \left(\sigma_u^2 \ + \ u_0^2\right)$.

Synchrotron radiation in quadrupoles III

- This result is generally valid for probability density functions $f(x)$ normalised to $\int_a^b \mathrm{d}x\ f(x) = 1$, if mean value μ and variance σ_μ^2 exist.
- In general, $a = -\infty$ and $b = +\infty$. In real accelerators a and b are limited by the vacuum chamber.
- The mean value of a distribution is

$$\int_a^b \mathrm{d}x\ x\ f(x) = \mu$$

- From the definition of the variance, one immediately obtains the result derived for the special case

$$\sigma_\mu^2 = \int_a^b \mathrm{d}x\ (x-\mu)^2\ f(x) = \int_a^b \mathrm{d}x\ x^2 f(x) - 2\mu \int_a^b \mathrm{d}x\ x\ f(x) + \mu^2 \int_a^b \mathrm{d}x\ f(x)$$

$$= \int_a^b \mathrm{d}x\ \left\{x^2 f(x)\right\} - \mu^2 \qquad \Rightarrow \int_a^b \mathrm{d}x\ x^2 f(x) = \mu^2 + \sigma_\mu^2$$

Synchrotron radiation in quadrupoles IV

- This can be generalized to two dimensions: Because of the rotational symmetry in x-y ($P_\gamma \propto k^2(x^2 + y^2)$) one can always find a transformation, so that the 2D function factorizes: $h(x, y) = f(x)g(y)$, where g such as f is a probability density function with mean value ν and variance σ_ν^2.
- The integral is then

$$\int_a^b \int_c^d \mathrm{d}x \ \mathrm{d}y \ \left(x^2 \ + \ y^2\right) \ h(x, y) = \int_a^b \int_c^d \mathrm{d}x \ \mathrm{d}y \ x^2 f(x) \ g(y) \ + \int_a^b \int_c^d \mathrm{d}x \ \mathrm{d}y \ y^2 \ f(x) \ g(y)$$

$$= \int_a^b \mathrm{d}x \ x^2 f(x) \ + \int_c^d \mathrm{d}y \ y^2 \ g(y)$$

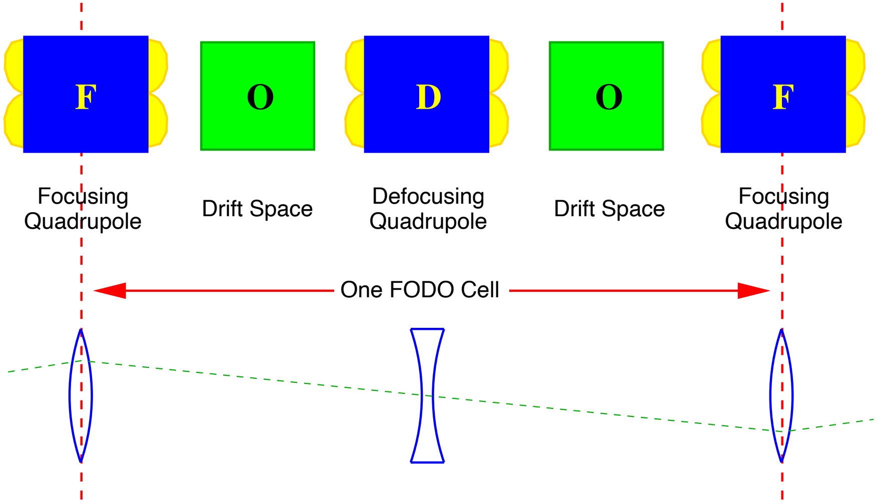
$$= \mu^2 + \sigma_\mu^2 \ + \ \nu^2 + \sigma_\nu^2 \, .$$

- Thus, the energy loss with transverse offset and finite beam size in x and y is

$$\Delta E_i = \frac{C_\gamma \ l_i}{2\pi} k^2 \ E^4 \left(x_0^2 \ + \ y_0^2 \ + \ \sigma_x^2 \ + \ \sigma_y^2\right).$$

The FODO lattice

- The alternating arrangement of (horizontally) focusing and (horizontally) defocusing quadrupoles allows to build compact systems.
  A frequently used arrangement is the periodic so-called **"FODO structure"**.



- F and D quadrupoles have comparable strength and the distance between two equal lenses is $\lesssim 2f$.

Equations of motion

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- For a (horizontally) focusing quadrupole is $k < 0$.
- A particle with vertical offset y in a thin quadrupole of length $\mathrm{d}s$ and strength k undergoes a vertical deflection of

$$\mathrm{d}y' = -\ k\ y\ \mathrm{d}s.$$

- With this, a differential equation for the motion can be written immediately, called "Hill's differential equation" with the periodic coefficient $k(s)$:

$$y'' + k(s)\ y = 0$$

- In general, for the coordinate $u(s)$ and the length of the periodic structure l we write

$$\boxed{u'' + K(s)\ u = 0} \qquad \text{with} \quad K(s+l) = K(s)$$

$$K(s) = \begin{cases} -k(s) + \dfrac{1}{\rho(s)^2} & \text{horizontal} \\ +k(s) & \text{vertical} \end{cases}$$

Solutions of Hill's differential equation

- Structure of a simple harmonic oscillator — but with variable reset constant $K(s)$
- The independent solutions are

$$u(s) = a\sqrt{\beta(s)}\; e^{\pm i\left(\Phi(s) + \Phi_0\right)}$$

with $\Phi'(s) = \dfrac{1}{\beta(s)}$ and $a = \text{const}\,.$

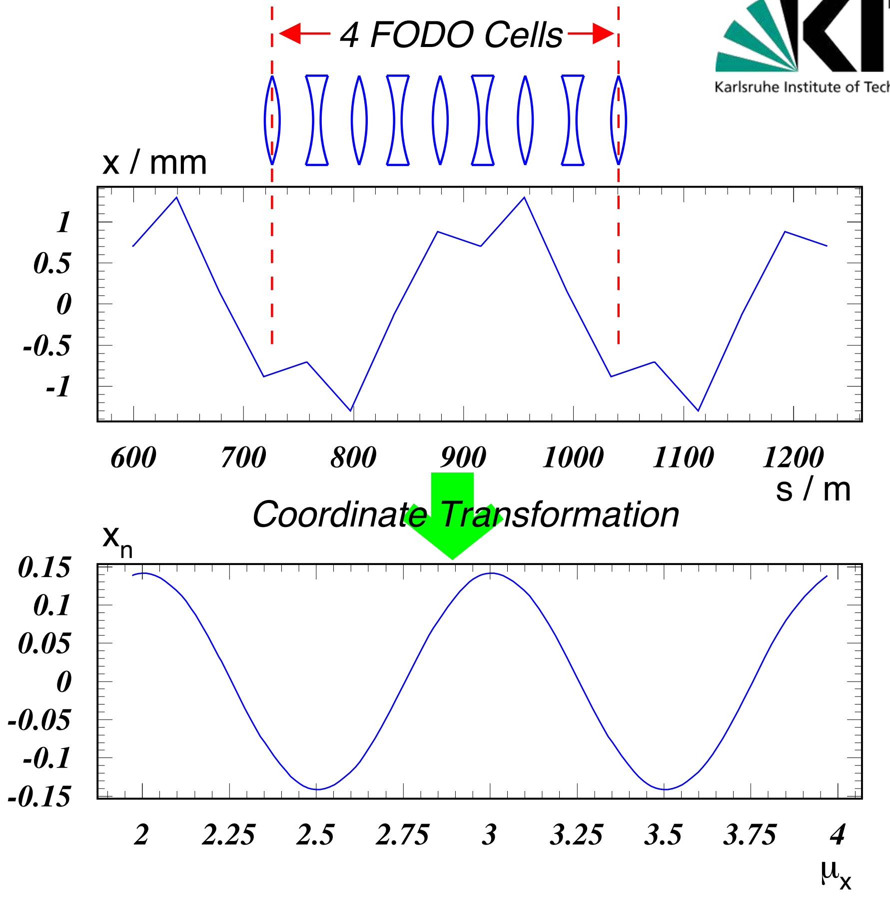
- The phase function $\Phi(s)$ increases non-linearly with time or longitudinal position s, but — as the amplitude function $\beta(s)$ — must have the same periodicity as the magnetic lattice.
- The phase advance μ per period length l is derived from the betafunction $\beta(s)$:

$$\boxed{\mu = \mu(s,l) = \Phi(s+l) - \Phi(s) = \int_s^{s+l} \frac{1}{\beta(t)}\mathrm{d}t}$$

Particle trajectory

Illustration:

- Particle trajectory in regular FODO cells
- Periodicity of 4 FODO cells, $\Rightarrow \mu(s,l) = \pi/2$
- Normalised representation: $X(s) = x(s)/\sqrt{\beta(s)}$



Equations of motion II

- The general solutions of the equation $u'' + K(s)u = 0$ are "Cosinelike" and "Sinelike" trajectories:

$$C(s) = \cos(\sqrt{K}s) \quad \text{and} \quad S(s) = \frac{1}{\sqrt{K}}\sin(\sqrt{K}s) \quad \text{for} \quad K > 0$$

$$C(s) = \cosh(\sqrt{|K|}s) \quad \text{and} \quad S(s) = \frac{1}{\sqrt{|K|}}\sinh(\sqrt{|K|}s) \quad \text{for} \quad K < 0$$

- These linearly independent solutions fulfill following initial conditions:

$$C(0) = 1, \qquad C'(0) = \frac{\mathrm{d}C}{\mathrm{d}s} = 0, \qquad S(0) = 0, \qquad S'(0) = \frac{\mathrm{d}S}{\mathrm{d}s} = 1$$

- Any solution $u(s)$ can be written as linear combination :

$$u(s) = C(s)u_0 + S(s)u_0'$$

$$u'(s) = C'(s)u_0 + S'(s)u_0'$$

Wronskian

- The solutions of the equation of motion can be written as a (transfer) matrix:

$$\begin{pmatrix} u(s) \\ u'(s) \end{pmatrix} = \begin{pmatrix} C(s) & S(s) \\ C'(s) & S'(s) \end{pmatrix} \begin{pmatrix} u_0(s) \\ u_0'(s) \end{pmatrix}$$

- Consider a general homogeneous linear differential equation of second order:

$$u'' + v(s)u' + w(s)u = 0$$

- For two two linearly independent solutions the Wronski determinant $\mathbb{W}$ can be formed:

$$\mathbb{W} = \begin{vmatrix} u_1(s) & u_2(s) \\ u_1'(s) & u_2'(s) \end{vmatrix} = u_1u_2' - u_2u_1'$$

- Combination of the two solutions:

$$\begin{aligned} u_1'' + v(s)u_1' + w(s)u_1 &= 0 && |(-u_2) \\ u_2'' + v(s)u_2' + w(s)u_2 &= 0 && |u_1 \\ (u_1u_2'' - u_2u_1'') + v(s)(u_1u_2' - u_2u_1') &= 0 && \end{aligned}$$

- From this follows a differential equation for $\mathbb{W}$: $\frac{\mathrm{d}\mathbb{W}}{\mathrm{d}s} + v(s)\mathbb{W} = 0$

Wronskian II

- Differential equation for $\mathbb{W}$: $\dfrac{\mathrm{d}\mathbb{W}}{\mathrm{d}s} + v(s)\mathbb{W} = 0$
- Integration immediately yields: $\mathbb{W}(s) = \mathbb{W}_0 e^{-\int_{s_0}^{s} \mathrm{d}\tilde{s}\ v(\tilde{s})}$
- Without acceleration or energy loss due to synchrotron radiation, in linear beam dynamics $v(s) = 0$ and thus $\mathbb{W}(s) = \mathbb{W}_0 = \text{const}\,.$. With Sinelike and Cosinelike solutions

$$\mathbb{W}_0 = C_0 S_0' - C_0' S_0 = 1$$

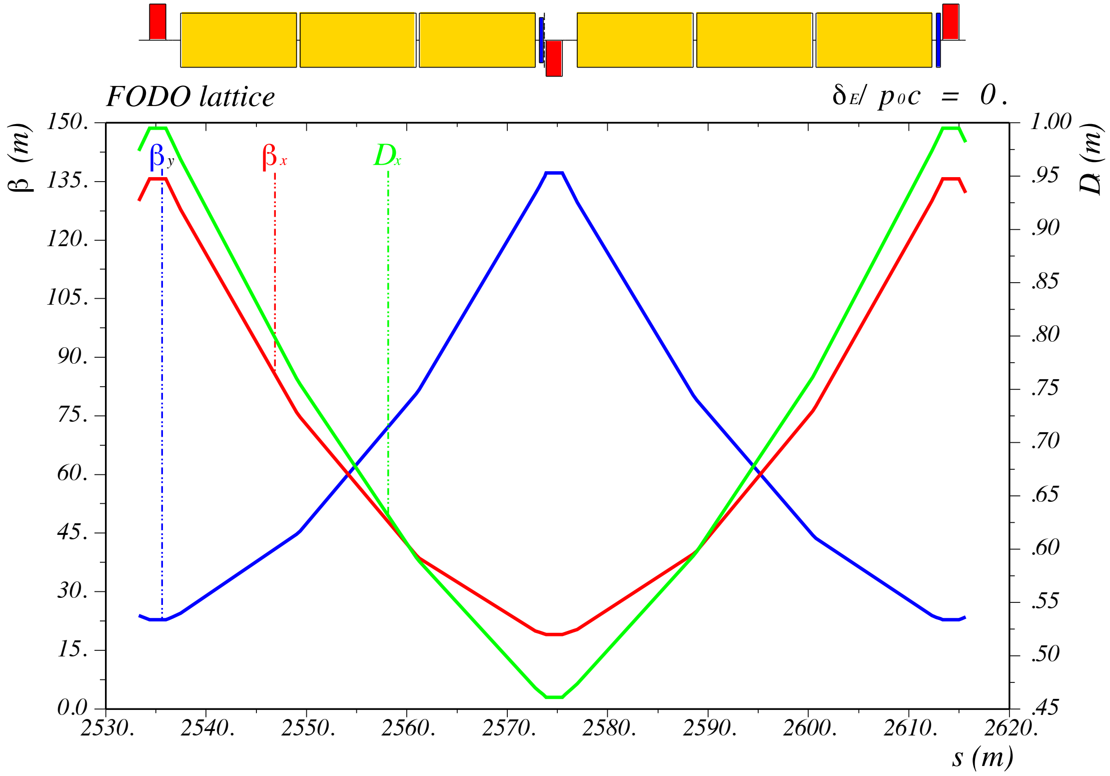
- For the transfer matrix applies in general

$$\mathbb{W}(s) = \begin{vmatrix} C(s) & S(s) \\ C'(s) & S'(s) \end{vmatrix} = 1$$

$\Rightarrow$ Phase space density and energy are conserved

- This result is valid for any beam guiding system as long as $v(s) = 0$ and $w(s) = K(s)$.

Beam optics example



$$u(s) = a\boxed{\sqrt{\beta(s)}}\,e^{\pm i\left(\Phi(s)+\Phi_0\right)}$$

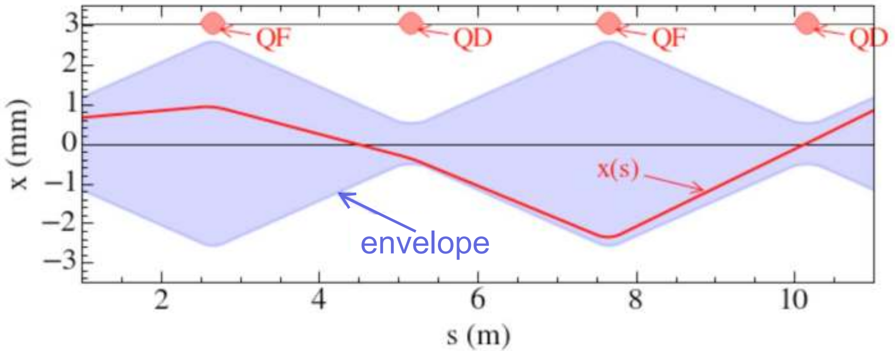
- Beam optics functions of a LEP FODO cell calculated by “MAD”

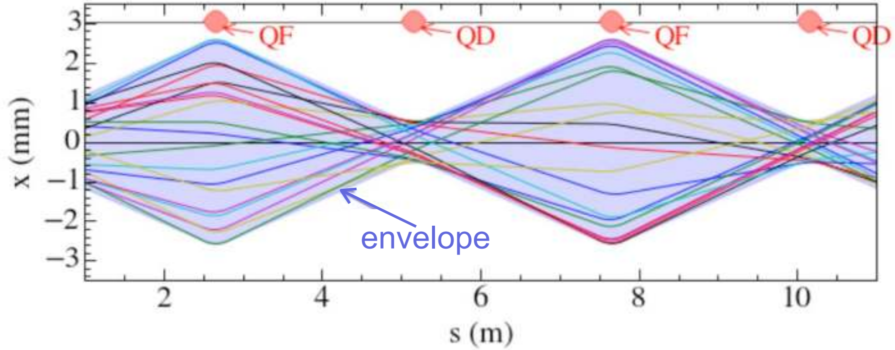
Trajectories and β

- The trajectories $x_i(s)$ are individual solutions of the equation of motion and have different initial conditions.

