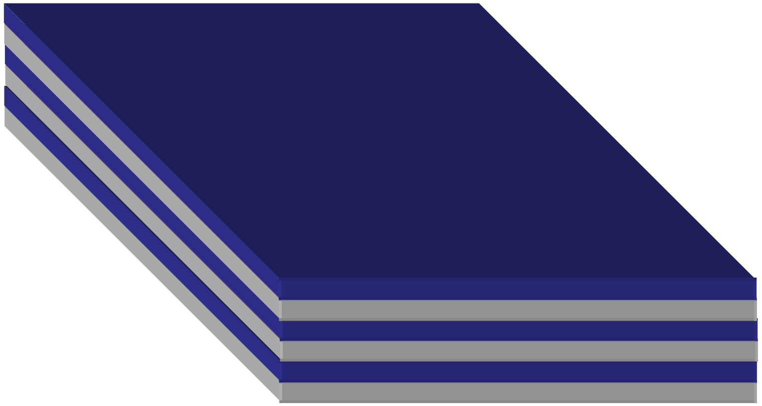
Computational Photonics

Waves in stratified media

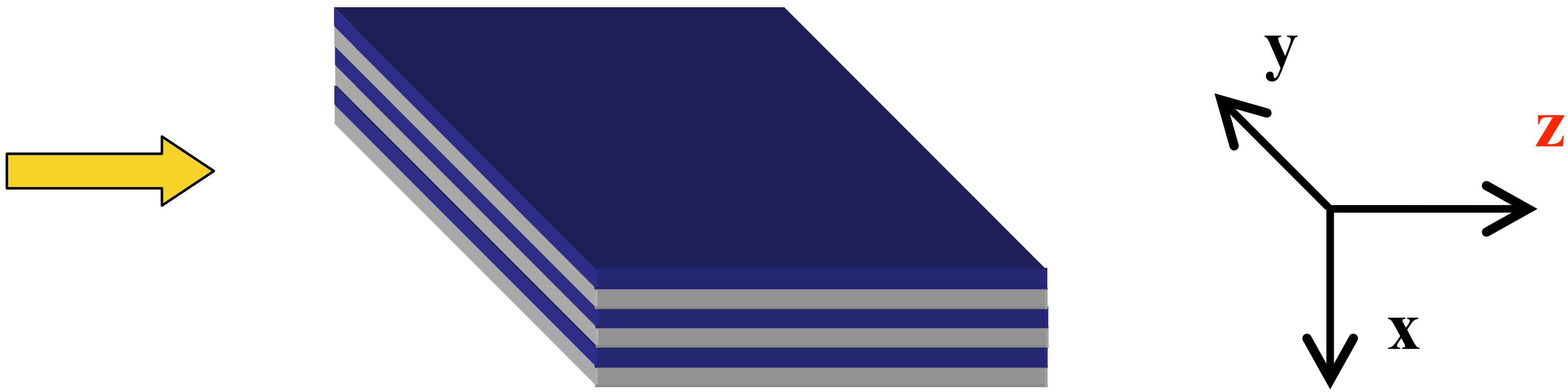
Matrix method for stratified (layered) media



- waves in homogeneous media
- single interface and then a stratified media
- deriving general expressions for transmission and reflection
- detailing how the dispersion relation of guided modes can be extracted from the transmission coefficients

