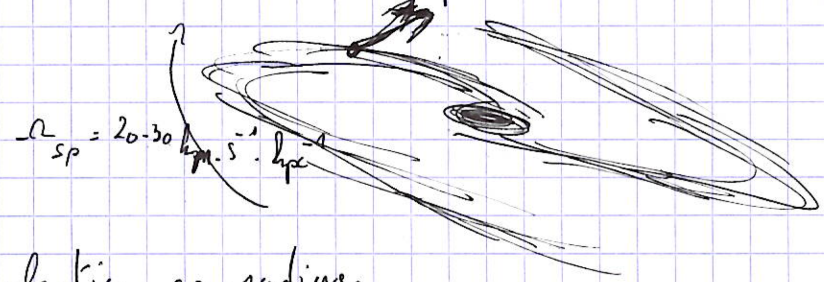
Cosmic Rays - Exercises set 2

1. Milky Way

$v_\phi = 220\ km/s$



Galactic co-radius.

$$D_{sp} = \frac{v_\phi}{\Omega_{sp}} \Rightarrow \begin{cases} \Omega_{sp} = 20\ km.s^{-1}.kpc^{-1} \Rightarrow D_{sp} = 11\ kpc \\ \Omega_{sp} = 30\ km.s^{-1}\ kpc^{-1} \Rightarrow D_{sp} = 7.33\ kpc. \end{cases}$$

$R_{\oplus} \simeq 8.3\ kpc.$ if $D_{sp} = 11\ kpc \rightarrow \oplus$ faster than arm.

if $D_{sp} = 7.33\ kpc \rightarrow \oplus$ slower ———

2. Diffusion equation

1D diffusion equation:

$$\frac{\partial n}{\partial t} = D \frac{\partial^2 n}{\partial x^2}$$

lets $$n(t,x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} \quad \text{with } \sigma^2 = 2Dt$$

$$= \frac{e^{-\frac{x^2}{4Dt}}}{\sqrt{4\pi D t}}$$

$$\frac{\partial n}{\partial t} = \frac{\frac{x^2}{4Dt^2}\sqrt{4\pi Dt}\, e^{-\frac{x^2}{4Dt}} - \frac{\sqrt{4\pi D}}{2\sqrt{t}}\, e^{-\frac{x^2}{4Dt}}}{4\pi Dt}$$

$$\boxed{\frac{\partial n}{\partial t} = \frac{e^{-\frac{x^2}{4Dt}}}{\sqrt{4\pi Dt}}\left(\frac{x^2}{4Dt^2} - \frac{1}{2t}\right)} \quad (*)$$

$$\frac{\partial n}{\partial x} = -\frac{2x}{4Dt}\,\frac{e^{-\frac{x^2}{4Dt}}}{\sqrt{4\pi Dt}}$$

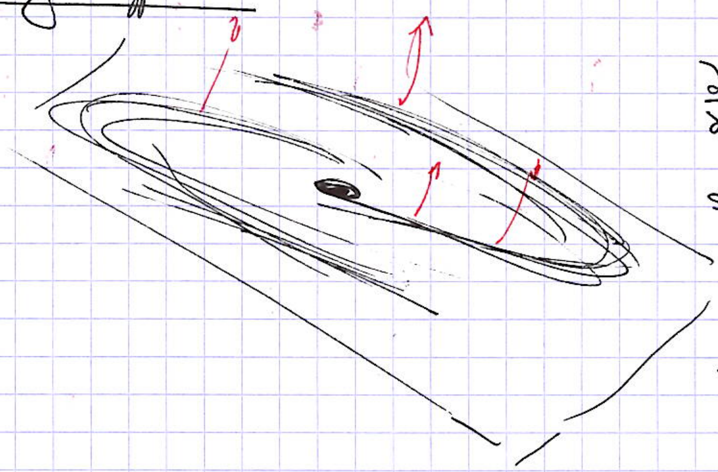
$$\frac{\partial^2 n}{\partial x^2} = \frac{e^{-\frac{x^2}{4Dt}}}{\sqrt{4\pi Dt}}\left[-\frac{2}{4Dt} + \left(\frac{\cancel{2x}}{4Dt}\right)^2\right]$$

$$= \frac{e^{-\frac{x^2}{4Dt}}}{\sqrt{4\pi Dt}}\left[-\frac{1}{2Dt} + \frac{x^2}{4D^2t^2}\right]$$

$$D\frac{\partial^2 n}{\partial x^2} = \frac{e^{-\frac{x^2}{4Dt}}}{\sqrt{4\pi Dt}}\left[\frac{x^2}{4Dt^2} - \frac{1}{2t}\right] = \frac{\partial n}{\partial t} \quad \text{see } (*)$$