- The envelope $E(s)$ of all trajectories is proportional to the beta function:

$$E(s) \propto \sqrt{\beta(s)}$$





Matrix formalism

- $u(s) \;=\; a\sqrt{\beta(s)}\;\cos\left(\Phi(s) - \delta\right)$
- Full description of particle state by vector $\quad U = (u, u')$
- $\dfrac{\mathrm{d}u(s)}{\mathrm{d}s} \;=\; u'(s) \;=\; \dfrac{1}{2\sqrt{\beta(s)}}\,\dfrac{\mathrm{d}}{\mathrm{d}s}\beta(s)\; a\cos(\Phi(s) - \delta) \;-\; a\sqrt{\beta(s)}\,\sin(\Phi(s) - \delta)\,\dfrac{\mathrm{d}}{\mathrm{d}s}\,\Phi(s)$
- $\;=\; -\dfrac{a}{\sqrt{\beta(s)}}\left(\sin(\Phi(s) - \delta) \;-\; \dfrac{1}{2}\beta'(s)\;\cos(\Phi(s) - \delta)\right)$
- Linear differential equations can be transferred from location s_1 to location s_2 with a transfer matrix:
- $\begin{pmatrix} u(s_2) \\ u'(s_2) \end{pmatrix} \;=\; \begin{pmatrix} b & c \\ f & g \end{pmatrix}\;\begin{pmatrix} u(s_1) \\ u'(s_1) \end{pmatrix} \;=\; \boldsymbol{M}(s_2\,|\,s_1)\begin{pmatrix} u(s_1) \\ u'(s_1) \end{pmatrix}$
- For simplification we write $\quad w = \sqrt{\beta},\;\; u(s_i) = u_i,\;\; \phi_i = \Phi_i - \delta$
- There are two classes of solutions: the “Cosinelike with $\delta = 0$ and the “Sinelike” with $\delta = \pi/2$.
- Four equations are obtained: $\begin{aligned} u_i &= a\; w_i\;\cos\phi_i & \quad u_i' &= a\; w_i'\;\cos\phi_i - (a/w_i)\;\sin\phi_i \\ \tilde{u}_i &= a\; w_i\;\sin\phi_i & \quad \tilde{u}_i' &= a\; w_i'\;\sin\phi_i + (a/w_i)\;\cos\phi_i \end{aligned}$

Matrix formalism II

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- The matrix equation results in

$$u_2 = bu_1 + cu_1' \quad \text{und} \quad u_2' = fu_1 + gu_1'$$
$$\tilde{u}_2 = b\tilde{u}_1 + c\tilde{u}_1' \quad \text{und} \quad \tilde{u}_2' = f\tilde{u}_1 + g\tilde{u}_1'$$

- Substituting the solutions of u_1 and u_2 results in
- (1) $\quad a\ w_2 \cos\phi_2 = b\ \left(a\ w_1 \cos\phi_1\right) + c\ \left(a\ w_1' \cos\phi_1 - (a/w_1)\sin\phi_1\right)$
- (2) $\quad a\ w_2 \sin\phi_2 = b\ \left(a\ w_1 \sin\phi_1\right) + c\ \left(a\ w_1' \sin\phi_1 + (a/w_1)\cos\phi_1\right)$

- Summation of (1) $\cdot\ w_1 \cos\phi_2$ and (2) $\cdot\ w_1 \sin\phi_2$ gives
- (3) $\quad w_1 w_2 \cos^2\phi_2 = b\ \left(w_1^2 \cos\phi_1 \cos\phi_2\right) + c\ \left(w_1 w_1' \cos\phi_1 \cos\phi_2 - \sin\phi_1 \cos\phi_2\right)$
- (4) $\quad w_1 w_2 \sin^2\phi_2 = b\ \left(w_1^2 \sin\phi_1 \sin\phi_2\right) + c\ \left(w_1 w_1' \sin\phi_1 \sin\phi_2 + \cos\phi_1 \sin\phi_2\right)$

- (3)+(4) $\quad w_1 w_2 = (bw_1^2 + cw_1 w_1')\left(\cos\phi_1 \cos\phi_2 + \sin\phi_1 \sin\phi_2\right) + c\left(\cos\phi_1 \sin\phi_2 - \sin\phi_1 \cos\phi_2\right)$
- (3)+(4) $\quad w_1 w_2 = (bw_1^2 + cw_1 w_1')\ \cos\Delta\phi + c\ \sin\Delta\phi$

Matrix formalism III

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- Analogous addition of (1) $\cdot\, w_1 \cos\phi_1$ and (2) $\cdot\, w_1 \sin\phi_1$
- (5) $$w_1 w_2 \cos\phi_1 \cos\phi_2 \;=\; b\;\left(w_1^2 \cos^2\phi_1\right) \;+\; c\;\left(w_1 w_1' \cos^2\phi_1 - \sin\phi_1 \cos\phi_1\right)$$
- (6) $$w_1 w_2 \sin\phi_1 \sin\phi_2 \;=\; b\;\left(w_1^2 \sin^2\phi_1\right) \;+\; c\;\left(w_1 w_1' \sin^2\phi_1 + \sin\phi_1 \cos\phi_1\right)$$
- (5)+(6) $$w_1 w_2 \cos\Delta\phi \;=\; b\; w_1^2 + c\; w_1 w_1'$$

- With the previous result: two equations for the determination of b and c
- (3)+(4) $$w_1 w_2 \;=\; (b\, w_1^2 + c\, w_1 w_1')\;\cos\Delta\phi \;+\; c\;\sin\Delta\phi$$
  $$c\;\sin\Delta\phi \;=\; w_1 w_2\;\sin^2\Delta\phi \quad\rightarrow\quad c \;=\; w_1 w_2\;\sin\Delta\phi$$

- Substitution of c in (5)+(6)
  $$b\; w_1^2 \;=\; w_1 w_2\;\cos\Delta\phi - w_1^2 w_2 w_1'\;\sin\Delta\phi \quad\rightarrow\quad b \;=\; (w_2/w_1)\;\cos\Delta\phi \;-\; w_1' w_2\;\sin\Delta\phi$$

- After analogous procedure for f and g is
  $$\boldsymbol{M}(s_2\,|\,s_1) \;=\; \begin{pmatrix} \frac{w_2}{w_1}\cos\Delta\phi - w_1' w_2 \sin\Delta\phi & w_1 w_2 \sin\Delta\phi \\ -\frac{1 + w_1 w_1' w_2 w_2'}{w_1 w_2}\sin\Delta\phi - \left(\frac{w_1'}{w_2} - \frac{w_2'}{w_1}\right)\cos\Delta\phi & \frac{w_1}{w_2}\cos\Delta\phi + w_1 w_2' \sin\Delta\phi \end{pmatrix}$$

Twiss matrix

- General transfer matrix from s_1 to s_2:

$$M(s_2\,|\,s_1) \;=\; \begin{pmatrix} \frac{w_2}{w_1}\cos\Delta\phi - w_1' w_2 \sin\Delta\phi & w_1 w_2 \sin\Delta\phi \\ -\frac{1 + w_1 w_1' w_2 w_2'}{w_1 w_2}\sin\Delta\phi - \left(\frac{w_1'}{w_2} - \frac{w_2'}{w_1}\right)\cos\Delta\phi & \frac{w_1}{w_2}\cos\Delta\phi + w_1 w_2' \sin\Delta\phi \end{pmatrix}$$

- Simplification with periodic boundary conditions: $w_1 \;=\; w_2, \quad w_1' \;=\; w_2', \quad \Delta\phi \to \mu$

$$M \;=\; \begin{pmatrix} \cos\mu - w\,w' \sin\mu & w^2 \sin\mu \\ -\frac{1 + w^2 w'^2}{w^2}\sin\mu & \cos\mu + w\,w' \sin\mu \end{pmatrix}$$

- Definition of new functions: $\alpha \;=\; -\,w\,w' \;=\; -\frac{\beta'}{2} \qquad \beta \;=\; w^2 \qquad \gamma \;=\; \frac{1 + (w\,w')^2}{w^2} \;=\; \frac{1+\alpha^2}{\beta}$

- Thus, the Twiss matrix for periodic structures has the form

$$M = \begin{pmatrix} b & c \\ f & g \end{pmatrix} = \begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu - \alpha\sin\mu \end{pmatrix} = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix}$$

Symplectic matrices

- A $2n \times 2n$ dimensional matrix is "symplectic", if

$$A^{\dagger} S A = S \quad \text{with} \quad S = \begin{pmatrix} \emptyset & \mathbb{I} \\ -\mathbb{I} & \emptyset \end{pmatrix} \quad \text{for real matrices} \quad A^{\dagger} = A^{T}$$

$\rightarrow$ A is symplectic, if $A^{-1} = S^{T} A^{\dagger} S$

- If a symplectic matrix consists of the $(n \times n)$ matrices P, Q, R, S

$$A = \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \quad \text{then follows for} \quad A^{-1} = \begin{pmatrix} S^{\dagger} & -R^{\dagger} \\ -Q^{\dagger} & P^{\dagger} \end{pmatrix}$$

$\rightarrow$ The set of symplectic $(2n \times 2n)$ matrices is closed under multiplication and inversion and therefore form a group.

$\rightarrow$ A is symplectic, if it preserves the symplectic form $\vec{x}^{\dagger} S \vec{y}$:

$$(A\ \vec{x})^{\dagger}\ S\ (A\ \vec{y}) = \vec{x}^{\dagger}\ S\ \vec{y} \qquad \forall \quad \vec{x}, \vec{y}$$

This is analogous to the case of preserving the scalar product by a unitary matrix U:

$$(U\ \vec{x})^{\dagger}\ (U\ \vec{y}) = \vec{x}^{\dagger}\ \vec{y}$$

Symplecticity of the Twiss matrix

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- Transfer matrices for beam steering systems in general are symplectic.
  Example: Twissmatrix

$$\boldsymbol{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu - \alpha\sin\mu \end{pmatrix}$$

- As $\boldsymbol{M}$ is real $\quad \boldsymbol{M}^\dagger = \boldsymbol{M}^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$
- Condition for symplecticity: $\quad \boldsymbol{M}^T\ \boldsymbol{S}\ \boldsymbol{M} = \boldsymbol{S}$, also

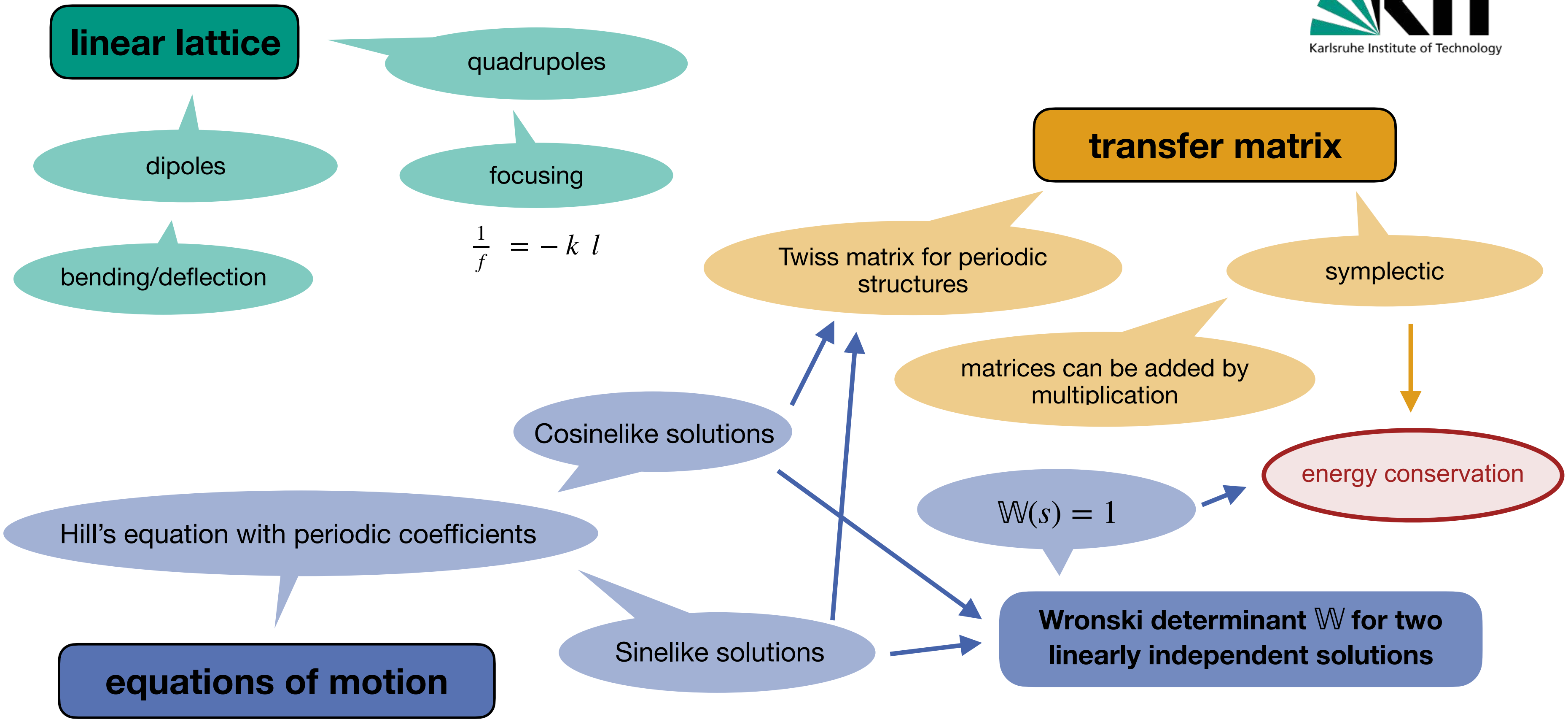
$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} c & d \\ -a & -b \end{pmatrix}$$

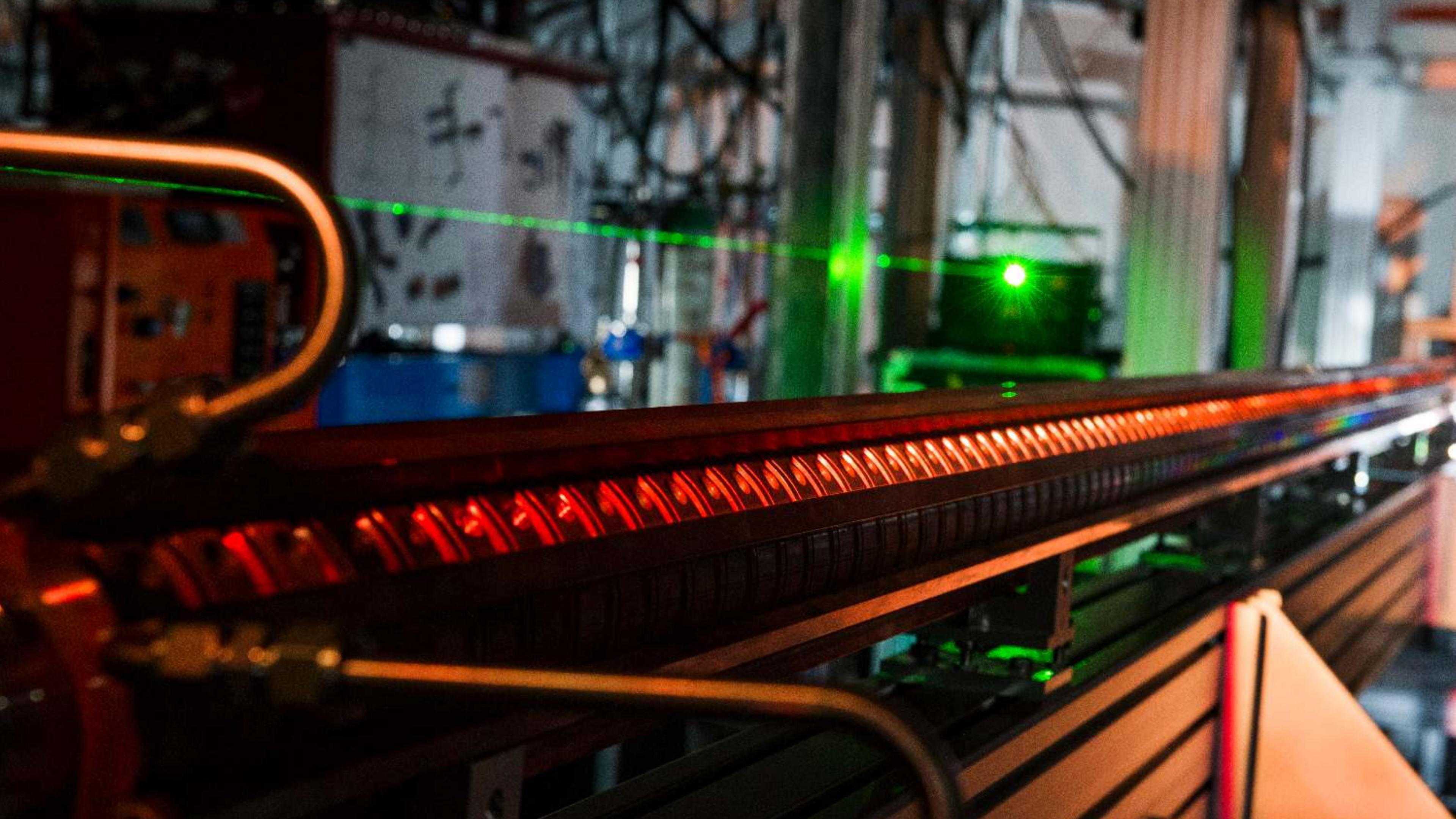
$$= \begin{pmatrix} ac - ca & ad - bc \\ bc - ad & bd - db \end{pmatrix} = \begin{pmatrix} 0 & ad - bc \\ -(ad - bc) & 0 \end{pmatrix}$$

- Using $\beta\gamma - \alpha^2 = \beta\,\frac{(1+\alpha^2)}{\beta} - \alpha^2 = 1$ yields:

$$ad - bc = \cos^2\mu - \alpha\cos\mu\sin\mu + \alpha\cos\mu\sin\mu - \alpha^2\sin^2\mu \; - \; (-\beta\gamma\sin^2\mu)$$

$$= \cos^2\mu \; + \; \sin^2\mu(\beta\gamma - \alpha^2) = 1 \qquad \text{q.e.d.}$$


linear lattice
quadrupoles
dipoles
focusing
bending/deflection
$\frac{1}{f} = -k\ l$
transfer matrix
Twiss matrix for periodic structures
symplectic
matrices can be added by multiplication
Cosinelike solutions
energy conservation
$\mathbb{W}(s) = 1$
Hill's equation with periodic coefficients
Wronski determinant $\mathbb{W}$ for two linearly independent solutions
Sinelike solutions
equations of motion



Particle Accelerator Physics

Anke-Susanne Müller, Axel Bernhard, Bastian Härer, Bennet Krasch, Nathan Ray

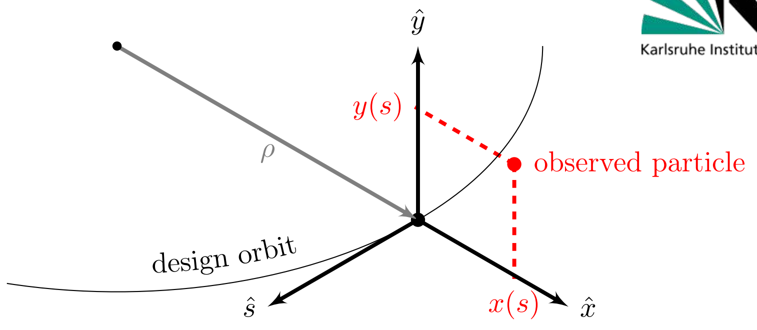
III - Beam optics & beam dynamics

- Basics of transverse beam optics
  - Magnetic lenses ✓
  - Equations of motion and transfer matrices ✓
  - Optic functions and emittance
  - Tune, chromaticity and resonances
  - Dispersion and beam size

- Basics of longitudinal beam dynamics
  - Longitudinal oscillations
  - RF buckets and stable phase

- Oscillations and damping
- Many-particle systems

Recap



Frenet-Serret coordinate

- Right-handed orthogonal coordinate system
  Simplification: $\vec{v} = (0,0,v_s)$ and $\vec{B} = (B_x, B_y, 0)$

- Taylor expansion of the magnetic field in the vicinity of the nominal trajectory ($x = 0$):

$$\frac{e}{p} B_y(x) = \frac{e}{p} B_{y0} + \frac{e}{p} \frac{dB_y}{dx} x + \frac{1}{2!} \frac{e}{p} \frac{d^2 B_y}{dx^2} x^2 + \frac{1}{3!} \frac{e}{p} \frac{d^3 B_y}{dx^3} x^3 + \ldots$$

$$= \frac{1}{\rho} + k\ x + \frac{1}{2!} m\ x^2 + \frac{1}{3!} o\ x^3 + \ldots$$

$\Rightarrow$ dipole + quadrupole + sextupole + octupole + …

- Structures that only consist of **dipole and quadrupole fields** are called **“linear lattices”**. The linear lattice determines trajectory (“orbit”) and focusing properties, higher-order field components are used for correction and error compensation.