General idea

definition of a principal propagation direction

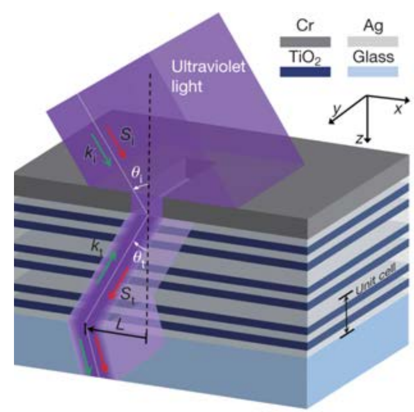


Wave guides

$$\mathbf{E}(x,y,z) = \mathbf{A}(x,y)e^{\imath k z}$$

- multi layer waveguides

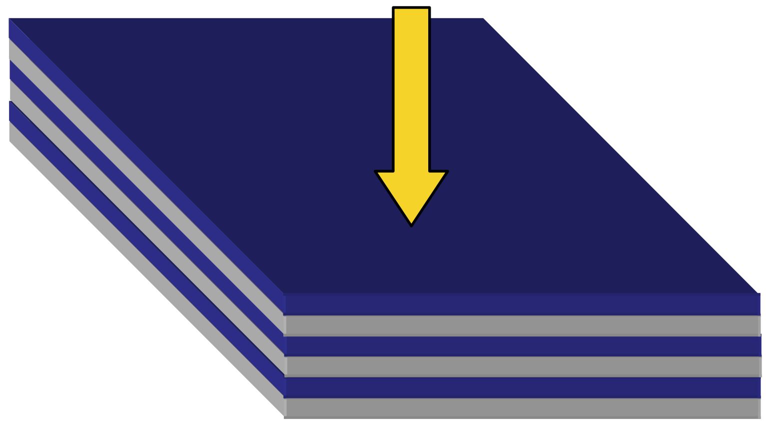
- Bragg waveguides

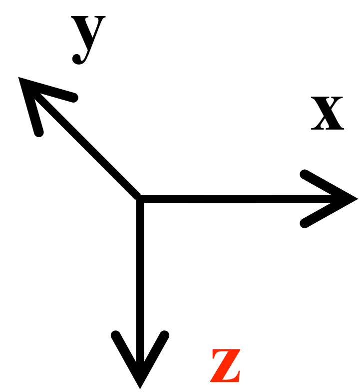


Nature **497**, 470–474

General idea

definition of a principal propagation direction

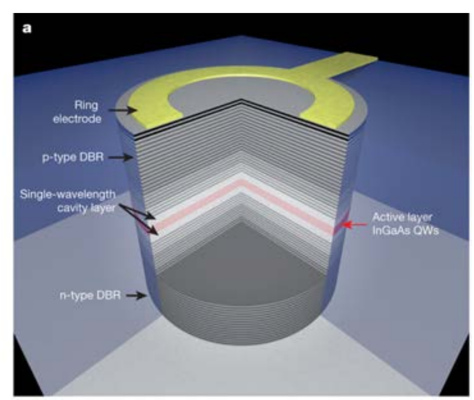




surface coatings

$$\mathbf{E}(x, y, z) = \mathbf{A}(x, y)e^{\imath k z}$$

- Bragg mirrors
- chirped mirrors for dispersion compensation
- interferometers



Nature **497**, 348–352

General idea

- separating the domains in regions for which an analytical solution for the wave equation exist = mode expansion (free space ⟹ plane wave)

- expanding the field into a superposition of those modes

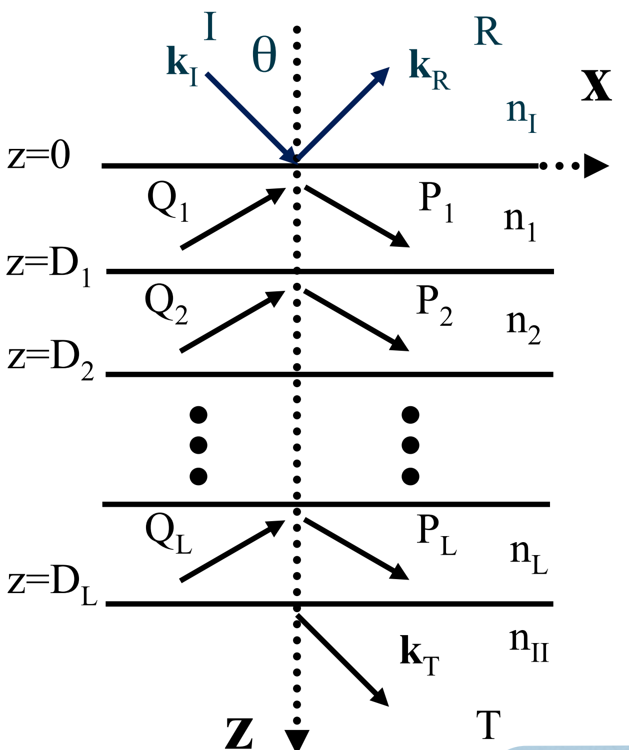
  ⟹ Adjusting the amplitudes of each mode, such that boundary conditions are met (exact or approximately)

- modes should be adopted to the geometry

- assumptions/prerequisites
  – stationary
  – layers in y-z-plane
  – incident fields in x-z-plane
  – full invariance in y-direction

