Please submit via ILIAS by **Monday January 16th**. Solutions will be discussed Jan. 19th 9:45 in room 10/1.

1. Rigidity, kinetic energy and velocity

The relativistic energy-momentum relation for a cosmic-ray nucleus can be written as

$$A^2(T+E_0)^2 = p^2c^2 + A^2E_0^2$$

where A denotes the number of nucleons, T is the kinetic energy per nucleon, E_0 is the nucleon rest mass of about 1 GeV and p is the total momentum of the nucleus. Please derive the following relations:

(a) $R(T) = (A/Ze)\sqrt{T(T+2E_0)}$

(b) $\beta(T) = \sqrt{T(T+2E_0)}/(T+E_0)$

(c) $\beta(R) = R/\sqrt{R^2 + (A/Ze)^2E_0^2}$

where $\beta = v/c$ is the dimensionless speed of the cosmic-ray nucleus, R denotes the rigidity $R = pc/(Ze)$ and Ze is the charge of the nucleus. **(3 points)**

2. Cosmic rays at low energies

Diffuse shock acceleration predicts that the number density of primary cosmic rays follows a power law in rigidity. Yet, the observed flux of cosmic rays at Earth deviates from a power law at low energies. Please discuss the influence of the following effects on the flux of cosmic rays between 0.01 and 100 GeV by plotting the resulting fluxes $J_i(T)$ in comparison with a power-law flux $J(T) \propto T^{-\gamma}$. Which effects dominate the observed deviation from a power law at low energies?

(a) derive the energy flux $J_a(T) = \mathrm{d}^4N/\mathrm{d}T\,\mathrm{d}t\,\mathrm{d}A\,\mathrm{d}\Omega$ given a power-law density in rigidity $U(R) = \mathrm{d}^2N/\mathrm{d}R\,\mathrm{d}V \propto R^{-\gamma}$. Hint: first derive $U(T)$ using $U(T)\mathrm{d}T = U(R)\mathrm{d}R \rightarrow U(T) = U(R(T))|\mathrm{d}R/\mathrm{d}T|$, then convert to flux using $U(T) = 4\pi/\beta c\, J_a(T)$, cf. lecture 1. $R(T)$ and $\beta(T)$ were derived in problem 1. **(1 point)**

(b) estimate the influence of ionization losses in the ISM for proton and iron nuclei by assuming an energy-independent column-depth $X = \lambda_{\mathrm{esc}}$ using the value derived from the boron-to-carbon ratio at 1 GeV (lecture 5) and an energy loss $\mathrm{d}E/\mathrm{d}X$ of minimum ionizing particles (lecture 2), i.e. $T' = T - \mathrm{d}E/\mathrm{d}X\,\lambda_{\mathrm{esc}}$. Hint: use again $J_b(T') = J_a(T(T'))|\mathrm{d}T/\mathrm{d}T'|$ to derive the flux as a function of T'. **(1 point)**

(c) plot the spectrum J_c after taking into account losses due to fragmentation (cf. lecture 5)? The interaction length of protons is $\lambda_{\mathrm{p}} \sim 55\ \mathrm{g/cm^2}$ and for iron it is $\lambda_{\mathrm{Fe}} \sim 2.3\ \mathrm{g/cm^2}$. Hint: assume J_a is the spectrum of primary cosmic rays without fragmentation, i.e. $Q \cdot \tau_{\mathrm{esc}} = J_a$. **(1 point)**

(d) cosmic-ray spectra at Earth deviate from the ones in the local ISM due to "solar modulation", i.e. the diffusion and drift of particles in the irregularities of the heliospheric magnetic field. The effect on the flux can be approximated as

$$J_d(T) = \frac{T(T+2E_0)}{(T+\Phi)(T+\Phi+2E_0)}\,J_{\mathrm{a}}(T+\Phi)$$

where $\tau = Ze\phi$ and the so-called "force-field" ϕ depends on the solar cycle. How important is solar modulation for the AMS-2 data for which the time-averaged $\phi \sim 0.5$ GV? **(1 point)**

3. Maximum Energy in Supernova Remnants

Calculate the maximum energy $E(t)$ of cosmic rays by integrating the energy gain in diffuse shock acceleration with a time-dependent shock velocity. Assume a sweep-up time of $t_{sw} = 500$ years, a free streaming speed of 10^4 km/s and a magnetic field at the shock of 100 μG. What are the maximum energies of a proton injected at $t_0 = 0$ and accelerated until a) $t = t_{sw}$ and b) until the end of the Sedov-Taylor phase ($t_{\mathrm{sd}} = 10^5$ years)? What is the maximum energy of a proton injected at $t_0 = t_{sw}$ and accelerated until $t = t_{\mathrm{sd}}$? **(3 points)**