Recap II

- **Magnets of the linear lattice:**
  - Dipole $$\frac{1}{\rho}[\mathrm{m}^{-1}] = \frac{e}{p}B_0 = 0.2998\,\frac{B_0[\mathrm{T}]}{p[\mathrm{GeV/c}]}$$
  - Quadrupole $$k[\mathrm{m}^{-2}] = \frac{e}{p}\frac{\mathrm{d}B_y}{\mathrm{d}x} = 0.2998\,\frac{g[\mathrm{T/m}]}{p[\mathrm{GeV/c}]}$$ $\Rightarrow$ "Thin lenses": $\frac{1}{f} = -k\,l$

- **Equations of motion**
  - Hill's differential equation with periodic coefficients

$$\boxed{u'' + K(s)\,u = 0} \quad \text{with} \quad K(s+l) = K(s) \quad \text{and} \quad K(s) = \begin{cases} -k(s) + \frac{1}{\rho(s)^2} & \text{horizontal} \\ +k(s) & \text{vertical} \end{cases}$$

  - General solutions: "Cosinelike" and "Sinelike" functions
  - Wronski determinant $\mathbb{W}$ for two linearly independent solutions
    - Differential equation for $\mathbb{W}$; for transfer matrices applies $\mathbb{W}(s) = 1$

Recap III

- **Transfer matrices** (map the trajectory vector from s_1 to s_2)
  - "Twiss matrix" for periodic structures determined from solutions of the equation of motion

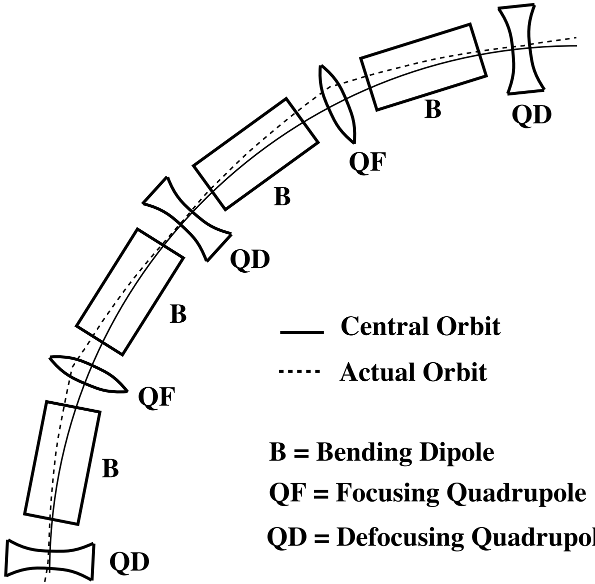
$$\boldsymbol{M} = \begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu - \alpha\sin\mu \end{pmatrix} = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix}$$

with the optical functions: $\alpha(s) = -\frac{\beta(s)'}{2}$, $\beta(s)$, $\gamma(s) = \frac{1+\alpha(s)^2}{\beta(s)}$

  - are symplectic: form a group (multiplication of transfer matrices yields another transfer matrix)
    $\Rightarrow$ conservation of energy

Composite structures

- The individual components of an accelerator can each be represented as transfer matrices $\boldsymbol{M}_i$.
- The entire structure can be written as a product of the individual component matrices: $\boldsymbol{M} = \boldsymbol{M}_n \ldots \boldsymbol{M}_3 \, \boldsymbol{M}_2 \, \boldsymbol{M}_1$



- Drift space: $\boldsymbol{M}_{\text{drift}} = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}$
- Thin quadrupole with $-1/f = kl$: $\boldsymbol{M}_{\text{quad}} = \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ kl & 1 \end{pmatrix}$
- Thick quadrupole: $\boldsymbol{M}_{\text{QF}} = \begin{pmatrix} \cos l\sqrt{|k|} & (1/\sqrt{|k|})\sin l\sqrt{|k|} \\ -\sqrt{|k|}\,\sin l\sqrt{|k|} & \cos l\sqrt{|k|} \end{pmatrix}$

$$\boldsymbol{M}_{\text{QD}} = \begin{pmatrix} \cosh l\sqrt{|k|} & (1/\sqrt{|k|})\sinh l\sqrt{|k|} \\ \sqrt{|k|}\,\sinh l\sqrt{|k|} & \cosh l\sqrt{|k|} \end{pmatrix}$$

- Sector dipole: $\boldsymbol{M}_{\text{dipole}} = \begin{pmatrix} \cos\theta & \rho\sin\theta \\ -(1/\rho)\sin\theta & \cos\theta \end{pmatrix}$

Stability criterion

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- Be $\boldsymbol{M}_p$ the transfer matrix for one period length, then the movement is only stable if the product matrix $\boldsymbol{M} = (\boldsymbol{M}_p)^{Nk}$ for N periods and k turns does not diverge.
- $\boldsymbol{M}$ is given by

$$\boldsymbol{M} = \begin{pmatrix} b & c \\ f & g \end{pmatrix} = \begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu - \alpha\sin\mu \end{pmatrix}$$

- With $\boldsymbol{U} = (u, u')$, the eigenvalues of $\boldsymbol{M}$ are given by $\boldsymbol{MU} = \lambda\boldsymbol{U}$ with $\det(\boldsymbol{M} - \lambda\boldsymbol{I}) = 0$. Since $\det\boldsymbol{M} = 1$ one obtains

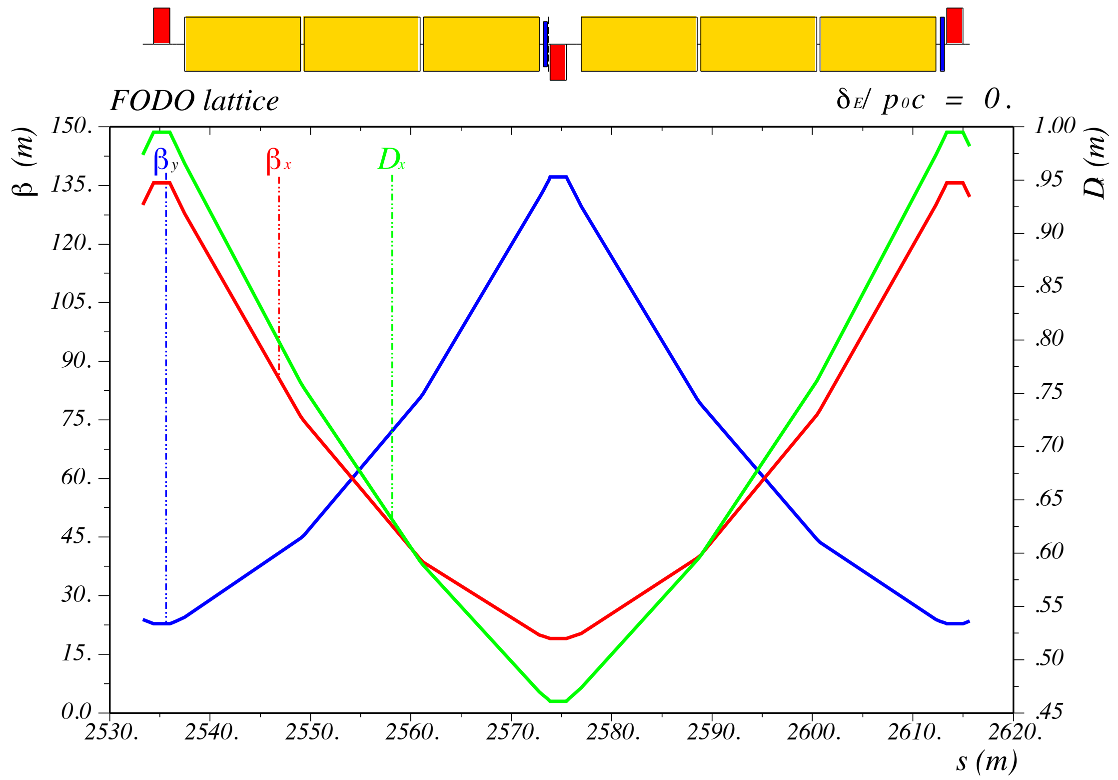
$$0 = (b-\lambda)(g-\lambda) - cf = (bg - cf) - \lambda(g+b) + \lambda^2 = 1 - \lambda(b+g) + \lambda^2$$

- The trace of $\boldsymbol{M}$ is $\mathrm{tr}\,\boldsymbol{M} = 2\cos\mu = (b+g)$. Substitution yields the quadratic equation $0 = 1 - 2\lambda\cos\mu + \lambda^2$.
- The eigenvalues are thus

$$\lambda = \cos\mu \pm \sqrt{\cos^2\mu - 1} = \cos\mu \pm \sqrt{-\sin^2\mu} = \cos\mu \pm i\sin\mu = e^{\pm i\mu}$$

- Stability is ensured for real μ ($|\cos\mu| \leq 1$). That implies $\left|\frac{1}{2}\mathrm{tr}\,\boldsymbol{M}\right| \leq 1$ and $|\lambda| = 1$

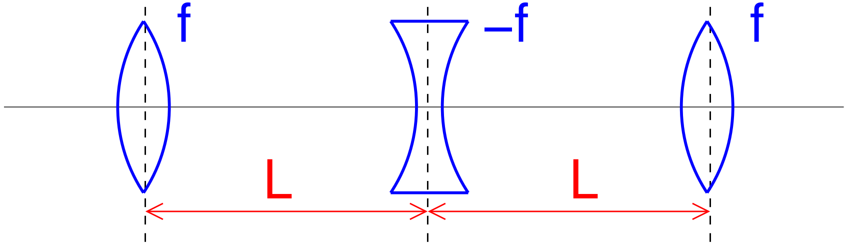
The LEP FODO cell



- Beam optics functions of a LEP FODO cell calculated by “MAD”

Example: FODO structure

- As an example of a simple composite structure, we consider a regular FODO structure in thin lens approximation.



Remark: *L is* *half* *the cell length and*
f is the focal length of a *half* *quadrupole*

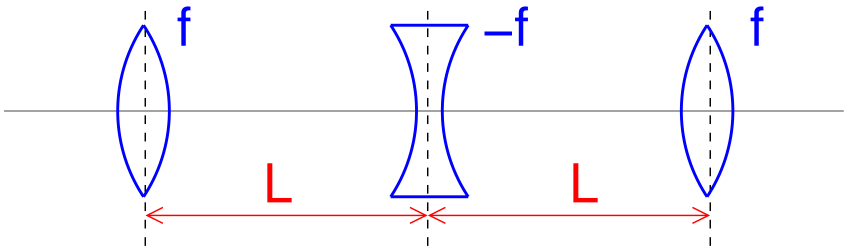
Example: FODO structure

- As an example of a simple composite structure, we consider a regular FODO structure in thin lens approximation.
- From the center of the F quadrupole to the center of the D quadrupole the transfer matrix is

$$\boldsymbol{M}_{\mathrm{FD}} = \begin{pmatrix} 1 & 0 \\ -1/f_2 & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1/f_1 & 1 \end{pmatrix} = \begin{pmatrix} 1 - L/f_1 & L \\ -1/f^* & 1 - L/f_2 \end{pmatrix} \quad \text{mit} \quad \frac{1}{f^*} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{L}{f_1 f_2}$$

- In order to complete the period, the "backward" structure must be traversed, i.e.

$$\boldsymbol{M}_{\mathrm{FD}} = \boldsymbol{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{M}_r = \begin{pmatrix} d & b \\ c & a \end{pmatrix}$$



- The overall matrix is then

$$\boldsymbol{M}_{\mathrm{tot}} = \boldsymbol{M}_r \, \boldsymbol{M} = \begin{pmatrix} ad + bc & 2bd \\ 2ac & ad + cb \end{pmatrix}$$

- Einsetzen mit $f_1 = -f_2 = f$ ergibt

$$\boldsymbol{M}_{\mathrm{FODO}} = \boldsymbol{M}_r \, \boldsymbol{M} = \begin{pmatrix} 1 - 2L^2/f^2 & 2L(1 + L/f) \\ -2(1 - L/f)(L/f^2) & 1 - 2L^2/f^2 \end{pmatrix}$$

Remark: *L is <u>half</u> the cell length and f is the focal length of a <u>half</u> quadrupole*

FODO II

- In the general case, $f_1 \neq -f_2$, the transfer matrix is

$$M_{\text{FODO}} = M_r\, M = \begin{pmatrix} 1 - 2L/f^* & 2L(1 - L/f_2) \\ -(2/f^*)(1 - L/f_1) & 1 - 2L/f^* \end{pmatrix}$$

- From the trace, the stability criterion results in

$$\mathrm{tr}\, M = \left| 2 - \frac{4L}{f^*} \right| < 2 \quad \text{and thus} \quad 0 < \frac{L}{f^*} < 1$$

- For $u = L/f_1$ and $v = L/f_2$ and with $L/f^* = L/f_1 + L/f_2 - L^2/(f_1 f_2)$ we get

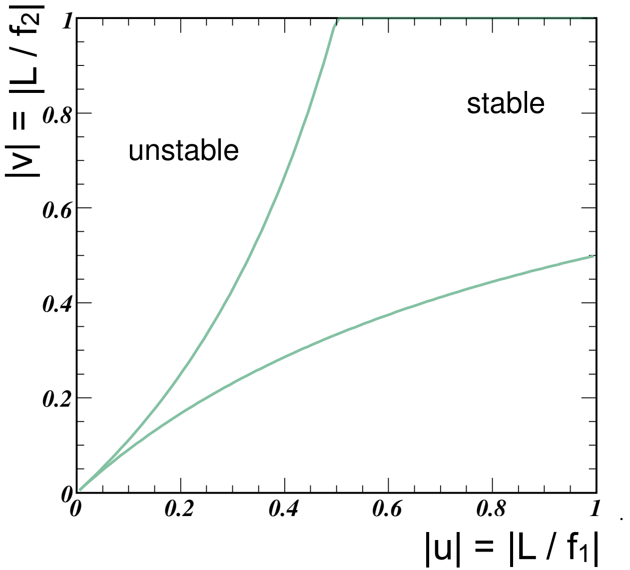
$$0 < u + v - uv < 1$$

- If one solves these inequalities, one obtains the boundaries of the stable region:

$$|u| = 1, \qquad |v| = \frac{|u|}{1 - |u|},$$

$$|v| = 1, \qquad |u| = \frac{|v|}{1 + |v|}$$

„necktie diagram“



In phase space

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- The equations of the particle trajectory (solution of Hill's differential equation) and its derivative with respect to s form a parametric representation of an ellipse in u-u' at location s:

$$u(s) = a\sqrt{\beta(s)}\ \cos\left(\Phi(s) - \delta\right)$$

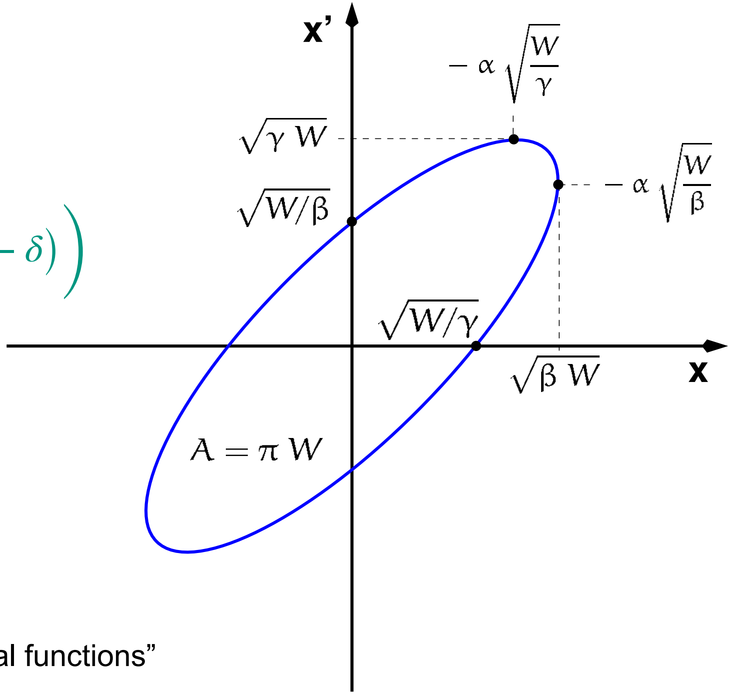
$$u'(s) = -\frac{a}{\sqrt{\beta(s)}}\left(\sin\left(\Phi(s)-\delta\right)\ -\ \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}\beta(s)\ \cos\left(\Phi(s)-\delta\right)\right)$$

- In absence of dissipative forces, the surface of this ellipse, πA^2, is independent of s (Liouville's theorem).
- The parameter a^2 is referred to as the "Courant Snyder invariant".

$$\gamma u^2 + 2\alpha u u' + \beta u'^2 = W_u$$

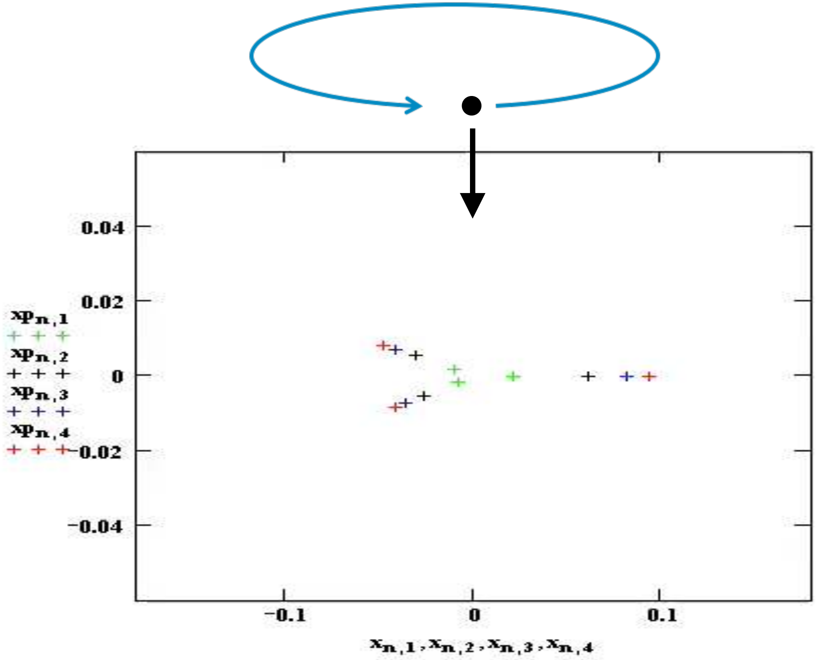
$$\alpha = -\frac{1}{2}\beta' \quad \text{and} \quad \gamma = \frac{1+\alpha^2}{\beta}$$

- The location dependent functions $\alpha(s)$, $\beta(s)$ and $\gamma(s)$ are called the "optical functions" (formerly often known as "Twiss parameters").