Reflection / Transmission at a stack of layers

assumption: slab consist of an arbitrary number of layers (TE polarisation)



→ fields in homogenous space have to be a solution to the wave equation

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}+\frac{\omega^2}{c^2}\epsilon_f(x)\right]\mathbf{E}(x,z,\omega)=0$$

→ invariant in x-direction justifies the use of plane waves in this direction in each region

ansatz: $$\mathbf{E}(x,z)=\mathbf{E}(z)e^{ik_x x}$$

$$\mathbf{H}(x,z)=\mathbf{H}(z)e^{ik_x x}$$

→

$$\left[\frac{\partial^2}{\partial z^2}+\frac{\omega^2}{c^2}\epsilon_f(\omega)-k_x^2\right]\mathbf{E}(z)=0$$

$$\mathbf{H}_{\mathrm{TE}}(x,z)=-\frac{i}{\omega\mu_0}\mathbf{rot}\mathbf{E}_{\mathrm{TE}}(x,z)$$

Separating the problem into 2 polarisations

⇒ continuity of the tangential electric and magnetic field

tangential fields (TE): $E_y = E \quad H_x$

$$\left[\frac{\partial^2}{\partial z^2} + k_{fz}^2(k_x,\omega)\right] E(z) = 0 \qquad k_{fz}^2(k_x,\omega) = \frac{\omega^2}{c^2}\epsilon_f(\omega) - k_x^2$$

$$H_x(z) = -\frac{i}{\omega\mu_0}\frac{\partial}{\partial z}E(z) \qquad i\omega\mu_0 H_x(z) = \frac{\partial}{\partial z}E(z)$$

tangential fields (TM): $H_y = H \quad E_x$

$$\left[\frac{\partial^2}{\partial z^2} + k_{fz}^2(k_x,\omega)\right] H(z) = 0$$

$$E_x(z) = \frac{i}{\omega\epsilon_0\epsilon_f}\frac{\partial}{\partial z}H(z) \qquad -i\omega\epsilon_0 E_x(z) = \frac{1}{\epsilon_f}\frac{\partial}{\partial z}H(z)$$

Polarisation independent formulation

- calculating the fields and their normal derivatives at $z = d$
- need to know the respective values at $z = 0$
- equations are the same ⇨ simultaneous treatment

$$E, H \longrightarrow F$$

$$i\omega\mu_0 H_x, -i\omega\epsilon_0 E_x \longrightarrow G$$

$$\left[\frac{\partial^2}{\partial z^2} + k_{fz}^2(k_x, \omega)\right] F(z) = 0$$

$$G(z) = \alpha_f \frac{\partial}{\partial z} F(z)$$

$$\alpha_{f\mathrm{TE}} = 1$$

$$\alpha_{f\mathrm{TM}} = \frac{1}{\epsilon_f}$$

Solving the initial value problem

$$F(z) = C_1 e^{ik_{fz}z} + C_2 e^{-ik_{fz}z}$$

$$G(z) = \alpha_f \frac{\partial}{\partial z} F(z) = i\alpha_f k_{fz} \left[C_1 e^{ik_{fz}z} - C_2 e^{-ik_{fz}z} \right]$$

need to know constants C_1 and C_2

$F(0)$ and $G(0)$ are known $\Rightarrow$ C_1 and C_2

$$F(0) = C_1 + C_2 \qquad G(0) = i\alpha_f k_{fz} \left[C_1 - C_2 \right]$$

$$C_1 = \frac{1}{2}\left[F(0) - \frac{i}{\alpha_f k_{fz}} G(0) \right] \quad C_2 = \frac{1}{2}\left[F(0) + \frac{i}{\alpha_f k_{fz}} G(0) \right]$$

A single transfer matrix

$$F(z) = \cos(k_{fz} z)\, F(0) + \frac{1}{\alpha_f k_{fz}} \sin(k_{fz} z)\, G(0)$$

$$G(z) = -\alpha_f k_{fz} \sin(k_{fz} z)\, F(0) + \cos(k_{fz} z)\, G(0)$$

writing the equations in matrix form

$$\begin{Bmatrix} F(z) \\ G(z) \end{Bmatrix} = \hat{\mathbf{m}} \begin{Bmatrix} F(0) \\ G(0) \end{Bmatrix}$$

$$\hat{\mathbf{m}}(z) = \begin{Bmatrix} \cos(k_{fz} z) & \frac{1}{k_{fz}\alpha_f} \sin(k_{fz} z) \\ -k_{fz}\alpha_f \sin(k_{fz} z) & \cos(k_{fz} z) \end{Bmatrix}$$