CQFD

3. Cosmic-ray diffusion



$$\frac{\partial n}{\partial t} = D\frac{\partial^2 n}{\partial z^2} + Q(z,t)$$

sources $Q(z,t) = Q_0\,\delta(z)$

all in plane $z=0$

1. steady state $\frac{\partial n}{\partial t} = 0 \Rightarrow \frac{\partial^2 n}{\partial z^2} = -\frac{Q(z,t)}{D} = -\frac{Q_0}{D}\delta(z)$

$$\int_{-\infty}^{z}\delta(s)\,ds = \mathcal{H}(z) \quad \text{Heavyside function}$$

$$\Rightarrow \frac{\partial n}{\partial z} = -\frac{Q_0}{D}\mathcal{H}(z) + A$$

$$\Rightarrow \boxed{n(z) = -\frac{Q_0}{D}\mathcal{H}(z)\,z + Az + B} \quad (*)$$

condition $n(-H) = n(H) = 0$

$$n(-H) = 0 \quad \Rightarrow (*) \quad -AH + B = 0 \quad \Rightarrow \quad B = AH$$

$$(*) \Leftrightarrow \quad n(z) = -\frac{Q_0}{D} \mathcal{H}(z)\, z + A(z+H)$$

$$n(H) = 0 \quad \Rightarrow (*) \quad -\frac{Q_0}{D} H + A\, 2H = 0$$

$$\Rightarrow A = \frac{Q_0}{2D}$$

then $(*)$ $$\boxed{n(z) = -\frac{Q_0}{D} \mathcal{H}(z)\, z + \frac{Q_0}{2D}(z+H)}$$

$$n(z) = \frac{Q_0}{2D}\left[\underbrace{-2\mathcal{H}(z)\, z + z + H}_{H - |z|}\right]$$

$$\boxed{n(z) = \frac{Q_0}{2D}\left[H - |z|\right]}$$

2. $$N = \int_{-H}^{H} n(z)\, dz = \frac{Q_0}{2D} \int_{-H}^{H} (H - |z|)\, dz$$

$$= \frac{Q_0}{\cancel{2}D} \cancel{2} \int_0^H (H - z)\, dz$$

$$= \frac{Q_0}{D} \left[Hz - \frac{z^2}{2} \right]_0^H$$

$$= \frac{Q_0}{D} \left[H^2 - \frac{H^2}{2} \right]$$

$$\boxed{N = \frac{Q_0}{2D} H^2}$$

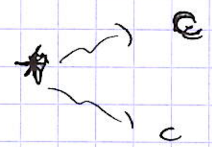
$$N = Q_0 \frac{H^2}{2D} = Q_0 \, \tau_{res}$$

↳ residense time $\boxed{\tau_{res} = \frac{H^2}{2D}}$

$H = 500$ pc $D = 0.05$ pc^2 yr^{-1}

$$\Rightarrow \boxed{\tau_{res} = 25 \cdot 10^5 \text{ yr}}$$

4. Particle propagation



$$\frac{\partial N(t)}{\partial t} = - \sigma_{abs} \frac{\rho_b \beta c}{m_b} N(t)$$

↳ solution: $N(t) = N_0 \, e^{-t/\tau}$

with $\tau = \dfrac{m_b}{\sigma_{abs} \rho_b \beta c}$

t_{90} such as $N(t_{90}) = 0.1 \, N_0 \Rightarrow 0.1 = e^{-t_{90}/\tau}$

$$\Rightarrow t_{90} = \tau \ln 10$$

$$\boxed{t_{90} \simeq 2.3 \, \tau}$$

proton density $\frac{\rho_b}{m_b} = 1$ cm^{-3}

cross section $\sigma_{abs} = 250$ mb. $= 250 \; 10^{-27}$ cm^2

$\beta \sim 1$ $c = 2.9979 \; 10^{10}$ cm.s^{-1}

$$\Rightarrow \tau = 4.23 \; 10^6 \text{ yr.}$$

$$\boxed{t_{90} = 9.7 \; 10^6 \text{ yr.}}$$

$\cancel{\langle P^2 \rangle} =$

1D diffusion : $\rightarrow$ solution gaussian. $\langle x \rangle = 0$

$$\sigma^2 = \langle x^2 \rangle - \cancel{\langle x \rangle^2} = 2Dt$$

here $\langle x_{90}^2 \rangle = 2 D t_{90}$

$$D = 0.05 \ pc^2/yr \rightarrow X_{90} =$$

$$X_{90} = \sqrt{\langle x_{90}^2 \rangle} = \sqrt{2 D t_{90}}$$

$$\boxed{X_{90} = 9{,}867 \ pc}$$

5. Production of anti-protons

(a) (I) $\underset{+1}{p} + \underset{+1}{p} \longrightarrow \underset{+1}{p} + \underset{+1}{p} + \underset{-1}{\bar{p}} + \underset{+1}{p}$