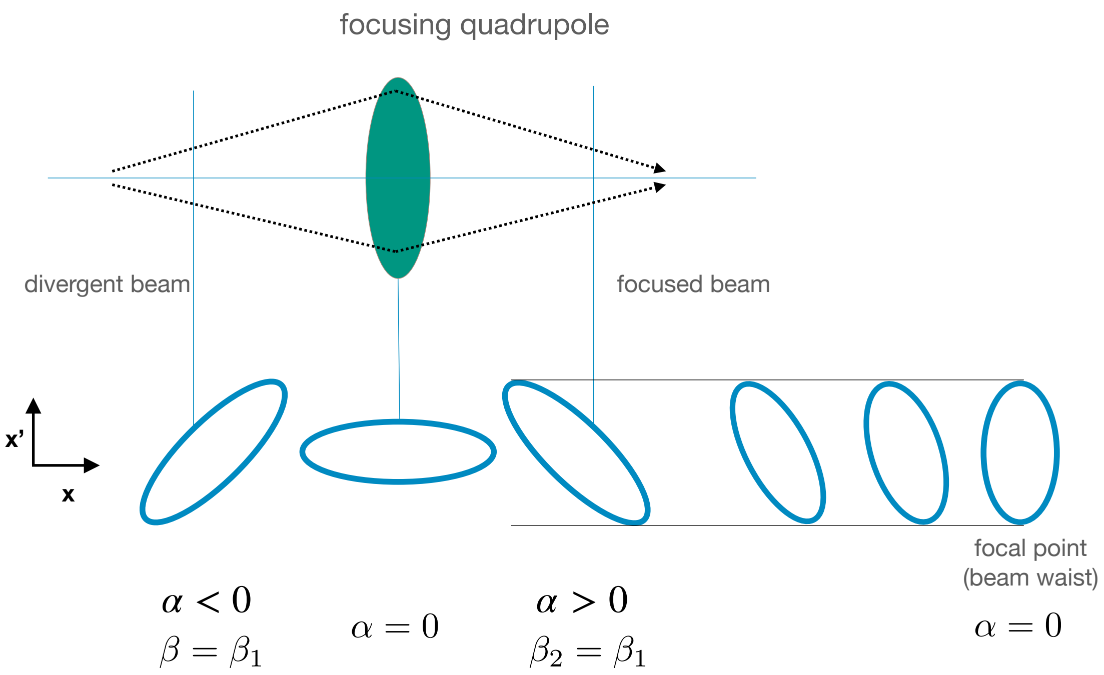


Phase space ellipse II

- In the case of a ring or matched cell, the periodicity imposes equality on the input and output α and β values.
- This means that the particle returns after each turn to the same ellipse but with a different phase.



Evolution of phase space ellipse along the lattice



Beta matrix

- Transformation of the solutions of Hill's equation gives (remember $\phi(s) = \Phi(s) - \delta$):

$$\cos(\phi(s)) = \frac{u}{a\,\sqrt{\beta(s)}} \quad \text{und} \quad -\sin(\phi(s)) = \frac{\sqrt{\beta(s)}\;u'}{a} + \frac{\alpha(s)\;u}{a\sqrt{\beta(s)}}$$

- With $\sin^2\phi(s) + \cos^2\phi(s) = 1$ and $\gamma = (1+\alpha^2)/\beta$ follows

$$\frac{u^2}{\beta(s)} + \left(\beta(s)u'^2 + \frac{2\sqrt{\beta(s)}\alpha(s)uu'}{\sqrt{\beta(s)}} + \frac{\alpha^2(s)u^2}{\beta(s)}\right) = a^2 = W_u$$

$$\gamma(s)u^2 + 2\alpha(s)uu' + \beta(s)u'^2 = a^2$$

- The elliptic equation can also be written in the following way:

$$W_u = \begin{pmatrix} u & u' \end{pmatrix} \begin{pmatrix} \gamma & \alpha \\ \alpha & \beta \end{pmatrix} \begin{pmatrix} u \\ u' \end{pmatrix} = \boldsymbol{U}^T\,\boldsymbol{B}^{-1}\,\boldsymbol{U}$$

- The "beta matrix" $\boldsymbol{B}$ with $\det \boldsymbol{B} = 1$ is given by $\quad \boldsymbol{B} = \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}$

Transformation of the beta matrix

- The trajectory vector transforms like $U_2 \;=\; M\; U_1$. For matrix M applies $M^{-1}M = 1$ and $M^T(M^T)^{-1} = 1$.
- Using the matrix relations $A^T B^T = (BA)^T$ and $A^{-1}B^{-1} = (BA)^{-1}$ the definition of the beta matrix can be re-written:

$$W_u \;=\; U_1^T\; B_1^{-1}\; U_1 \;=\; U_1^T\big(M^T(M^T)^{-1}\big)\; B_1^{-1}\big(M^{-1}M\big)\; U_1$$

$$=\; U_1^T M^T\big((M^T)^{-1}\; B_1^{-1}M^{-1}\big)M\; U_1 \;=\; U_1^T M^T\big((M^T)^{-1}\;(MB_1)^{-1}\big)M\; U_1$$

$$=\; U_1^T M^T\big(\; MB_1M^T\big)^{-1}M\; U_1 \quad=\; (MU_1)^T\big(\; MB_1M^T\big)^{-1}M\; U_1$$

with the transformation relation $U_2^T \;=\; (MU_1)^T$ follows immediately:

$$W_u =\; U_2^T\big(\; MB_1M^T\big)^{-1}\; U_2$$

Because W_u is independent of s also applies:

$$W_u =\; U_2^T\; B_2^{-1}\; U_2$$

By comparison we obtain the transformation relation of the beta matrix:

$$B_2 \;=\; MB_1M^T$$

Transport of β

- **Example:** With the transformation relation $\boldsymbol{B}_2 = \boldsymbol{M}\boldsymbol{B}_1\boldsymbol{M}^T$ one can observe the evolution of the beta function in the area of a symmetry point.
- In a symmetry point $s = 0$ following conditions apply: $\beta = \beta^*$ and $\alpha^* = 0$.

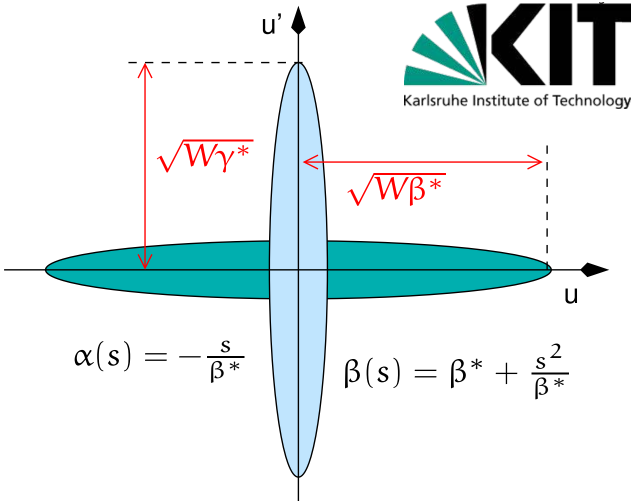
  A simple translation over a distance l is given by

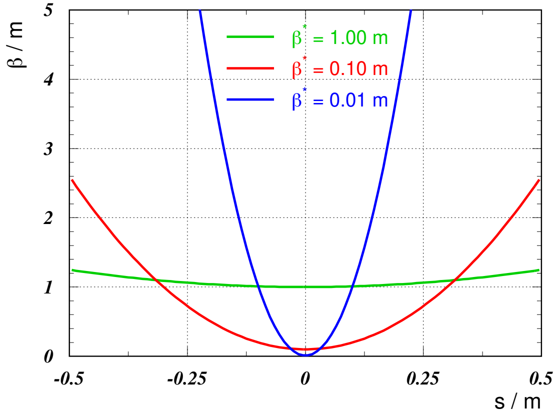
$$(u, u') \;\rightarrow\; (u + lu', u')$$

- As a consequence the beta matrix is transformed like

$$\boldsymbol{B}_2 = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \beta^* & 0 \\ 0 & 1/\beta^* \end{pmatrix} \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}$$
$$= \begin{pmatrix} \beta^* + (s^2/\beta^*) & s/\beta^* \\ s/\beta^* & 1/\beta^* \end{pmatrix}$$

- The smaller the beta function, the more divergent is the beam (Consequence of “Liouville’s theorems”).





β and aperture

- For the symmetry point at $s = 0$ applies $\beta = \beta^*$ and $\alpha^* = 0$. By definition $\gamma = (1+\alpha^2)/\beta$.
- For a simple drift space of length L the value of the beta function expressed as a function of L and β^* is:

$$\boxed{\beta(s = L) \;=\; \beta^* + \frac{L^2}{\beta^*}}$$

Example: Be $\beta^* = \ 1\,\mathrm{cm}$ at the symmetry point and the RMS beam size $\sigma_u = \ 0.1\,\mathrm{mm}$.

- The wall of the vacuum chamber should be at least 10 standard deviations (beam sizes) from the center of the beam.
- At a distance of 5 m from the symmetry point β is

$$\beta_L \;=\; 10^{-2}\ \mathrm{m} + \frac{25\ \mathrm{m}^2}{10^{-2}\ \mathrm{m}} \;\approx\; 2.5\ \mathrm{km}$$

β and aperture II

- For the beam size applies $\sigma \propto \sqrt{\beta}$. That consequently means

$$\sigma_L \propto \sqrt{\beta_L} \quad \text{und} \quad \sigma^* \propto \sqrt{\beta^*}$$

$$\frac{\sigma_L}{\sigma^*} = \frac{\sqrt{\beta_L}}{\sqrt{\beta^*}} \quad \text{and therefore} \quad \sigma_L = \sigma^* \sqrt{\frac{\beta_L}{\beta^*}} = 100 \cdot 10^{-6}\ \text{m} \sqrt{\frac{2500}{10^{-2}}}$$

$$= 10^{-4}\ \text{m} \sqrt{25 \cdot 10^4} = 5 \cdot 10^{-2}\ \text{m}$$

- The minimum diameter ("$10\,\sigma$") of the vacuum chamber at a distance of 5 m from the symmetry point is thus

$$d = 2\ 10\ \sigma_L = 1\ \text{m}\ !$$

- In the large detectors at interaction (symmetry) points the beam increases rapidly towards both sides
  - $\rightarrow$ Luminosity reduction due to the "hourglass effect"
  - $\rightarrow$ Large beam in strong final focus quadrupoles $\Rightarrow$ synchrotron radiation, background, ...

Liouville's theorem

- *"Under the influence of conservative forces, the phase volume remains constant."*

- Liouville can be used to describe the properties of the beam as a whole.
- Starting point: Time evolution of a 6d phase space element
  Number of particles in the phase space element of density Ψ:

$$\Psi(x, y, s, p_x, p_y, p_s)\ dx\ dy\ ds\ dp_x\ dp_y\ dp_s$$

- The movement of the particle generates the current $\vec{j} = (\Psi\dot{x}, \Psi\dot{y}, \Psi\dot{s}, \Psi\dot{p}_x, \Psi\dot{p}_y, \Psi\dot{p}_s)$,
  where the time derivative is taken after the time τ along the trajectory of the phase space element: $\dot{x} = dx/d\tau$
- The current in phase space must fulfil the continuity equation:

$$\nabla\vec{j} + \frac{\partial\Psi}{\partial\tau} = 0$$

Liouville's theorem II

- Assuming that location and momentum are independent of each other, the following is true

$$-\frac{\partial \Psi}{\partial \tau} = \nabla_r \left(\Psi\, \dot{\vec{r}}\right) + \nabla_p \left(\Psi\, \dot{\vec{p}}\right) = \dot{\vec{r}}\, \nabla_r \Psi + \Psi \left(\nabla_r \dot{\vec{r}}\right) + \dot{\vec{p}}\, \nabla_p \Psi + \Psi \left(\nabla_p \dot{\vec{p}}\right)$$

- The temporal derivative does not depend of the location $\vec{r}$ itself ($\beta = pc/E$)

$$\frac{\dot{\vec{r}}}{c} = \frac{c\,\vec{p}}{\sqrt{c^2p^2 + m^2c^4}} \quad \text{and therefore} \quad \nabla_r \dot{\vec{r}} = 0$$

- From the Lorentz force equation follows

$$\nabla_p \dot{\vec{p}} = \frac{e}{c} \nabla_p [\dot{\vec{r}} \times \vec{B}] = \frac{e}{c} \vec{B} \left(\nabla_p \times \dot{\vec{r}}\right) - \frac{e}{c} \dot{\vec{r}} \left(\nabla_p \times \vec{B}\right)$$

- The second term vanishes, because $\vec{B}$ does not depend on the momentum. In the first term is $\nabla_p \times \dot{\vec{r}} = 0$, as

$$\left(\nabla_p \times \dot{\vec{r}}\right)_x = \frac{\partial \dot{s}}{\partial p_y} - \frac{\partial \dot{y}}{\partial p_s} \quad \text{and} \quad \left(p^2 = p_x^2 + p_y^2 + p_s^2\right)$$

$$\frac{\partial \dot{s}}{\partial p_y} = \frac{\partial}{\partial p_y} \frac{c\, p_s}{(p^2 + m^2c^2)^{1/2}} = \frac{c\, p_y\, p_s}{(p^2 + m^2c^2)^{3/2}} = \frac{\partial \dot{y}}{\partial p_s}$$

- Analog calculation for remaining components in the end provides $\nabla_p \dot{\vec{p}} = 0$

Liouville's theorem III

- Therefore we can write
$$\frac{\partial\Psi}{\partial\tau} + \nabla_r \Psi \dot{\vec{r}} + \nabla_p \Psi \dot{\vec{p}} = \frac{d\Psi}{d\tau} = 0$$
→ Thus, the phase space density is invariant in time.
- The invariance can also be shown using the properties of the transfer matrix:
→ For a determinant consisting of the components of six vectors $\vec{x}$ defining a 6d phase space volume and transforming like $y_i = \boldsymbol{M}\, x_i$, one can show that
$$|\vec{y_1}, \vec{y_2}, \ldots, \vec{y_6}| = |\boldsymbol{M}|\, |\vec{x_1}, \vec{x_2}, \ldots, \vec{x_6}|$$
→ For beam steering systems $|\boldsymbol{M}|$ is the Wronski determinant $\mathbb{W}$ with
$$\mathbb{W} = |\boldsymbol{M}| = 1.$$
→ Thus, the phase space volume is constant.

Emittance

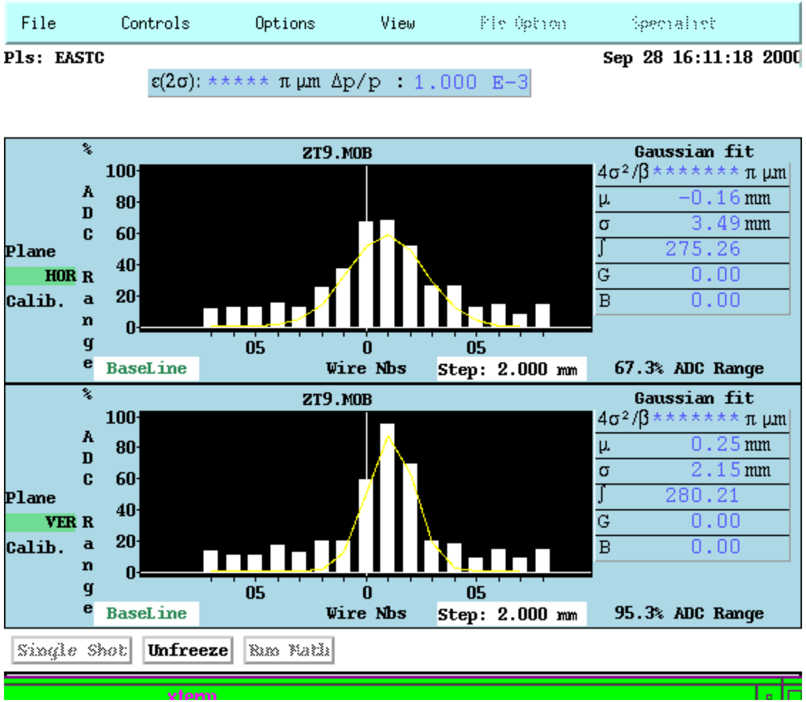
- As a consequence of Liouville's theorem the W is locationally invariant. W corresponds to a "single particle emittance".
- In good approximation, the transverse charge density distribution in a particle beam is Gaussian.
- The beam size is defined as the standard deviation of the charge density distribution:

$$\sigma_u(s) = \sqrt{\varepsilon\beta(s)}$$

- The (equilibrium) emittance is thus given by

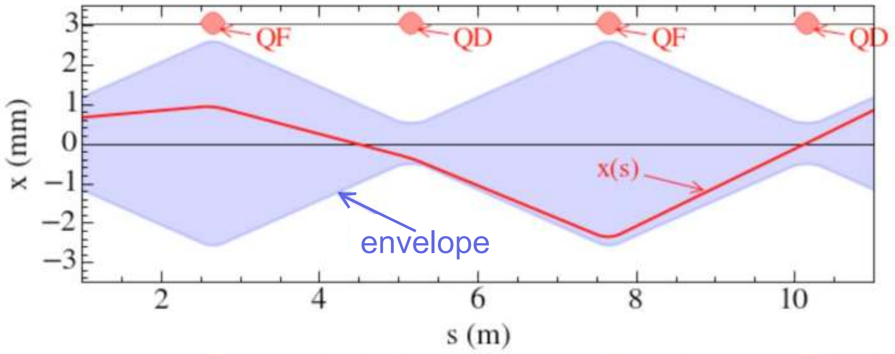
$$\varepsilon = \frac{\sigma_u^2(s)}{\beta(s)}.$$

- The maximum possible emittance limited by the aperture is called the acceptance.