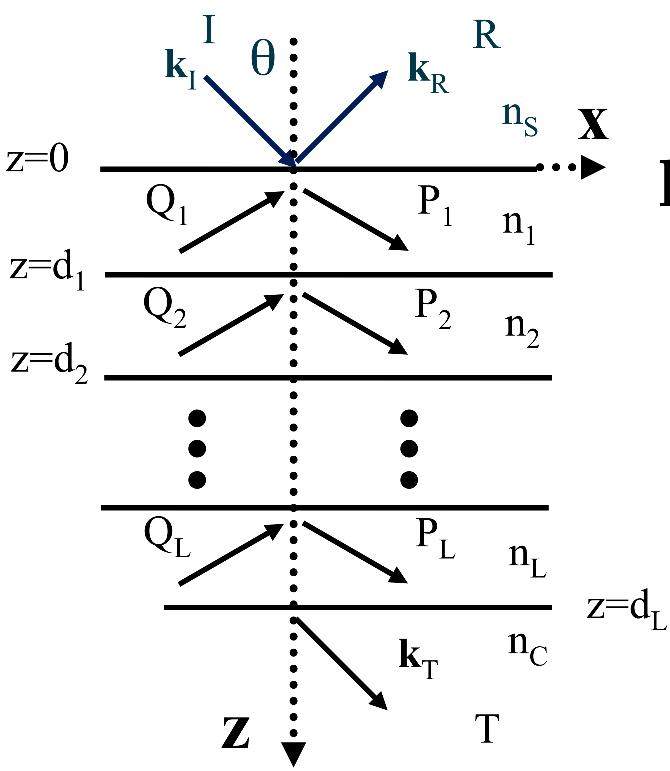
single transfer matrix

Transfer matrix of a stack

N-layers:

$$\begin{pmatrix} F \\ G \end{pmatrix}_{d_1+d_1+..+d_N=D} = \prod_{i=1}^{N} \hat{\mathbf{m}}_{\mathbf{i}}\left(d_i\right) \begin{pmatrix} F \\ G \end{pmatrix}_0 = \hat{\mathbf{M}} \begin{pmatrix} F \\ G \end{pmatrix}_0$$

Incident and transmitted field



$$\mathbf{k}_I = \begin{pmatrix} k_x \\ 0 \\ k_{sz} \end{pmatrix} \quad \mathbf{k}_R = \begin{pmatrix} k_x \\ 0 \\ -k_{sz} \end{pmatrix} \quad \mathbf{k}_T = \begin{pmatrix} k_x \\ 0 \\ k_{cz} \end{pmatrix}$$

$$k_{cz} = \sqrt{\frac{\omega^2}{c^2}\epsilon_c(\omega) - k_x^2}$$

$$k_{sz} = \sqrt{\frac{\omega^2}{c^2}\epsilon_s(\omega) - k_x^2}$$

Coupling an incident field

ansatz for the field in the substrate:

$$F_s(x,z) = e^{ik_x x}\left[F_I e^{ik_{sz}z} + F_R e^{-ik_{sz}z}\right]$$

$$G_s(x,z) = i\alpha_s k_{sz} e^{ik_x x}\left[F_I e^{ik_{sz}z} - F_R e^{-ik_{sz}z}\right]$$

ansatz for the field in the stratified media:
(known from matrix method)

$$F_f(x,z) = e^{ik_x x}F(z) \qquad G_f(x,z) = e^{ik_x x}G(z)$$

$$\begin{pmatrix} F \\ G \end{pmatrix}_z = \hat{\mathbf{M}} \begin{pmatrix} F \\ G \end{pmatrix}_0$$

ansatz for the field in the cladding:

$$F_c(x,z) = e^{ik_x x} F_T e^{ik_{cz}(z-D)}$$

$$G_c(x,z) = i\alpha_c k_{cz} e^{ik_x x} F_T e^{ik_{cz}(z-D)}$$

Coupling an incident field

$$\begin{pmatrix} F \\ G \end{pmatrix}_D = \hat{\mathbf{M}} \begin{pmatrix} F \\ G \end{pmatrix}_0$$

cladding field at $z = D$ substrate field at $z = 0$

$$\begin{pmatrix} F_T \\ i\alpha_c k_{cz} F_T \end{pmatrix} = \begin{Bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{Bmatrix} \begin{pmatrix} F_I + F_R \\ i\alpha_s k_{sz} \left(F_I - F_R\right) \end{pmatrix}$$

