Cosmic rays II - Exercises set 3

Exercice 1

$$A^2(E_0+T)^2 = p^2c^2 + A^2E_0^2$$

$$p^2c^2 = A^2E_0^2 + 2A^2E_0T + A^2T^2 - A^2E_0^2$$

$$\boxed{p^2c^2 = A^2T(2E_0+T)} \quad (*)$$

(a) $$\boxed{R(T) = \frac{pc}{Ze} \overset{(*)}{=} \frac{A}{Ze}\sqrt{T(T+2E_0)}} \quad (**)$$

(b) $\beta = \frac{v}{c}$ $\qquad p = \cancel{A}mv$ $\qquad$ with $m = \frac{m_0}{\sqrt{1-\beta^2}}$

mass of a nucleon.

$$p = \frac{A\,m_0}{\sqrt{1-\beta^2}}\,v$$

$$pc = \frac{A\,m_0c^2\,\beta}{\sqrt{1-\beta^2}} = \frac{AE_0\,\beta}{\sqrt{1-\beta^2}}$$

$$\Rightarrow \beta^2A^2E_0^2 = p^2c^2(1-\beta^2)$$

$$\beta^2 = \frac{p^2c^2}{p^2c^2+A^2E_0^2}$$

$$\overset{(*)}{=} \frac{\cancel{A^2}\,T(2E_0+T)}{\cancel{A^2}\,(E_0+T)^2}$$

$$\boxed{\beta = \frac{\sqrt{T(2E_0+T)}}{E_0+T}} \quad (***)$$

(c) $(**)$ $\sqrt{T(2E_0+T)} = \frac{R\,Ze}{A}$

$(**)$ $R^2\left(\frac{Ze}{A}\right)^2 = T^2 + 2E_0T$

$$\Rightarrow R^2\left(\frac{Ze}{A}\right)^2 + E_0^2 = (T+E_0)^2$$

$$\Rightarrow \boxed{\beta(R) = \frac{R\,\frac{Ze}{A}}{\sqrt{R^2\left(\frac{Ze}{A}\right)^2+E_0^2}} = \frac{R}{\sqrt{R^2+\frac{E_0^2A^2}{(Ze)^2}}}}$$

Exercise 2

(a) $U(T)\,dT = U(R)\,dR$

$$\Rightarrow \quad U(T) = U(R)\,\frac{dR}{dT}$$

(Ex 1) $$R(T) = \frac{A}{Ze}\sqrt{T(T+2E_0)}$$

$$\frac{dR}{dT} = \frac{A}{Ze}\,(2T+2E_0)\,\frac{1}{2\sqrt{T(T+2E_0)}}$$

$$= \frac{A}{Ze}\,\frac{T+E_0}{\sqrt{T(T+2E_0)}}$$

$$= \frac{A}{Ze}\,\frac{1}{\beta(T)} \qquad \beta(T) = \frac{\sqrt{T(T+2E_0)}}{T+E_0} \quad (Ex\,1)$$

$U(R) \propto R^{-\gamma}$

$$U(T) = R(T)^{-\gamma}\,\frac{A}{Ze\,\beta(T)}$$

$$J(T) = \frac{\beta c}{\cancel{\beta}4\pi}\,U(T)$$

$$= \frac{\cancel{\beta} c A}{Ze\,4\pi}\,\frac{1}{\cancel{\beta(T)}}\,R(T)^{-\gamma}$$

$$= \frac{c A}{Ze\,4\pi}\quad 1 \quad \left(\frac{A}{Ze}\right)^{-\gamma}\left[T(T+2E_0)\right]^{-\gamma/2}$$

$$\boxed{J(T) = \frac{c}{4\pi}\left(\frac{A}{Ze}\right)^{1-\gamma}\left[T(T+2E_0)\right]^{-\frac{\gamma}{2}}} \quad (*)$$

(b) lecture 5: boron-to-carbon ratio. @ Earth $\gamma_{\circ} = 2.7/3$

$$\lambda_{esc} \simeq 7\left(\frac{R}{10\,GV}\right)^{-1/3} g/cm^2$$

$E = 1$ GeV $\Rightarrow \lambda_{esc} \simeq 23.3\ g/cm^2$

lecture 1: ionization energy loss $\langle -\frac{dE}{dx}\rangle \sim Z^2 \rho$

$\Rightarrow \gamma = \frac{1}{3}$; $E_0 = \begin{cases} 1\ \text{GeV} & (\text{proton}) \\ 56\ \text{GeV} & (\text{Fe}) \end{cases}$

(b) Influence of ionization losses in ISM

B/C ratio $\rightarrow \lambda_{esc} \simeq 7 \left(\frac{R}{10\,GV}\right)^{-1/3}$ g/cm^2

for proton @T=1 GeV $\rightarrow R = 1\,GV \rightarrow \lambda_{esc} = 23.3$ g/cm^2

for iron @T=1 GeV/n $\Rightarrow R = \frac{56}{26}\,GV \sim 2.15\,GV \rightarrow \lambda_{esc} = 10.8$ g/cm^2