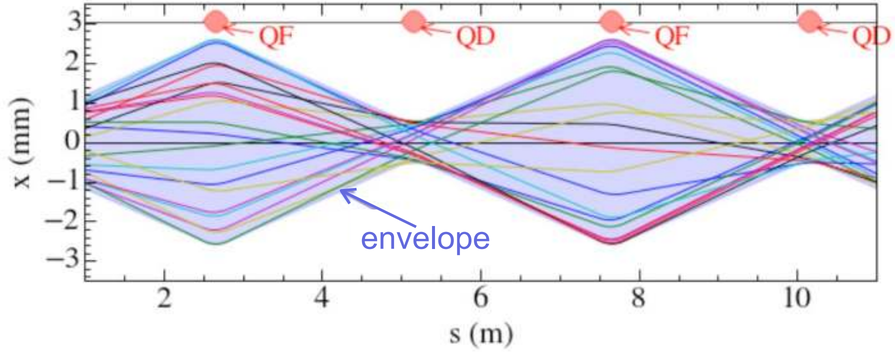
Trajectories and β

- The trajectories $x_i(s)$ are individual solutions of the equation of motion and have different initial conditions.



- The envelope $E(s)$ of all trajectories is proportional to beta function and emittance:

$$E(s) = \sqrt{\epsilon\,\beta(s)}$$



Optical functions

There are two ways of looking at the optics functions:

- The first is to regard them as a **parametric way of expressing the equation of motion and its solution**. This interpretation makes the bridge from tracking single particles to the wider view of calculating beam envelopes.

- The second is to regard them as purely **geometric parameters for defining ellipses and hence beam envelopes**. Dropping the strict correspondence to individual particles can lead to some interesting extensions such as the inclusion of scattering.

The beam matrix

- One can specify a covariance matrix $\boldsymbol{C}$ for the particle distribution in the phase space. The elements of this matrix are given by $C_{ij} = \langle ij \rangle - \langle i \rangle \langle j \rangle$, where i and j represent the coordinates.
- The covariance matrix

$$\boldsymbol{C} = \begin{pmatrix} C_{ii} & C_{ij} \\ C_{ji} & C_{jj} \end{pmatrix} = \begin{pmatrix} \varepsilon\beta & -\varepsilon\alpha \\ -\varepsilon\alpha & \varepsilon\gamma \end{pmatrix}$$

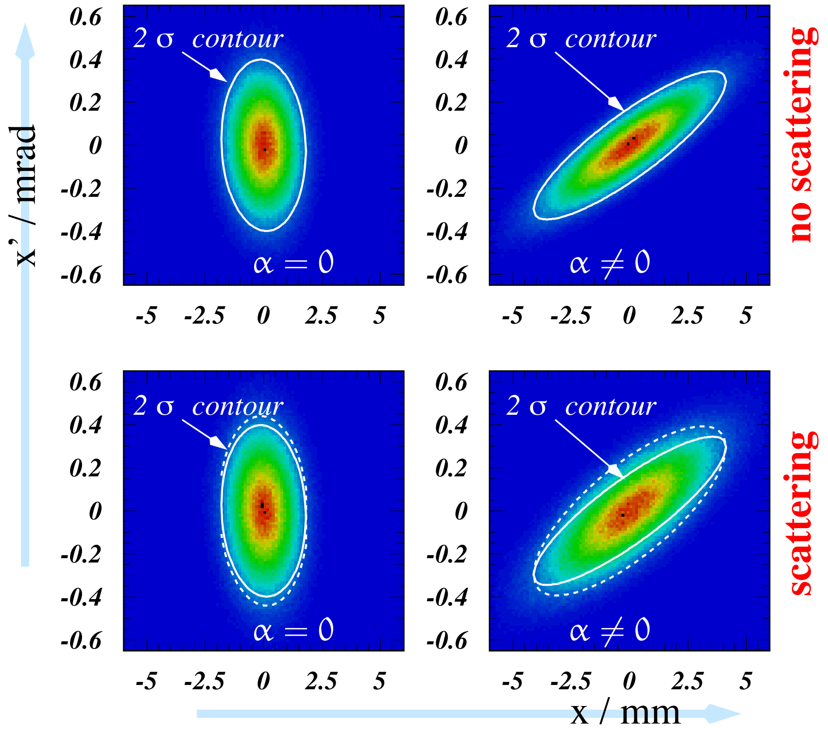
- With

$$\det \boldsymbol{C} = \varepsilon^2 \left(\beta\gamma - \alpha^2\right) = \varepsilon^2$$

- the RMS emittance is given by

$$\varepsilon_{\mathrm{RMS}}^2 \overset{\mathrm{def}}{=} \det \boldsymbol{C} = C_{xx}\, C_{x'x'} - C_{xx'}^2$$

- This definition is valid for any distribution and can easily be extended to 6D.



Emittance terms

- **Single particle emittance**
  - Courant Snyder invariant: $W_u \;=\; \gamma u^2 + 2\alpha u u' + \beta u'^2$
  - Area of the phase space ellipse delimited by the possible trajectory vectors $\boldsymbol{U}$ of a single particle at a certain location divided by π: $W_u = A/\pi$
- **Equilibrium emittance**
  - Effective emittance resulting from the effects of both radiation damping and quantum excitation by synchrotron radiation.
  - … is determined by the lattice design.
- **RMS emittance (“root mean square”)**
  - Derived from the charge density distribution of the beam
  - The “RMS beam size” is given by $\sigma_u^2 \;=\, < u^2 > - < u >^2$
  - Thus, the RMS emittance is $\varepsilon_{\mathrm{RMS}} \;=\; \sigma_u^2(s) \;/\; \beta(s)$
  - The RMS emittance can be equal to the equilibrium emittance.

Energy independent emittance

- All emittances defined so far are dependent on the beam energy.
  "Adiabatic damping" during acceleration process: $\varepsilon \;\propto\; 1/E_0$

  $\rightarrow$ Phase space elements in transverse beam dynamics:
  $\Delta u \Delta u'$ instead of correctly according to Liouville $\Delta u \Delta p_u$ with $\Delta p_u = p_0 u'$

  $\rightarrow$ During acceleration p_0 increases, thus the geometric ("physical") emittances
  $\Delta u \Delta u'$ decrease to keep the product $\Delta u \Delta p_u$ constant.

- Definition of a "normalized emittance":
  where $\beta = v/c$ and $\gamma = E/(m_0 c^2)$.

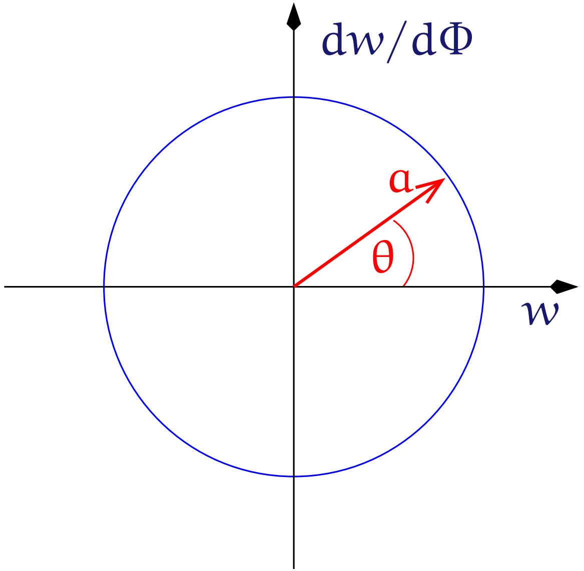
$$\boxed{\varepsilon_N \;=\; (\beta\,\gamma)\;\varepsilon}$$

Normalised coordinates

- Reminder: The particle trajectory is given by $u(s) = a\sqrt{\beta(s)}\,\cos(\Phi(s))$
- A practical normalization (“Floquet's coordinates”) transforms the phase space ellipse into a circle:

$$w(\theta) = \frac{u}{\sqrt{\beta}} = a\cos\Phi$$

$$\frac{dw}{d\Phi} = \sqrt{\beta}\,u' + \frac{\alpha}{\sqrt{\beta}}\,u = -a\sin\Phi$$

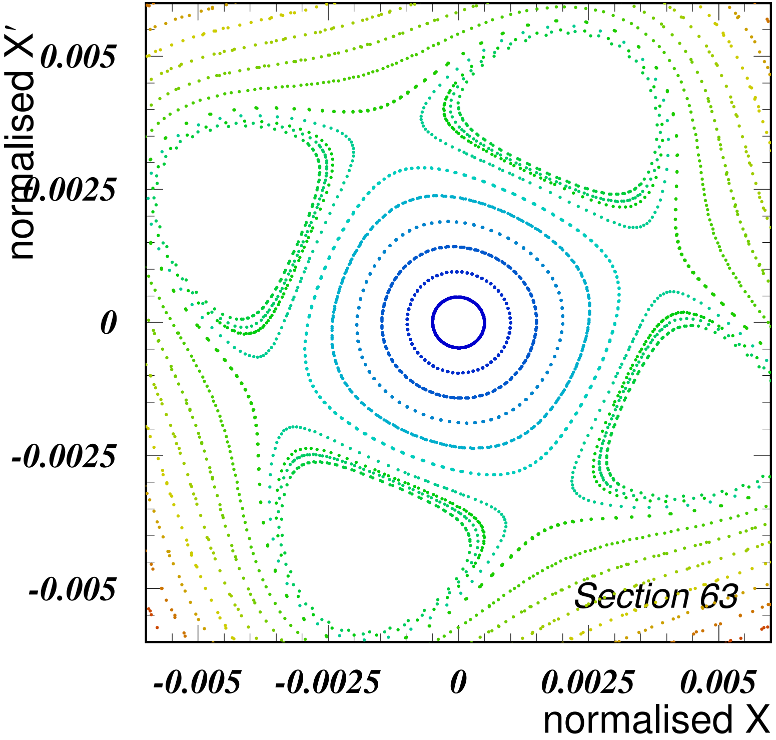


- The Courant Snyder invariant immediately becomes

$$w^2 + \left(\frac{dw}{d\Phi}\right)^2 = a^2$$

- Also convenient: Angle-action-coordinated
- Non-linearities lead to deviations of the circular shape

Normalised coordinates II



Example:

Simulation of the horizontal phase space in the CERN PS

Summary

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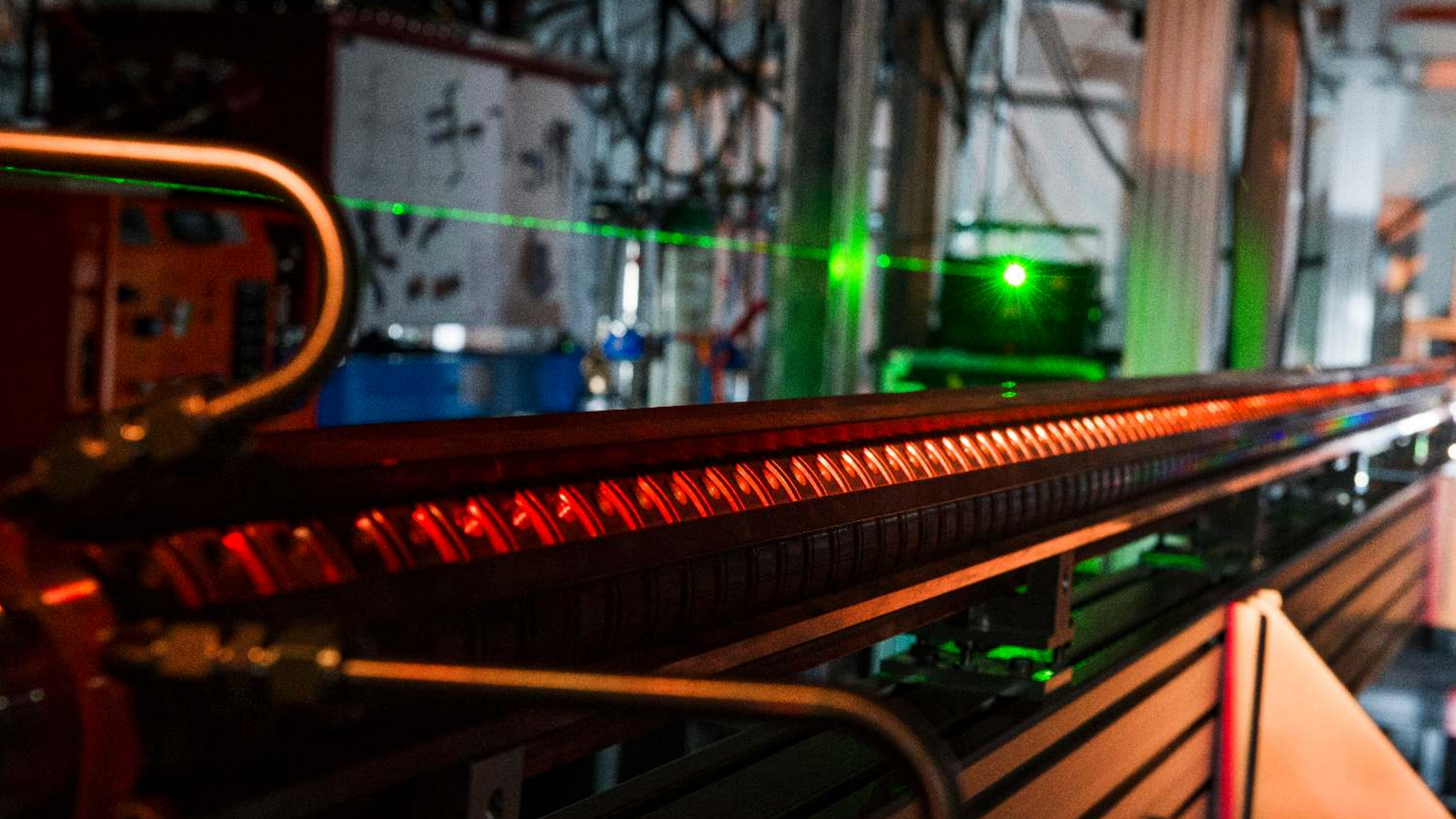
Matrix formalism

- Matrices can be derived either from solution of equation of motion or optical properties with the same numerical result
- The trajectory is stable, if stability criterion is fulfilled: $$\left|\frac{1}{2}\mathrm{tr}\,\boldsymbol{M}\right| \leq 1$$

Phase space

- Solutions of the equation of motion represent an ellipse in phase space.
- The area of the ellipse (“Courant Snyder invariant”) is conserved as a consequence of Liouville’s theorem.
- Transition for single particle to particle ensemble: phase space area occupied by the beam defines beam emittance.

$$\sigma(s) = \sqrt{\epsilon\,\beta(s)}$$



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Institute for Beam Physics and Technology

Particle Accelerator Physics

Anke-Susanne Müller, Axel Bernhard, Bastian Härer, Bennet Krasch, Nathan Ray

III - Beam optics & beam dynamics

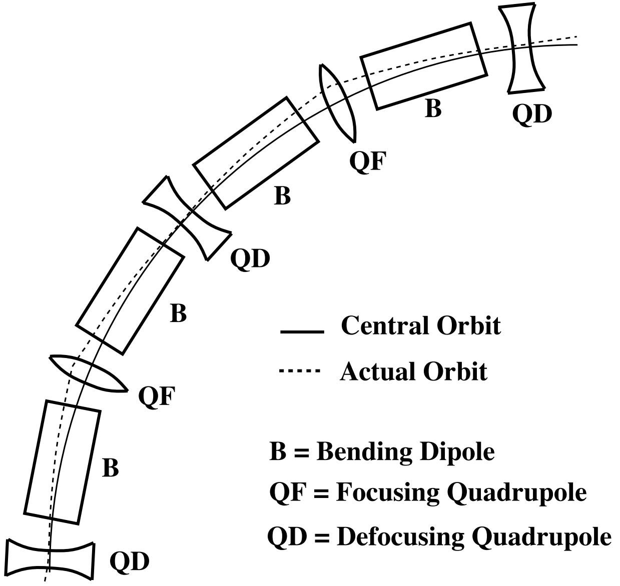
- Basics of transverse beam optics
  - Magnetic lenses ✓
  - Equations of motion and transfer matrices ✓
  - Optic functions and emittance ✓
  - Tune, chromaticity and resonances
  - Dispersion and beam size

- Basics of longitudinal beam dynamics
  - Longitudinal oscillations
  - RF buckets and stable phase

- Oscillations and damping
- Many-particle systems

Recap: Composite structures

- The individual components of an accelerator can each be represented as transfer matrices $\boldsymbol{M}_i$.
- The entire structure can be written as a product of the individual component matrices: $\boldsymbol{M} = \boldsymbol{M}_n \ldots \boldsymbol{M}_3 \boldsymbol{M}_2 \boldsymbol{M}_1$



- Drift space: $\boldsymbol{M}_{\text{drift}} = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}$
- Thin quadrupole with $-1/f = kl$: $\boldsymbol{M}_{\text{quad}} = \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ kl & 1 \end{pmatrix}$
- Thick quadrupole: $\boldsymbol{M}_{\text{QF}} = \begin{pmatrix} \cos l\sqrt{|k|} & (1/\sqrt{|k|})\sin l\sqrt{|k|} \\ -\sqrt{|k|}\,\sin l\sqrt{|k|} & \cos l\sqrt{|k|} \end{pmatrix}$

  $\boldsymbol{M}_{\text{QD}} = \begin{pmatrix} \cosh l\sqrt{|k|} & (1/\sqrt{|k|})\sinh l\sqrt{|k|} \\ \sqrt{|k|}\,\sinh l\sqrt{|k|} & \cosh l\sqrt{|k|} \end{pmatrix}$
- Sector dipole: $\boldsymbol{M}_{\text{dipole}} = \begin{pmatrix} \cos\theta & \rho\sin\theta \\ -(1/\rho)\sin\theta & \cos\theta \end{pmatrix}$

Recap: Phase space

- The equations of the particle trajectory (solution of Hill's differential equation) and its derivative with respect to s form a parametric representation of an ellipse in u-u' at location s:

$$u(s) = a\sqrt{\beta(s)}\ \cos\left(\Phi(s) - \delta\right)$$

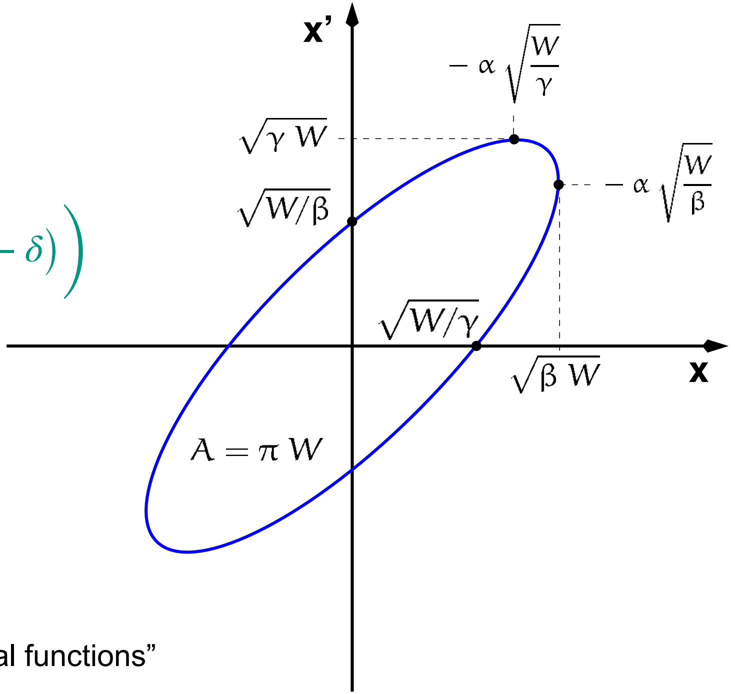
$$u'(s) = -\frac{a}{\sqrt{\beta(s)}}\left(\sin\left(\Phi(s) - \delta\right)\ -\ \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}\beta(s)\ \cos\left(\Phi(s) - \delta\right)\right)$$

- In absence of dissipative forces, the surface of this ellipse, πa^2, is independent of s (Liouville's theorem).
- The parameter a^2 is referred to as the "Courant Snyder invariant".

$$\gamma u^2 + 2\alpha u u' + \beta u'^2 = W_u$$

$$\alpha = -\frac{1}{2}\beta' \quad \text{and} \quad \gamma = \frac{1+\alpha^2}{\beta}$$

- The location dependent functions $\alpha(s)$, $\beta(s)$ and $\gamma(s)$ are called the "optical functions" (formerly often known as "Twiss parameters").