⇨ two equations for the two unknown amplitudes

R/T coefficients of the generalised variables

$$F_R = \frac{(\alpha_s k_{sz} M_{22} - \alpha_c k_{cz} M_{11}) - i\,(M_{21} + \alpha_s k_{sz} \alpha_c k_{cz} M_{12})}{(\alpha_s k_{sz} M_{22} + \alpha_c k_{cz} M_{11}) + i\,(M_{21} - \alpha_s k_{sz} \alpha_c k_{cz} M_{12})} F_I$$

$$F_T = \frac{2\alpha_s k_{sz}\,(M_{11} M_{22} - M_{12} M_{21})}{(\alpha_s k_{sz} M_{22} + \alpha_c k_{cz} M_{11}) + i\,(M_{21} - \alpha_s k_{sz} \alpha_c k_{cz} M_{12})} F_I$$

$$\mathrm{D} = (\alpha_s k_{sz} M_{22} + \alpha_c k_{cz} M_{11}) + i\,(M_{21} + \alpha_s k_{sz} \alpha_c k_{cz} M_{12})$$

⇨ absorptionless media $F_T = \frac{2\alpha_s k_{sz}}{\mathrm{D}} F_I$

$$\|\, \hat{\mathbf{m}}(x) \,\| = 1$$

R/T coefficients for TE polarisation

TE polarisation: $F = E = E_y \qquad \alpha_{TE} = 1$

reflected field: $E_R^{TE} = R_{TE} E_I^{TE}$

$$R_{TE} = \frac{(k_{sz} M_{22} - k_{cz} M_{11}) - i\,(M_{21} + k_{sz} k_{cz} M_{12})}{(k_{sz} M_{22} + k_{cz} M_{11}) + i\,(M_{21} - k_{sz} k_{cz} M_{12})}$$

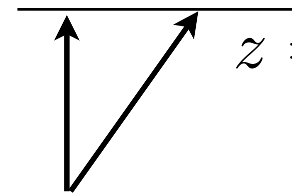
transmitted field: $E_T^{TE} = T_{TE} E_I^{TE}$

$$T_{TE} = \frac{2k_{sz}}{\mathrm{D}}$$

and similar for TM polarisation: $F = H = H_y \qquad \alpha_{TM} = \frac{1}{\epsilon}$

Calculating the efficiencies

- calculating the energy flux
- perpendicular through a surface



$$\langle \mathbf{S} \rangle \cdot \mathbf{e}_z = \frac{1}{2} \Re \left(\mathbf{E} \times \mathbf{H}^* \right) \cdot \mathbf{e}_z$$

using: $$\mathbf{H}^* = \frac{1}{\omega \mu_0} \left(\mathbf{k}^* \times \mathbf{E}^* \right)$$

$$\langle \mathbf{S} \rangle \cdot \mathbf{e}_z = \frac{1}{2\omega \mu_0} \Re \left(\mathbf{k}^* \cdot \mathbf{e}_z \right) \mid \mathbf{E} \mid^2 = \frac{1}{2\omega \mu_0} \Re \left(k_z \right) \mid \mathbf{E} \mid^2$$

$$\rho_{TE,TM} = \mid R_{TE,TM} \mid^2$$

$$\tau_{TE,TM} = \frac{\Re \left(k_{cz} \right)}{k_{sz}} \mid T_{TE,TM} \mid^2$$

Flow chart of a program

- ⇨ assuming a stack of layers each having a certain width and a relative permittivity
- ⇨ stack embedded in a substrate and a cladding medium and illuminated with a plane wave of a specified wavelength, polarisation, and angle of incidence