$\Downarrow$

T = 56 GeV

Then in (*) replace T by $T = T' + \frac{d\epsilon}{dx}\lambda_{esc}$

$$J(T') = J(T(T'))\left|\frac{dT}{dT'}\right| \qquad \left\langle -\frac{d\epsilon}{dx}\right\rangle_{min} \simeq 10\,MeV/g.cm^2\ Z^2$$

$$\frac{dT}{dT'} = \frac{d}{dT'}\left(T' + \underbrace{\frac{d\epsilon}{dx}\lambda_{esc}}_{cst.}\right)$$

$$\Rightarrow \frac{dT}{dT'} = 1.$$

$$J(T') = J(T(T'))$$

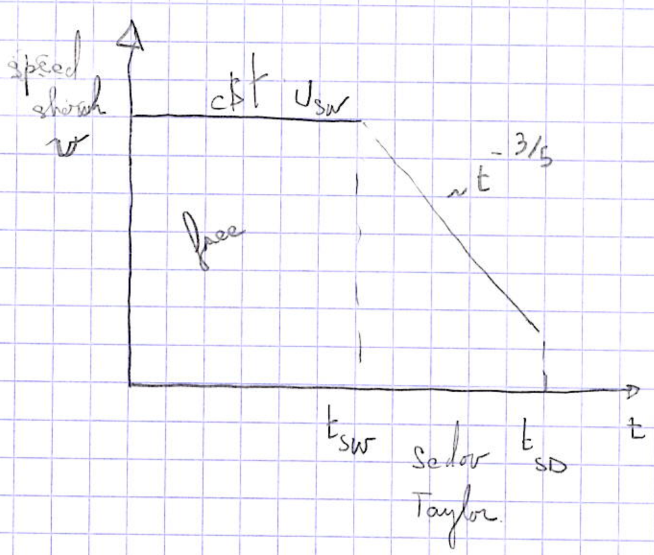
$$J(T') = cst\left[\left(T + \frac{d\epsilon}{dx}\lambda_{esc}\right)\left(T + \frac{d\epsilon}{dx}\lambda_{esc} + 2\epsilon_0\right)\right]^{-\gamma/2}$$

(c) loss due to fragmentation

$$n_p(\epsilon) = \frac{Q(E)\,\tau_{esc}}{1 + \lambda_{esc}(E)/\lambda_p} \quad \Rightarrow \quad J_p(T) = \frac{J_0}{1 + \lambda_{esc}/\lambda_p}$$

(d) Solar modulation seems extremely efficient!

Exercise 3



$$\frac{dE}{dt} = \frac{\varepsilon E}{T_{cycle}}.$$

$$T_{cycle} = \frac{4}{c}\left(\frac{D_1}{v_1} + \frac{D_2}{v_2}\right)$$

Bohm diffusion

$\hookrightarrow$ Ideal gas : $D_1 = D_2 = D_{min} = \frac{1}{3}\lambda_D v$

$$\lambda_D \approx R_L = \frac{P}{ZeB}$$

(+) $U_2 = \frac{U_1}{4}$ $\quad$ E

$$\Rightarrow T_{cycle} = \frac{4}{c}\left(\frac{1}{3}\frac{Pv}{ZeB}\right)\left(\frac{1}{U} + \frac{4}{U}\right)$$

$$\boxed{T_{cycle} = \frac{20}{3}\frac{E}{ZeBcU}}$$

thus $\frac{dE}{dt} = \frac{3}{20}\varepsilon ZeBcU$

$$\boxed{\frac{dE}{dt} = \frac{3}{20} ZeBU^2}$$

$\langle \varepsilon \rangle = \frac{4}{3}\beta$ $\quad$ Fermi 1st order

$v_2 = \frac{v_1}{4}$

$$= \frac{4}{3}\left(\frac{v_1}{c} - \frac{v_2}{c}\right)$$

$$= \frac{4}{3}\left(\frac{3}{4}\frac{v}{c}\right)$$

$$\boxed{\langle \varepsilon \rangle = \frac{v}{c}}$$

(a) free expansion $U = U_{sw} = cte = 10^7$ m/s

$$\boxed{E_{max}^{sw} = \int_0^{t_{sw}} \frac{dE}{dt}\,dt = \frac{3}{20}\, Ze\, B\, U_{sw}^2\, t_{sw}}$$

for proton ($Z = 1$), $B = 100\,\mu G$, $t_{sw} = 500$ yrs.

$$\boxed{E_{max}^{sw} = 2{,}37 \text{ PeV}}$$

(b) Sedov - Taylor phase $U = U_{sw}\left(\frac{t}{t_{sw}}\right)^{-3/5}$

$$E_{max}^{SD} = \int_{t_{sw}}^{t_{SD}} \frac{3}{20}\, ZeB\, U_{sw}^2 \left(\frac{t}{t_{sw}}\right)^{-6/5} dt$$

$$E_{max}^{SD} = \frac{3}{20}\, \frac{Z\, e\, B\, U_{sw}^2}{t_{sw}^{-6/5}} \left[-5\, t^{-1/5}\right]_{t_{sw}}^{t_{SD}}$$

$t^{-6/5}$

$$E_{max}^{SD} = \frac{15}{20}\, Z\, e\, B\, \frac{U_{sw}^2}{t_{sw}^{-6/5}} \left[t_{sw}^{-1/5} - t_{SD}^{-1/5}\right]$$

$$\boxed{E_{max}^{SD} = \frac{3}{4}\, Z\, e\, B\, U_{sw}^2\, t_{sw} \left[1 - \left(\frac{t_{SD}}{t_{sw}}\right)^{-1/5}\right]}$$

$$\boxed{E_{max} = 2{,}35 \text{ PeV}}$$

$t_{SD} = 10^5$ yrs !

(b) Accelerated from $t = 0$ to t_{SD}

$$E_{max} = \int_0^{t_{SD}} \frac{dE}{dt}\,dt = \int_0^{t_{sw}} \frac{dE}{dt}\,dt + \int_{t_{sw}}^{t_{SD}} \frac{dE}{dt}\,dt$$

$$= E_{max}^{sw} + E_{max}^{SD}$$

$$\boxed{E_{max} = 4{,}72 \text{ PeV}}$$