Recap: Emittance terms

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- **Single particle emittance**
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  - The RMS emittance can be equal to the equilibrium emittance.

Tunes and resonances

- The "tune" or working point of an accelerator with N periods is defined as the number of betatron oscillations per revolution.

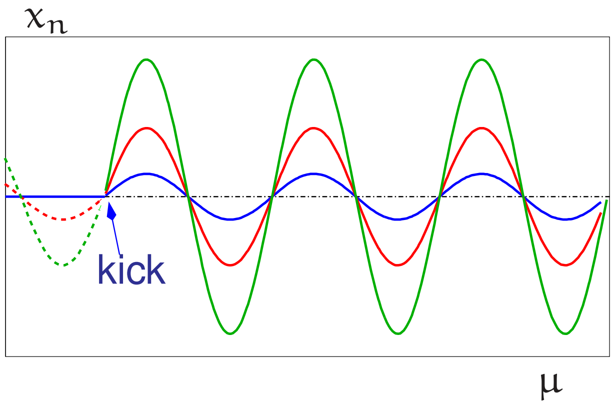
$$Q = \frac{N\,\mu}{2\pi} = \frac{1}{2\pi}\int_0^L \frac{\mathrm{d}s}{\beta(s)} = \frac{1}{2\pi}\oint \frac{\mathrm{d}s}{\beta(s)}$$

- If the tune reaches certain rational numbers, the movement becomes unstable.

- Condition for optical resonances:

$$mQ_x + nQ_y = p \quad \text{mit} \quad m, n, p \in \mathbb{Z}$$

- Order of the resonance: $|m| + |n|$



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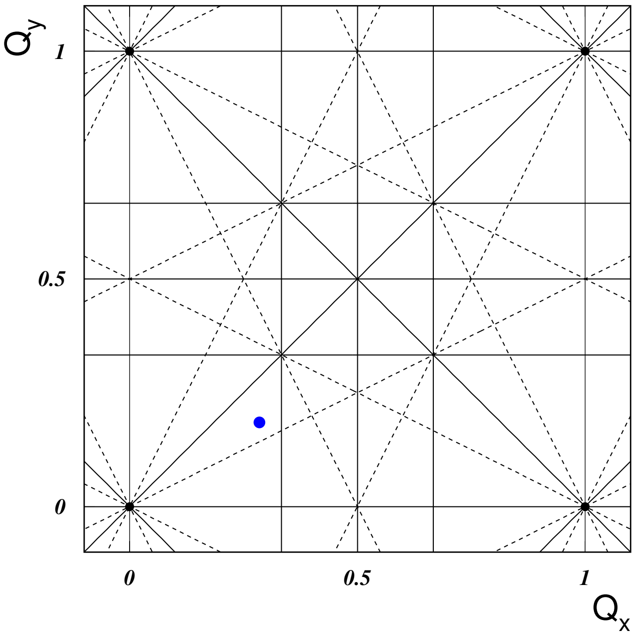
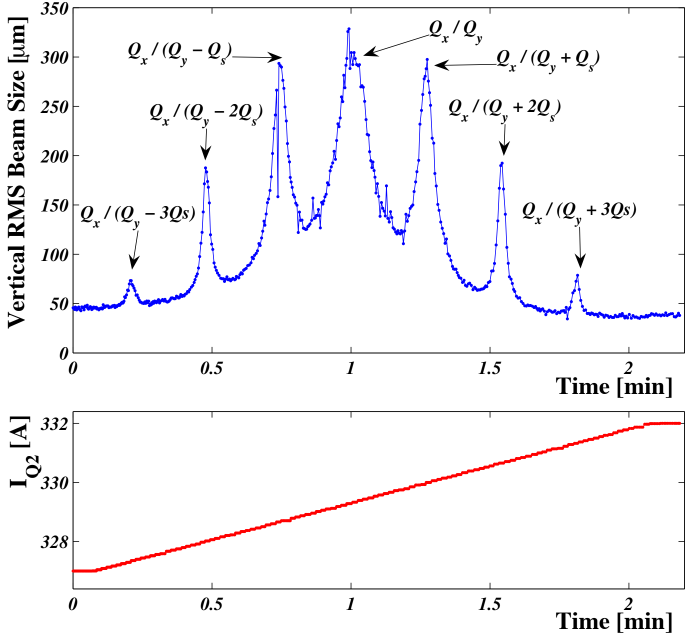


Synchro betatron resonances

- In a real accelerator, coupling resonances often occur between all three oscillation planes (hence the name “synchro-betatron resonances”):

$$mQ_x + nQ_y + lQ_s \;=\; p \text{ mit } m, n, l, p \in \mathbb{Z}$$

- **Example:** Vertical beam size when crossing synchro-betatron resonances by changing the quadrupole gradients:



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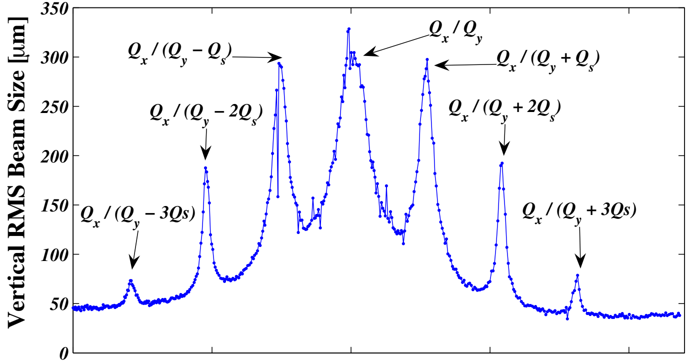
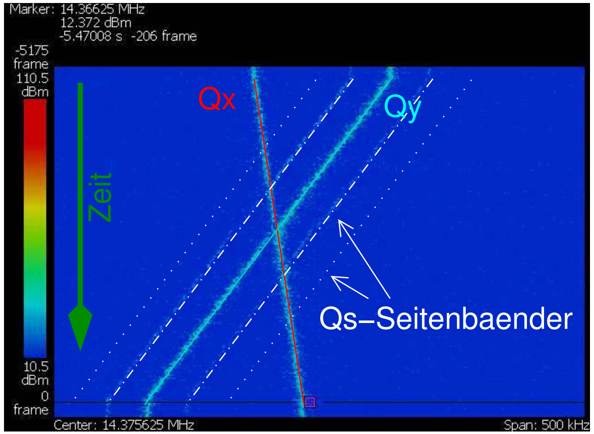




Gradient errors

- The unperturbed transport matrix for one turn is given by ($\Phi_0 = N\mu = 2\pi Q$)

$$\boldsymbol{M}_0 = \begin{pmatrix} \cos\Phi_0 + \alpha\sin\Phi_0 & \beta\sin\Phi_0 \\ -\gamma\sin\Phi_0 & \cos\Phi_0 - \alpha\sin\Phi_0 \end{pmatrix}$$

- Representations for an unperturbed and a perturbed quadrupole are

$$\boldsymbol{m}_0 = \begin{pmatrix} 1 & 0 \\ -k_0 \mathrm{d}s & 1 \end{pmatrix} \quad \text{und} \quad \boldsymbol{m} = \begin{pmatrix} 1 & 0 \\ -(k_0 + \Delta k)\mathrm{d}s & 1 \end{pmatrix}$$

- In order to incorporate the perturbation into the overall matrix, one goes "backwards" through the affected quadrupole:

$$\boldsymbol{M} = \boldsymbol{m}\boldsymbol{m}_0^{-1}\boldsymbol{M}_0 \quad \text{with} \quad \boldsymbol{m}\boldsymbol{m}_0^{-1} = \begin{pmatrix} 1 & 0 \\ -\Delta k \,\mathrm{d}s & 1 \end{pmatrix}$$

$$\boldsymbol{M} = \begin{pmatrix} \cos\Phi_0 + \alpha\sin\Phi_0 & \beta\sin\Phi_0 \\ -\Delta k\ \mathrm{d}s\ (\cos\Phi_0 + \alpha\sin\Phi_0) - \gamma\sin\Phi_0 & -\Delta k\ \mathrm{d}s\ \beta\sin\Phi_0 + \cos\Phi_0 - \alpha\sin\Phi_0 \end{pmatrix}$$

- Since $\mathrm{tr}\,\boldsymbol{M} = 2\cos\Phi$, the change in $\cos\Phi$ can be written as

$$\Delta(\cos\Phi) = -\Delta\Phi\ \sin\Phi_0 = (\mathrm{tr}\,\boldsymbol{M} - \mathrm{tr}\,\boldsymbol{M}_0)/2 = -\frac{\sin\Phi_0}{2}\ \beta\ \Delta k\ \mathrm{d}s$$

$$|\Delta\Phi| = \frac{\beta\ \Delta k\ \mathrm{d}s}{2} = 2\pi\Delta Q \quad \text{and} \quad \Delta Q = \frac{1}{4\pi}\oint \beta(s)\Delta k(s)\,\mathrm{d}s$$

Half-integer resonance

- A more detailed derivation based on the inhomogeneous equation of motion with perturbation term yields higher-order terms to the static tune displacement.
- The following approximation can be obtained for small perturbations:

$$\Delta Q = -\frac{1}{4\pi}\oint \beta(s)\ \Delta k(s)\ \mathrm{d}s$$
$$-\frac{1}{4\pi\ \sin(2\pi Q_0)}\oint \beta(s)\ \Delta k(s)\sin\left(2Q_0[\pi - \chi(s)]\right)\ \mathrm{d}s$$

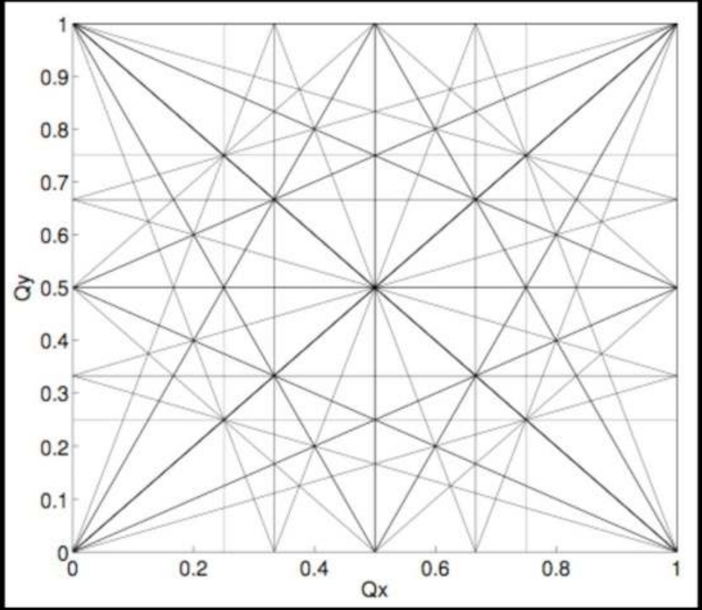
where $\mathrm{d}s = Q_0\beta(s)\mathrm{d}\chi$

- The first term corresponds to a tune shift, the second is an oscillation term that averages out over many turns unless Q_0 is half-integer.
- Resonance condition: $Q_0 \neq \frac{1}{2}n$

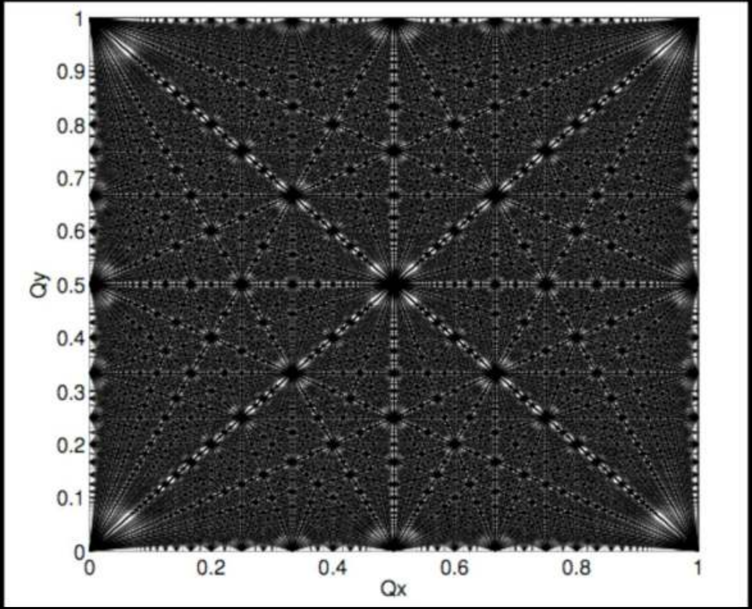
Tune-Diagramme

$$mQ_x + nQ_y + lQ_s \ = \ p \qquad\qquad \text{mit} \quad m, n, l, p \in$$

4. Ordnung



14. Ordnung



Momentum offset: dispersion orbit

- The deflection angle in the dipole magnets depends on the particle energy.
- In a storage ring, dipoles deflect horizontally, therefore mainly horizontal dispersion.
- Vertical dispersion is caused by field errors or misalignment for example.

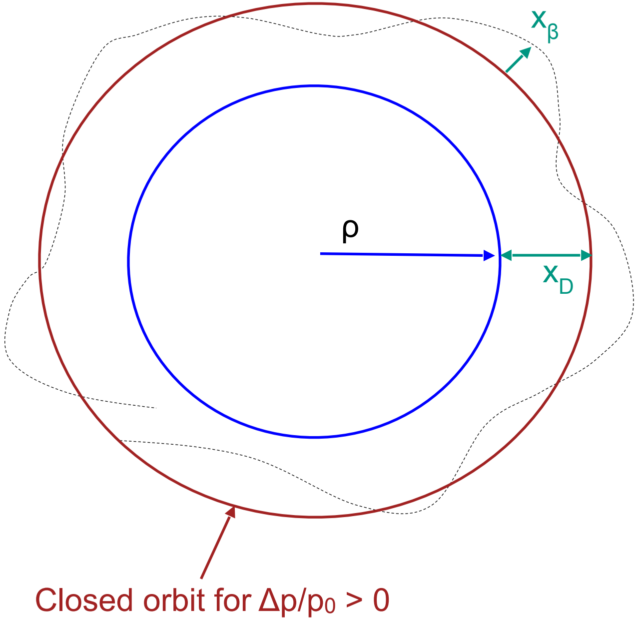
- As a consequence, the equation of motion becomes an inhomogeneous differential equation with an additional term on the right hand side:

$$x'' + K(s)\,x = \frac{\Delta p}{p_0}\frac{1}{\rho}$$

- The horizontal particle position at location s relative to the nominal orbit consists of two components:

$$x(s) \;=\; x_\beta(s) \;+\; x_\mathrm{D}(s)$$

Dispersion orbit with homogeneous dipole field



- x_D can be understood as a special orbit for the particle with momentum offset.
- Normalizing this orbit with respect to $\Delta p/p_0$ yields the dispersion function

$$\boxed{D_x(s) = \frac{\Delta x}{\Delta p/p_0}}$$

that describes the change of transverse position Δx due to a momentum deviation $\Delta p/p_0$.

- The total transverse offset for a particle with $\Delta p/p_0 \neq 0$ is then

$$x(s) = x_\beta(s) + x_\mathrm{D}(s) = x_\beta(s) + D_x(s)\frac{\Delta p}{p_0}.$$

- The beam size increases due to off-momentum particles in dispersive sections:

$$\sigma = \sqrt{\epsilon\beta + D^2(\Delta p/p_0)^2}$$

Momentum compaction factor

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- A particle with $\Delta p/p_0 > 0$ has a longer path length than the reference particle. The dependence of the relative change of orbit length $\Delta L/L$ on the momentum deviation $\Delta p/p_0$ defines the "momentum compaction factor":

$$\frac{\Delta L}{L} = \alpha_c \frac{\Delta p}{p_0}$$

- The change of orbit length of a particle that follows the closed dispersion orbit $x_D(s)$ is

$$\Delta L = \oint \mathrm{d}s \, \frac{x_D(s)}{\rho(s)} = \frac{\Delta p}{p_0} \oint \mathrm{d}s \, \frac{D(s)}{\rho(s)}.$$

- With this we can also write α_c as

$$\alpha_c = \frac{1}{L} \oint \mathrm{d}s \, \frac{D(s)}{\rho(s)}$$

- As a rough approximation, α_c can also be estimated from the (integer part of the) horizontal tune:

$$\alpha_c \approx \frac{1}{Q_x^2} \qquad \text{z. B. } Q_x \approx 100, \;\; \alpha_c \approx 1 \cdot 10^{-4}$$

Perturbation terms

- Ansatz for the particular solution of the inhomogeneous differential equation including perturbation $p(s)$:

$$P''(s) \;+\; K(s)\, P(s) \;=\; p(s)$$

- A solution can be written from the principal solutions of the homogeneous differential equation using Green's function $G(s,\tilde{s}) \;=\; S(s)C(\tilde{s}) \;-\; C(s)S(\tilde{s})$:

$$P(s) \;=\; \int_0^s p(\tilde{s})\; G(s,\tilde{s})\; \mathrm{d}\tilde{s}$$

- Insertion (with $\tilde{C} = C(\tilde{s})$, $\tilde{S} = S(\tilde{s})$, $\tilde{p} = p(\tilde{s})$, $C = C(s)$, etc.) yields

$$P(s) \;=\; S \int_0^s \tilde{p}\; \tilde{C}\; \mathrm{d}\tilde{s} \;-\; C \int_0^s \tilde{p}\; \tilde{S}\; \mathrm{d}\tilde{s}$$

- First derivative:

$$\begin{aligned} P'(s) &= S' \int_0^s \tilde{p}\; \tilde{C}\; \mathrm{d}\tilde{s} \;+\; S\, C\, p \;-\; C' \int_0^s \tilde{p}\; \tilde{S}\; \mathrm{d}\tilde{s} \;-\; S\, C\, p \\ &= S' \int_0^s \tilde{p}\; \tilde{C}\; \mathrm{d}\tilde{s} \;-\; C' \int_0^s \tilde{p}\; \tilde{S}\; \mathrm{d}\tilde{s} \end{aligned}$$

Perturbation terms II

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- Second derivative:

$$P'(s) = S' \int_0^s \tilde{p}\ \tilde{C}\ \mathrm{d}\tilde{s} - C' \int_0^s \tilde{p}\ \tilde{S}\ \mathrm{d}\tilde{s}$$

$$P''(s) = S'' \int_0^s \tilde{p}\ \tilde{C}\ \mathrm{d}\tilde{s} + S'\ C\ p - C'' \int_0^s \tilde{p}\ \tilde{S}\ \mathrm{d}\tilde{s} - C'\ S\ p$$

$$= p(CS' - C'S) + S'' \int_0^s \tilde{p}\ \tilde{C}\ \mathrm{d}\tilde{s} - C'' \int_0^s \tilde{p}\ \tilde{S}\ \mathrm{d}\tilde{s}$$

- Reminder: $(CS' - C'S) = \mathbb{W} = 1$. Furthermore, for the solutions of the homogeneous differential equation $S'' + KS = 0$ and $C'' + KC = 0$. From this follows

$$P''(s) = p - K\ S \int_0^s \tilde{p}\ \tilde{C}\ \mathrm{d}\tilde{s} + K\ C \int_0^s \tilde{p}\ \tilde{S}\ \mathrm{d}\tilde{s} = p + K \left[\int_0^s \tilde{p}\ C\ \tilde{S}\ \mathrm{d}\tilde{s} - \int_0^s \tilde{p}\ S\ \tilde{C}\ \mathrm{d}\tilde{s} \right]$$

$$= p + K \left[\int_0^s \tilde{p} \left(C\ \tilde{S} - S\ \tilde{C} \right) \mathrm{d}\tilde{s} \right] = p + K \left[\int_0^s \tilde{p} G(s, \tilde{s})\ \mathrm{d}\tilde{s} \right]$$

$$= p(s) - K(s)\ P(s) \qquad \Rightarrow \qquad p(s) = P''(s) + K(s)\ P(s)$$

- The general solution of the equations of motion is thus $u(s) = aC(s) + bS(s) + P(s)$.