⇩

calculate the transfer matrix $\hat{\mathbf{M}}$

⇩

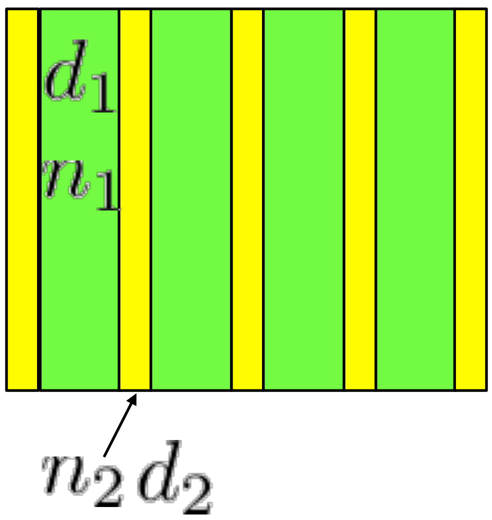
evaluate the coefficients F_R and F_T

⇩

evaluating the efficiencies

Example: Bragg-mirror

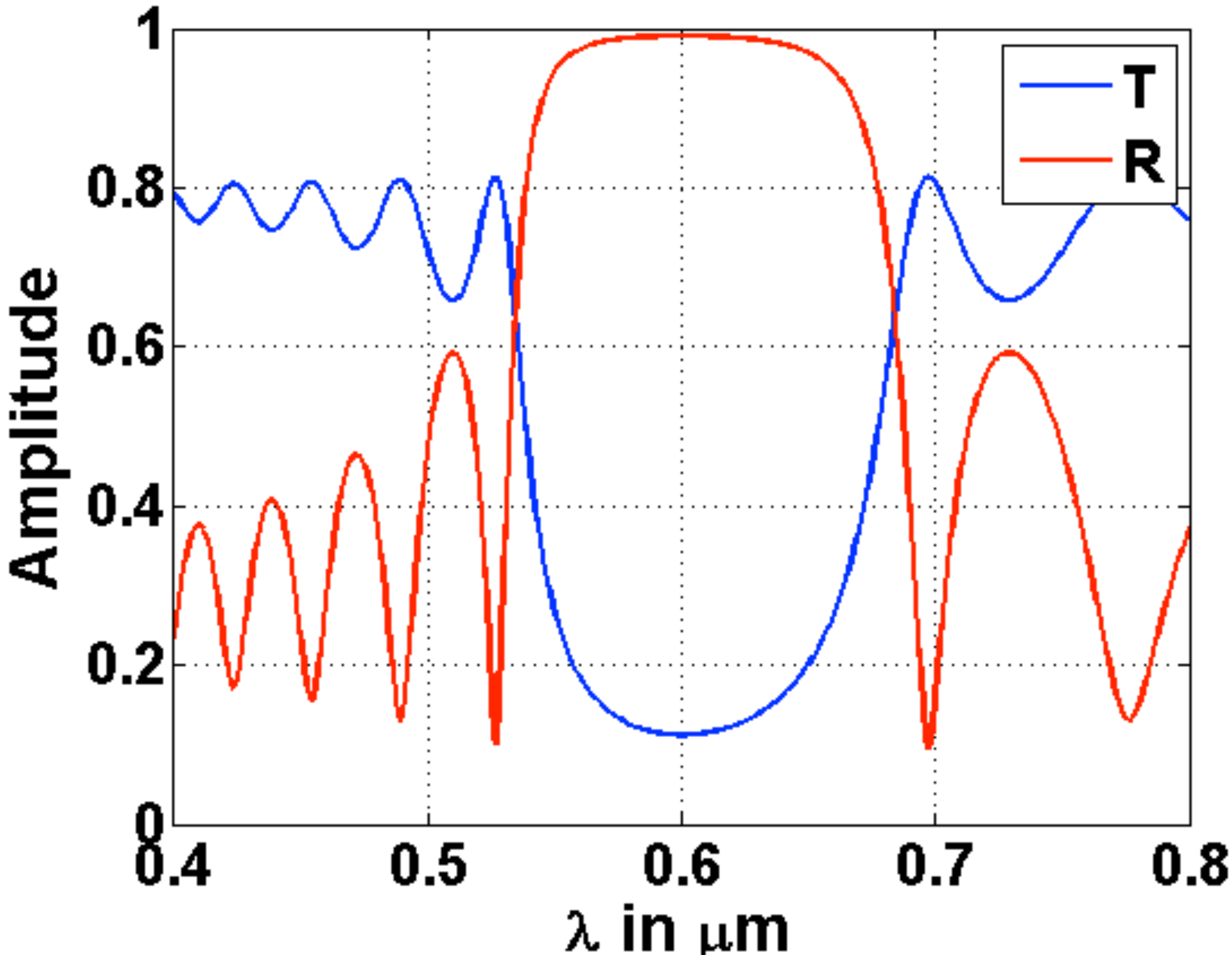
⇨ series of alternating dielectric layers with a chosen thickness, such that reflected light interferes constructively