(II) $p + p \longrightarrow \underset{+1}{p} + \underset{+1}{p} + \bar{d} + \underset{+1}{p} + \underset{0}{n}$

anti-deuteron

(III) $p + p \longrightarrow p + p + {}^3\overline{He} + p + p + n$

(b) Energy conservation:

center of mass: $s = (4 m_p)^2 = 16 m_p^4$

4 "protons" created

collision on fix' target $\longrightarrow$ proton: $\begin{pmatrix} \vec{0} \\ m_p \end{pmatrix} = \hat{p}_2$

proton $\hat{p}_1 = \begin{pmatrix} E_{th} \\ \vec{p}_{th} \end{pmatrix}$

$$\hat{p}_1 \hat{p}_2 = \vec{p} \cdot \vec{0} + E_{th} m_p$$

$$\hat{p} = E^2 - |\vec{p}|^2 = m^2 c^4$$

$$s = (\hat{p}_1 + \hat{p}_2)^2 = \hat{p}_1 + \hat{p}_2 + 2 \hat{p}_1 \hat{p}_2$$

$$= \cancel{E_{th}^2 - m_p^2 -}$$

$$= 2 m_p^2 + 2 E_{th} m_p$$

$$\Rightarrow 2\, E_{th}\, m_p + 2\, m_p^2 = 16\, m_p^2$$

(I) $\Rightarrow \boxed{E_{th} = 7\, m_p}$ 6.57 GeV

(II) COM $\rightarrow$ $s = (3\, m_p + m_n + m_d)^2 = 2\, E_{th}\, m_p + m_p^2$

$$E_{th} = \frac{(3\, m_p + m_n + m_d)^2 - 2\, m_p^2}{m_p}$$

$m_d = 1.875$ GeV / 2.014 Dalton (Da)

$E_{th} = 32.8$ GeV

(III) COM $\rightarrow$ $s = (4\, m_p + m_n + m_{He})^2 = 2\, E_{th}\, m_p + m_p^2$

$$m_{He} = \frac{(4\, m_p + m_n + m_{He})^2 - m_p^2}{m_p}$$

$E_{th} = 59.$ GeV

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com. antideuteron. (final state)

$p + p \rightarrow p + p + \bar{d} + d \longrightarrow 32$ GeV

$s = (2\, m_p + 2\, m_d)^2 = 2\, m_p^2 + 2\, E_{th}\, m_p$

com anti-He^4 (final state) $\longrightarrow$ 91.8 GeV

$p + p \rightarrow p + p + {}^4He$ $\quad s = (2\, m_p + 2\, m_{He})^2 = 2\, m_p^2$

anti He^3 $\longrightarrow$ 58.9 GeV

$$n(z) = \frac{Q_0}{2D}\left[H + z - 2z\,\Theta(z)\right]$$

cross-check for exercices 3

$$\frac{\partial n}{\partial z} = \frac{Q_0}{2D}\left[1 - 2\Theta(z) - 2z\,\delta(z)\right]$$

$$\frac{\partial^2 n}{\partial z^2} = \frac{Q_0}{2D}\left[-2\delta(z) - 2\delta(z) - 2z\frac{\partial\delta(z)}{\partial z}\right]$$

$$= -\frac{Q_0}{D}\left[2\delta(z) + z\frac{\partial\delta(z)}{\partial z}\right]$$

$$z\frac{\partial\delta(z)}{\partial z} = ?$$

derivative of the delta function $\boxed{f(x)\,\delta'(x) = -f'(x)\,\delta(x)}$

in our case: $z\frac{\partial\delta(z)}{\partial z} = -\delta(z)$

$$\Rightarrow \frac{\partial^2 n}{\partial z^2} = -\frac{Q_0}{D}\left[2\delta(z) - \delta(z)\right]$$

$$\boxed{\frac{\partial n}{\partial z^2} = -\frac{Q_0}{D}\delta(z)}$$ ← QED.

<u>Why?</u> Dirac δ function is not a function but a distribution!