Dispersion function

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- Application to the case of deflection of a particle with energy deviation in a dipole:

$$u'' + K(s)u = \frac{1}{\rho_0(s)}\frac{\Delta p}{p_0} = \frac{1}{\rho_0(s)}\,\delta$$

- General solution with perturbation term:

$$u(s) = aC(s) + bS(s) + \delta\, D(s)$$
$$u'(s) = aC'(s) + bS'(s) + \delta\, D'(s)$$

- Ansatz with Green's function:

$$D(s) = \int_0^s \frac{1}{\rho_0}(\tilde{s})\,\left[S\,\tilde{C} - C\,\tilde{S}\right]\,\mathrm{d}\tilde{s} = S(s)\int_0^s \frac{1}{\rho_0}(\tilde{s})\,C(\tilde{s})\,\mathrm{d}\tilde{s} - C(s)\int_0^s \frac{1}{\rho_0}(\tilde{s})\,S(\tilde{s})\,\mathrm{d}\tilde{s}$$

- For $D(s)$ to vanish at a location s_d, the quotient

$$\frac{S(s_d)}{C(s_d)} = \frac{\int_0^{s_d}\frac{1}{\rho_0}\,S\,\mathrm{d}\tilde{s}}{\int_0^{s_d}\frac{1}{\rho_0}\,C\,\mathrm{d}\tilde{s}}$$

- Adjustment by manipulation of the focusing structure

Achromatic structures

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- In an achromatic structure is at position s_d: $D(s_d) = D'(s_d) = 0$.
  This means that from s_d to the next dipole the dispersion disappears.
- With $I_C = \int_0^{s_d} \frac{1}{\rho_0} C \,\mathrm{d}\tilde{s}$ and $I_S = \int_0^{s_d} \frac{1}{\rho_0} S \,\mathrm{d}\tilde{s}$ this condition is satisfied if

$$D(s_d) = 0 = -S(s_d)\, I_C + C(s_d)\, I_S \quad \text{and} \quad D'(s_d) = 0 = -S'(s_d)\, I_C + C'(s_d)\, I_S$$

- Resolving to I_C and I_S gives

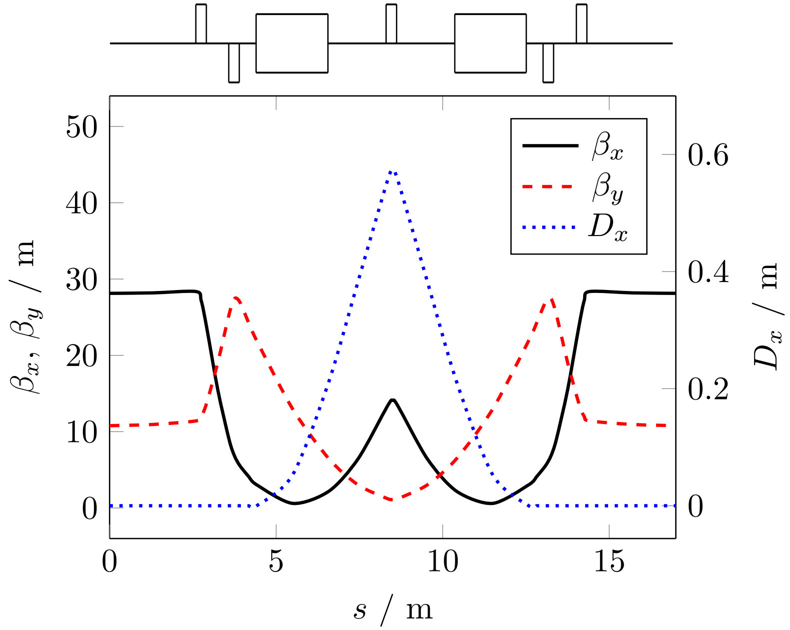
$$[C(s_d)\, S'(s_d) - S(s_d)\, C'(s_d)]\, I_C = 0 \quad \text{and} \quad [C(s_d)\, S'(s_d) - S(s_d)\, C'(s_d)]\, I_S = 0,$$

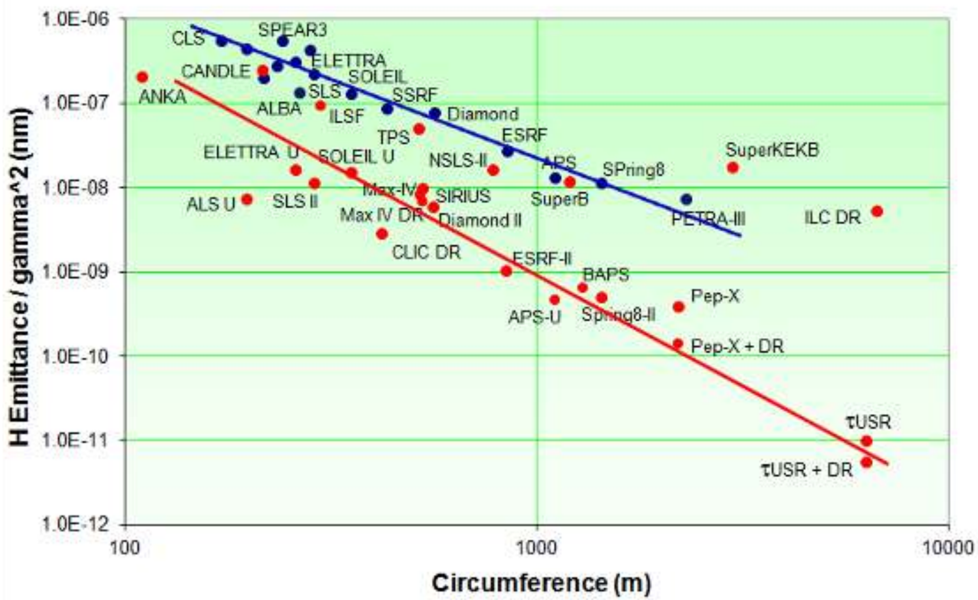
- Since $\mathbb{W} = 1$, the condition for vanishing dispersion is

$$I_C = \int_0^{s_d} \frac{1}{\rho_0} C \,\mathrm{d}\tilde{s} = 0 \quad \text{and} \quad I_S = \int_0^{s_d} \frac{1}{\rho_0} S \,\mathrm{d}\tilde{s} = 0.$$

- Such a structure is called a (first order) achromat.

Low emittance lattices





Chromaticity

- The focusing in the quadrupole that a particle experiences depends on its momentum.
- The tune therefore also depends on particle momentum (tune distribution in the beam) and one defines the so-called "chromaticity" as

$$Q' = p_0 \frac{\mathrm{d}Q}{\mathrm{d}p} \approx \frac{\Delta Q}{\Delta p / p_0}$$

- The natural chromaticity of a linear lattice is

$$Q'_u = -\frac{1}{4\pi} \oint \mathrm{d}s\ \beta_u(s)\ K(s)\,.$$

- Chromaticity can also be considered as a gradient error of the quadrupole.
- In general, one observes a superposition of the effects of Q' and D, which leads to a shift of the longitudinal and transverse position of the focal point.
- For large circular accelerators Q' can become very large (z.B. -150 at LEP, <-2000 at FCC-ee) and has to be compensated.

Sextupole magnets

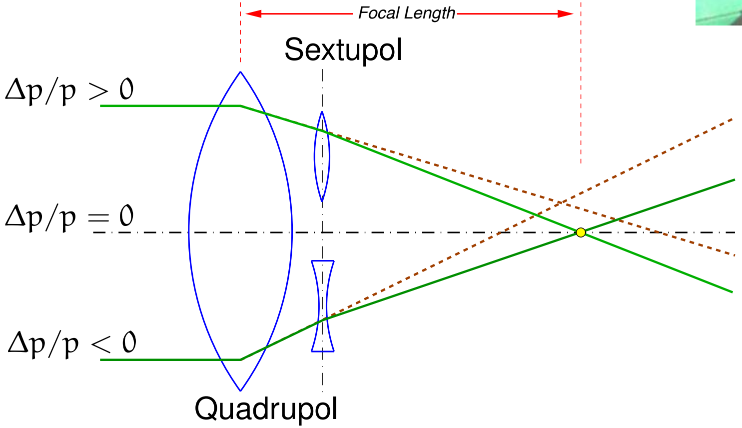
- Chromaticity is compensated by sextupole magnets in dispersive regions.

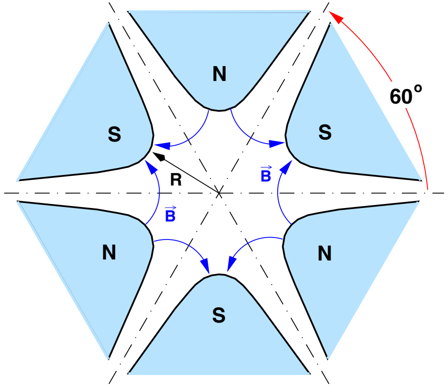
$$B_x(x,y) \sim x \cdot y \quad \text{and} \quad B_y(x,y) \sim \frac{1}{2}\,(x^2 - y^2)$$

$$\text{mit} \quad g' = 6\,\mu_0\,\frac{nI}{R^3}$$

- As in case of the quadrupole the normalized sextupole strength is

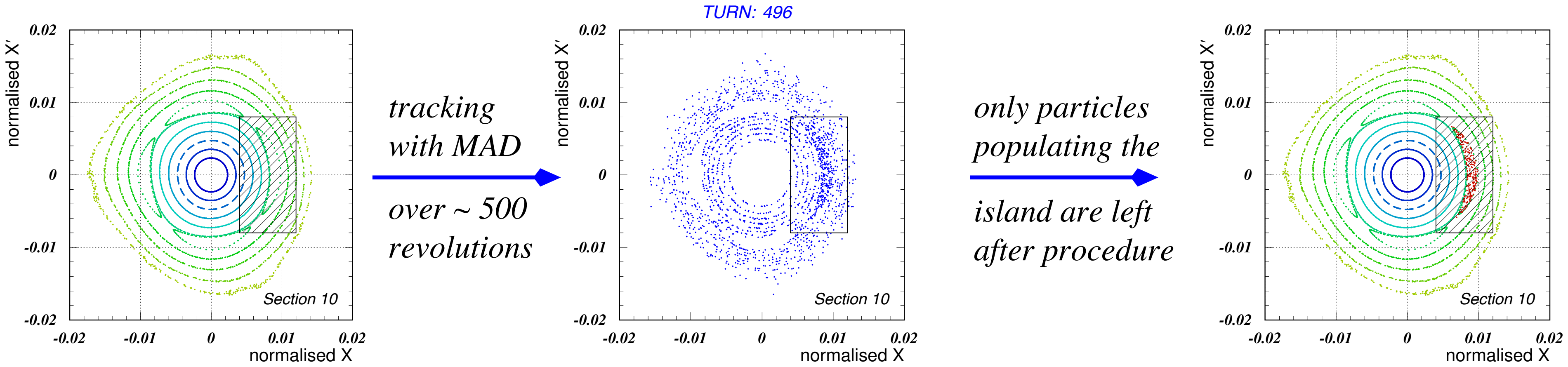
$$m = \frac{e}{p_0}\,g'$$





Phase space

- Non-linear fields cause non-linear oscillations.
- The frequency of such a n oscillation depends on the amplitude.
- Particle loss due to resonances — limitations of an accelerator's dynamic aperture.
- Observation in normalized phase space (Ellipse → Kreis): $X = x/\sqrt{\beta_x}$ and $X' = \sqrt{\beta_x}x' + \alpha_x x/\sqrt{\beta_x}$
- **Example**: iterative elimination to determine the size of a resonant island for protons (Monte Carlo methods and tracking)



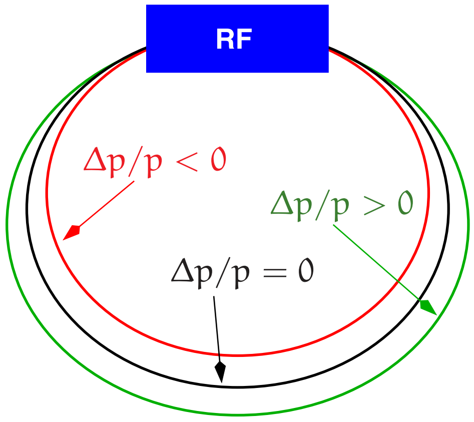
Synchrotron oscillations

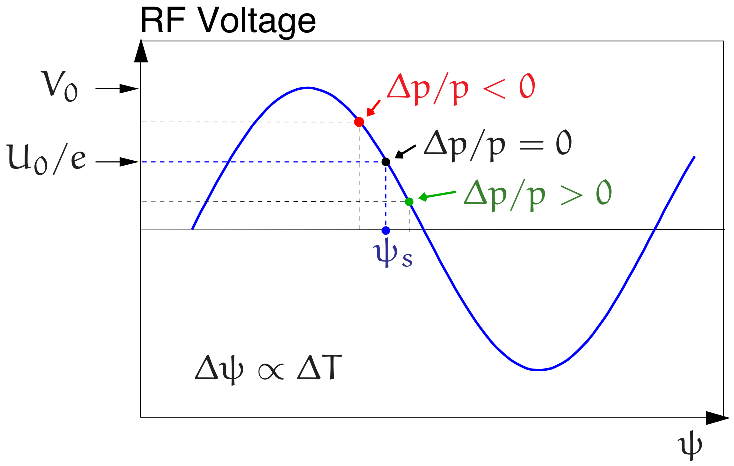
- Phase focusing: Particles in a circular accelerator with RF potential $V(\psi) = V_0 \sin\psi$ oscillate in phase.
- One turn of the reference particle lasts $T_0 = L_0/v_0$.
- For particles of arbitrary energy applies

$$T = \frac{L_0 + \Delta L}{v_0 + \Delta v} \approx (L_0 + \Delta L)\frac{v_0 - \Delta v}{v_0^2}$$
$$\approx \frac{1}{v_0^2}(L_0 v_0 + v_0 \Delta L - L_0 \Delta v)$$

- The relative temporal deviation is

$$\frac{\Delta T}{T_0} = \frac{T - T_0}{T_0} = \frac{v_0 \Delta L - L_0 \Delta v}{T_0 v_0^2}$$
$$= \frac{\Delta L}{L_0} - \frac{\Delta v}{v_0} = \boxed{\alpha_c \frac{\Delta p}{p_0}} - \frac{\Delta v}{v_0}$$





Synchrotron tune

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- With the relativistic relation $\Delta v/v_0 = (1/\gamma^2)(\Delta p/p_0)$ we get

$$\frac{\Delta T}{T_0} = \alpha_c \frac{\Delta p}{p_0} - \frac{\Delta v}{v_0} = \alpha_c \frac{\Delta p}{p_0} - \frac{1}{\gamma^2}\frac{\Delta p}{p_0} = \left(\alpha_c - \frac{1}{\gamma^2}\right)\frac{\Delta p}{p_0}$$

- Since $\Delta\psi = 2\pi f_{\mathrm{RF}}\Delta T$ and $f_{\mathrm{RF}} = h f_{\mathrm{rev}}$ the phase shift is

$$\boxed{\Delta\psi = 2\pi h\left(\alpha_c - \frac{1}{\gamma^2}\right)\frac{\Delta p}{p_0} = 2\pi h\left(\frac{1}{\gamma_{\mathrm{tr}}^2} - \frac{1}{\gamma^2}\right)\frac{\Delta p}{p_0} = -\, 2\pi h\ \eta_c \frac{\Delta p}{p_0}}$$

→ At $\gamma = \gamma_{\mathrm{tr}}$ there is no phase focusing.