thickness of each layer is chosen

$$d_i n_i = \frac{\lambda_0}{4}$$

Example: Bragg-mirror



10 layers, $n_1 = 1.5$, $n_2 = 2.0$ and $\lambda_{\text{Design}} = 0.6 \mu\text{m}$

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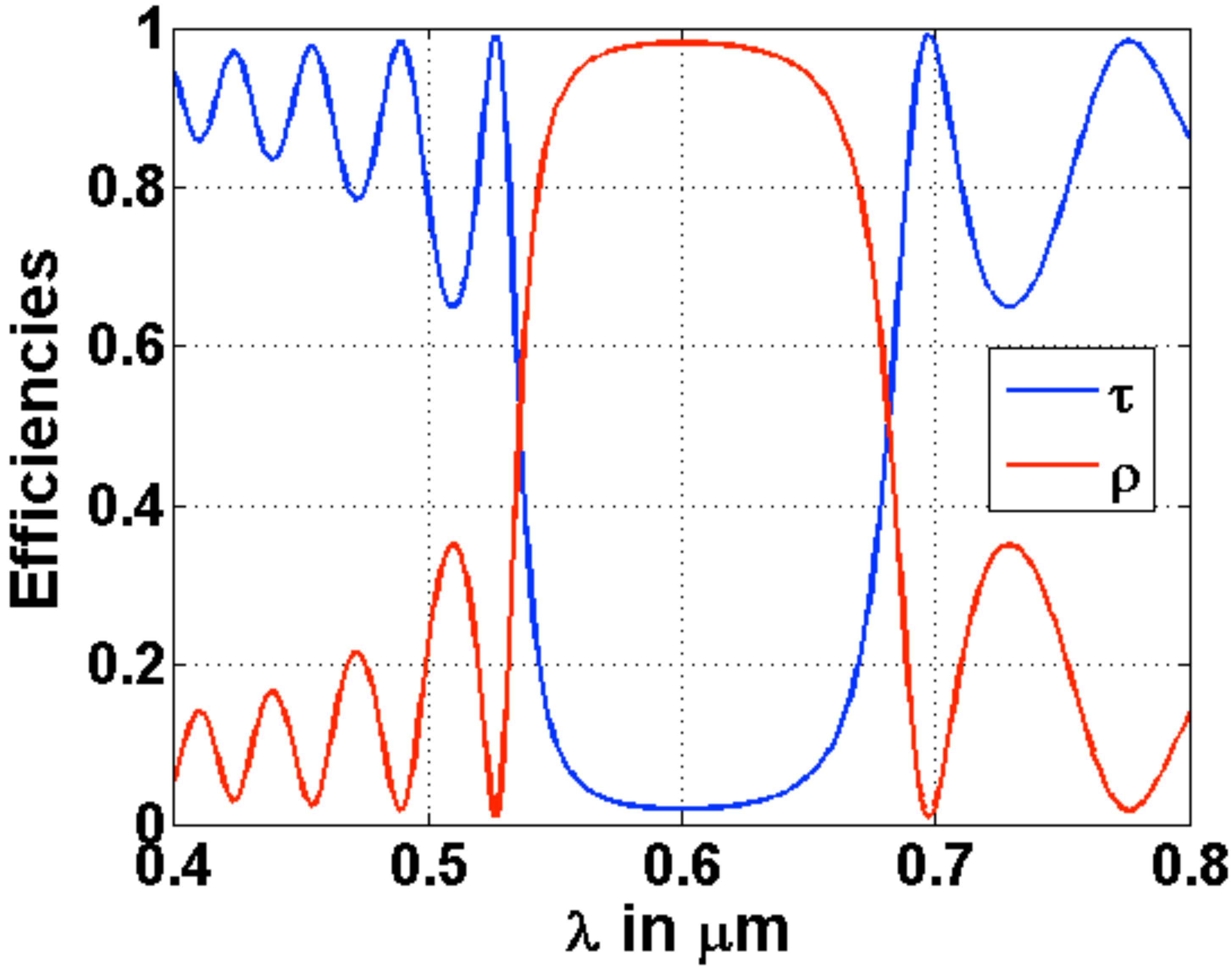