- For negligible radiation damping the equation for the oscillation is $\ddot{\psi} + \Omega^2\psi = 0$ with frequency

$$\Omega^2 = \omega_{\mathrm{rev}}^2\left(\frac{\eta_c h}{2\pi E_0}\right) e\,\frac{\mathrm{d}V(\psi_s)}{\mathrm{d}\psi}$$

- For small oscillation amplitude $\mathrm{d}V/\mathrm{d}\psi$ is constant. Using the condition for stable phase angle $U_0 = e\ V_0\ \sin\psi_s$, the synchrotron tune is defined as

$$\boxed{Q_s^2 = \left(\frac{\Omega}{\omega_{\mathrm{rev}}}\right)^2 = \left(\frac{\eta_c h}{2\pi E_0}\right) e\ V_0\ \cos\psi_s = \left(\frac{\eta_c h}{2\pi E_0}\right)\sqrt{e^2\ V_0^2 - U_0^2}}$$

Transition

- In many (old) proton synchrotrons of middle and high energy $E > 5\,\mathrm{GeV}$ the point of $\gamma = \gamma_{\mathrm{tr}}$ must be crossed.
- As phase focusing is paused at this point, the transition must be quick: "γ_{tr} jump"

  → Change of optics, since $\gamma_{\mathrm{tr}} = 1/\sqrt{\alpha_c}$ with $\alpha_c = \frac{1}{L} \oint \mathrm{d}s \, \frac{D(s)}{\rho(s)}$
- Reminder: Tune shift is $\Delta Q = \frac{1}{4\pi} \oint \beta(s) \Delta k(s) \, \mathrm{d}s$
- Tune neutral γ_{tr} manipulation, if $\beta_1 K_1 + \beta_2 K_2 = 0$

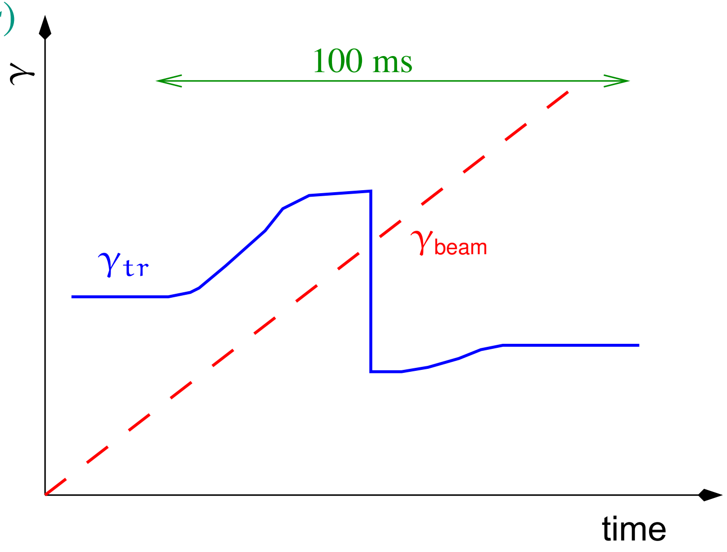
- **Example: CERN PS**

  → $2 \le E_0 \le 26\,\mathrm{GeV}$

  → $Q \approx 6.25$

  → $\gamma_{\mathrm{tr}} = 6.1$

  → $\mathrm{d}\gamma/\mathrm{d}t = 40\,\mathrm{s}^{-1}$



Longitudinal phase space

- For real values of the synchrotron frequency the phase can be written as $\psi = \psi_0 \cos(\Omega t + \chi_i)$, where χ_i is any phase for the i^th particle at time $t = 0$.
- The temporal derivative of the phase is $\dot{\psi} = -\psi_0\,\Omega\,\sin(\Omega t + \chi_i)$.
  Analogous to the transverse phase space, ψ and $\dot{\psi}$ describe the longitudinal motion in the phase space $(\psi, \dot{\psi})$.
- From $\dot{\psi} = \frac{\Delta\psi}{\Delta t} = \frac{\Delta\psi\,(\beta c)}{2\pi} = -\beta c\, h\, \eta_c \frac{\Delta p}{p_0}$ an oscillation equation can be derived for the energy deviation (with $\beta c = 2\pi/T_0 = \omega_{\mathrm{rev}}$): $\delta = \frac{\Delta p}{p_0} = -\frac{\dot{\psi}}{h\omega_{\mathrm{rev}}\eta_c} = \frac{\Omega\,\psi_0}{h\omega_{\mathrm{rev}}\eta_c}\sin(\Omega t + \chi_i)$

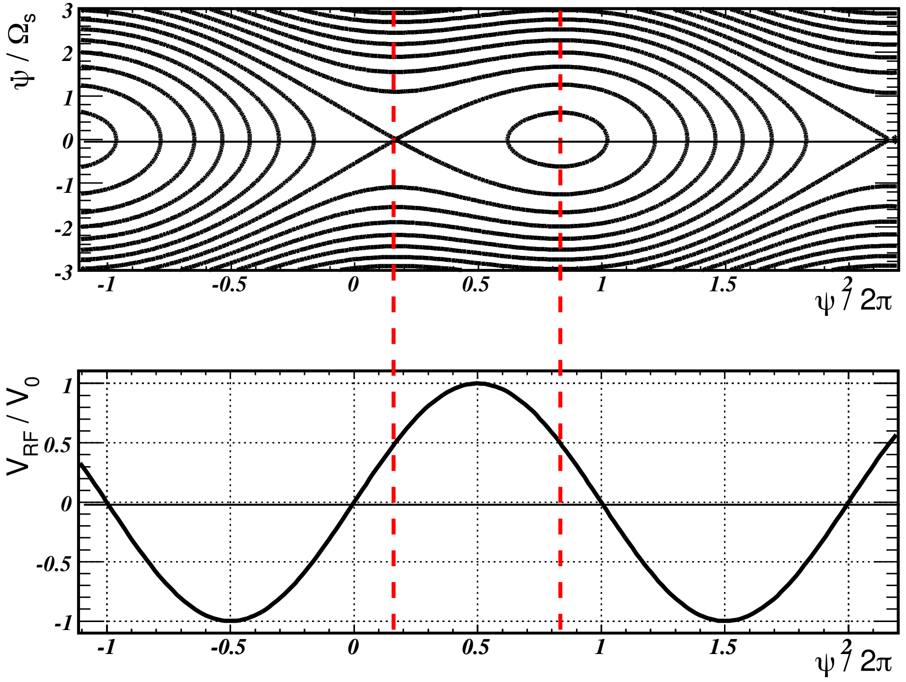
  → Particle energy oscillates with synchrotron frequency ψ_s.

  → The energy deviation is the conjugate variable to the phase ψ.
- In general, the equation for the phase is

$$\ddot{\psi} + \frac{\Omega^2}{\cos\psi_s}\left(\sin(\psi_s + \psi) - \sin\psi_s\right) = 0$$

Separatrix

- The boundary line between stable and unstable trajectories in the phase space is called "separatrix".
- The stable area enclosed by the separatrix is also called the "RF Bucket".



RF bucket and ψ_s

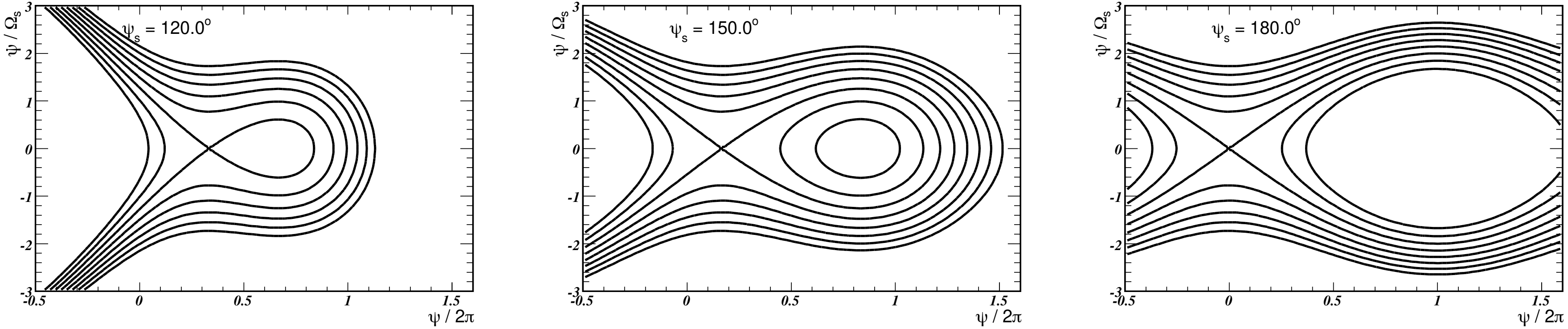
- The ratio of the available RF voltage and the energy loss per turn is defined as the "over voltage factor":

$$q \;=\; e\, V_0/U_0 = 1/\sin\psi_s$$

- Looking at the separatrix, the momentum acceptance can be written after some transformations as

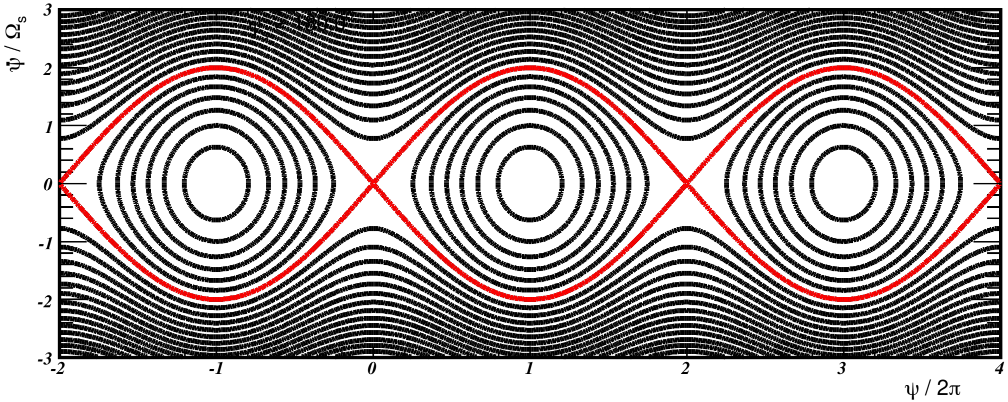
$$\left(\frac{\Delta p}{p_0}\right)^2_{\mathrm{acc}} = \frac{eV_0\ \sin\psi_s}{\pi\ h\ |\eta_c|\ cp_0}\, 2\left(\sqrt{q^2-1} - \arccos\frac{1}{q}\right)$$

- The momentum acceptance therefore depends on the selected RF voltage.

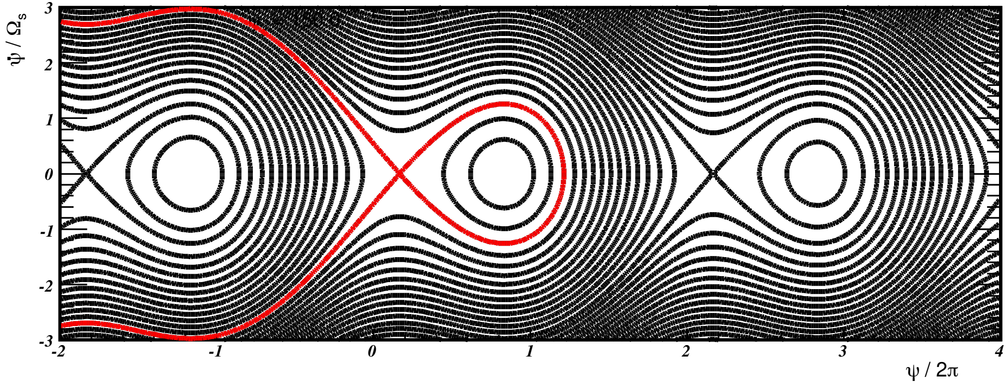


Adjacent buckets

- $\psi_s = 180^{\text{o}}$: stationary case

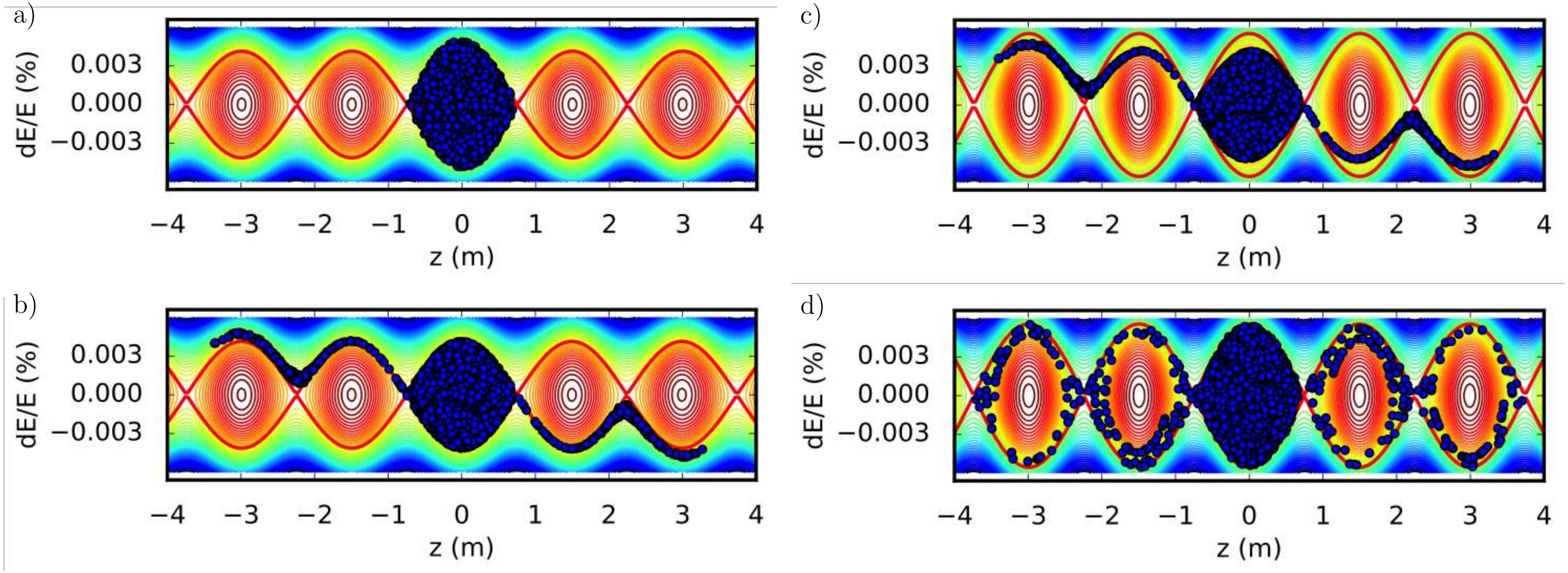


- $\psi_s = 150^{\text{o}}$: during acceleration



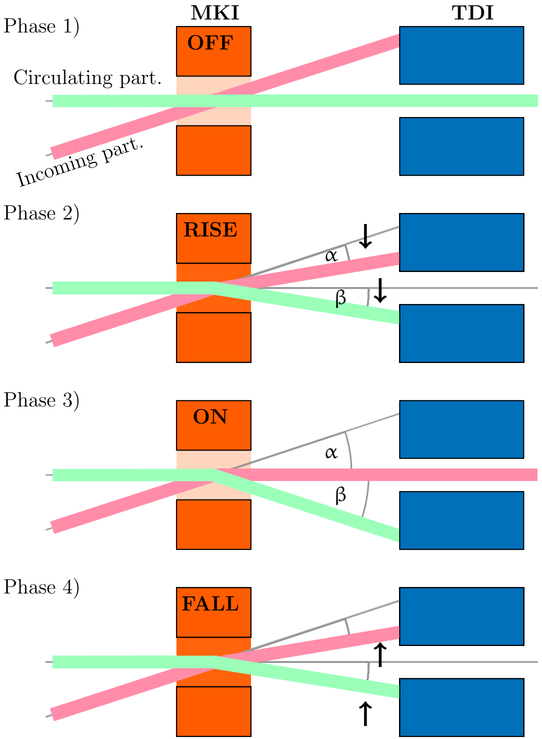
- Particle “migration” between adjacent buckets is possible

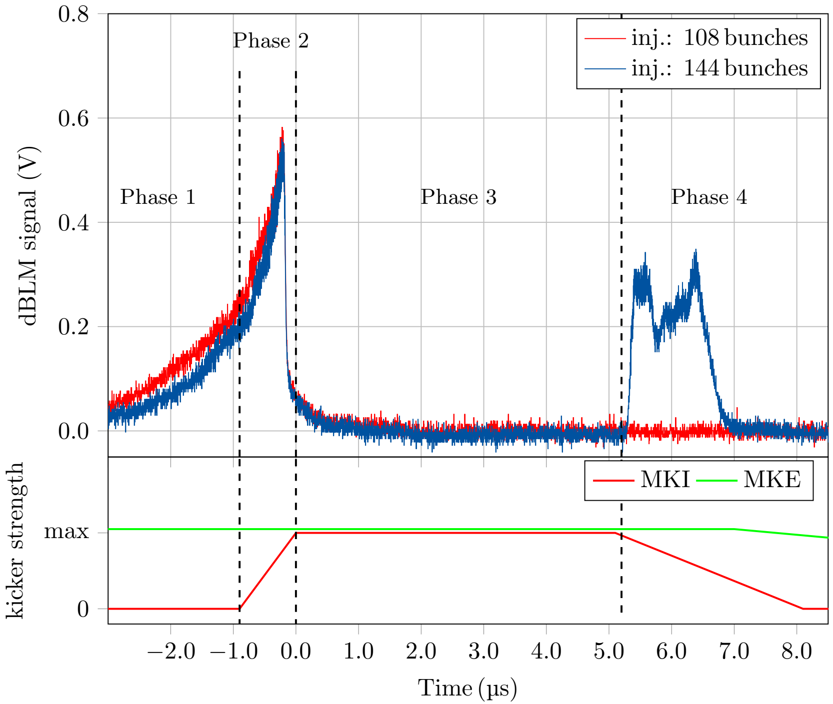
"Ghost bunches" in the SPS



CERN document server: Oliver Stein, PhD thesis

Injection losses in LHC due to ghost bunches





CERN document server: Oliver Stein, PhD thesis

Summary

- **Tune:** number of betatron oscillations per revolution $Q = \frac{N\,\mu}{2\pi} = \frac{1}{2\pi}\int_0^L \frac{\mathrm{d}s}{\beta(s)} = \frac{1}{2\pi}\oint \frac{\mathrm{d}s}{\beta(s)}$
  - Condition for optical resonances: $mQ_x + nQ_y = p$ mit $m, n, p \in \mathbb{Z}$
  - Order of the resonance: $|m| + |n|$
  - Chromaticity: Change of tune for particles with energy deviation $Q'_u = -\frac{1}{4\pi}\oint ds\; \beta_u(s)\; K(s)$
- **Dispersion function:** $D_u(s) = \frac{\Delta u}{\Delta p/p_0}$
  - The horizontal particle position at location s relative to the nominal orbit:

$$x(s) = x_\beta(s) + x_D(s) = x_\beta(s) + D_x(s)\,\frac{\Delta p}{p_0}$$

Summary II

- Momentum compaction factor $\alpha_\mathrm{c} = \frac{\Delta L}{L} / \frac{\Delta p}{p_0} = \frac{1}{L} \oint \mathrm{d}s \, \frac{D(s)}{\rho(s)}$
- Synchrotron oscillation: Longitudinal oscillation around reference particle
  - Particle energy oscillates with synchrotron frequency
  - Synchrotron tune: $Q_s^2 = \left(\frac{\alpha_c h}{2\pi E_0} \right) \sqrt{e^2 \, V_0^2 - U_0^2}$
- Separatrix: Boundary line between stable and unstable trajectories
- RF bucket: stable area enclosed by the separatfix