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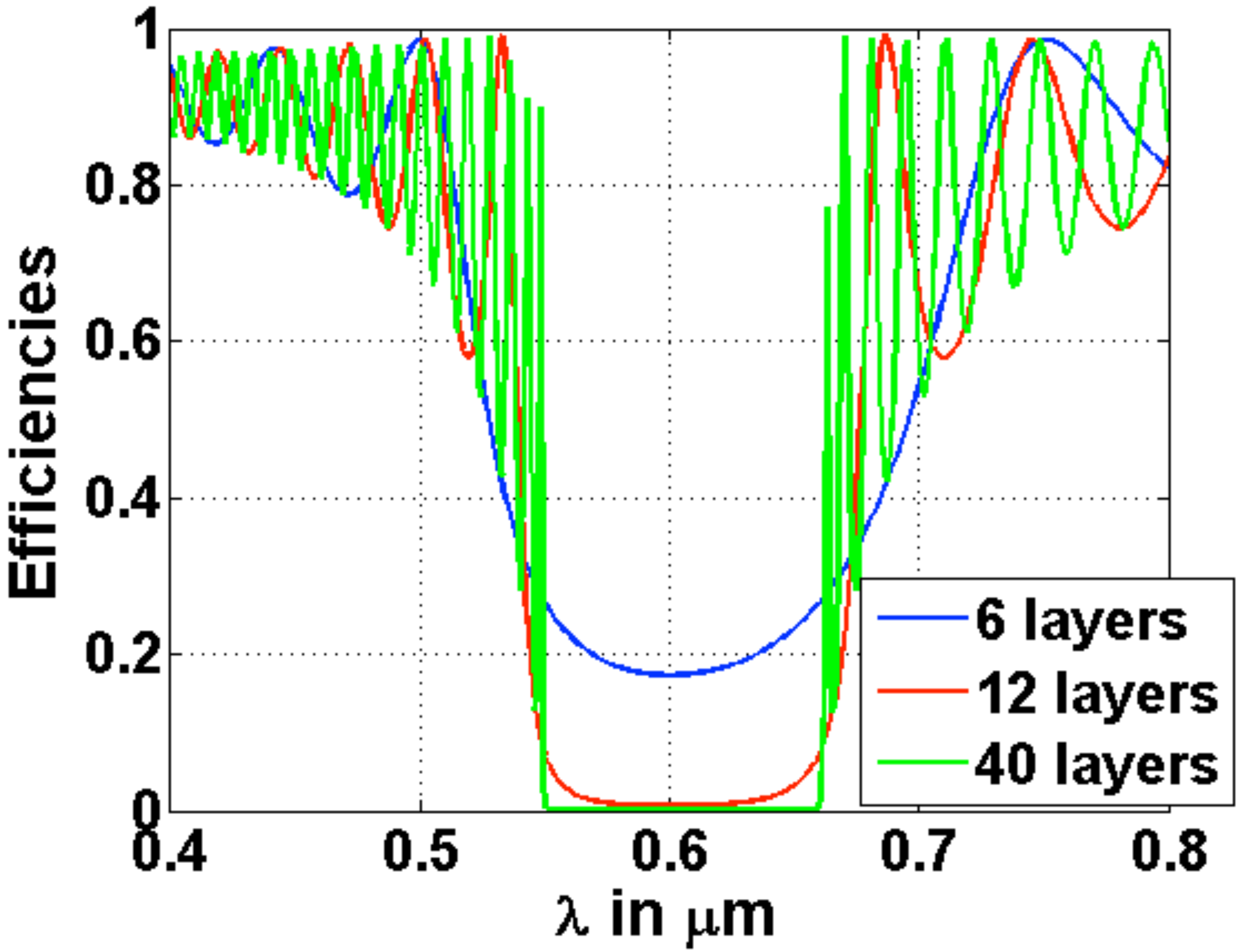

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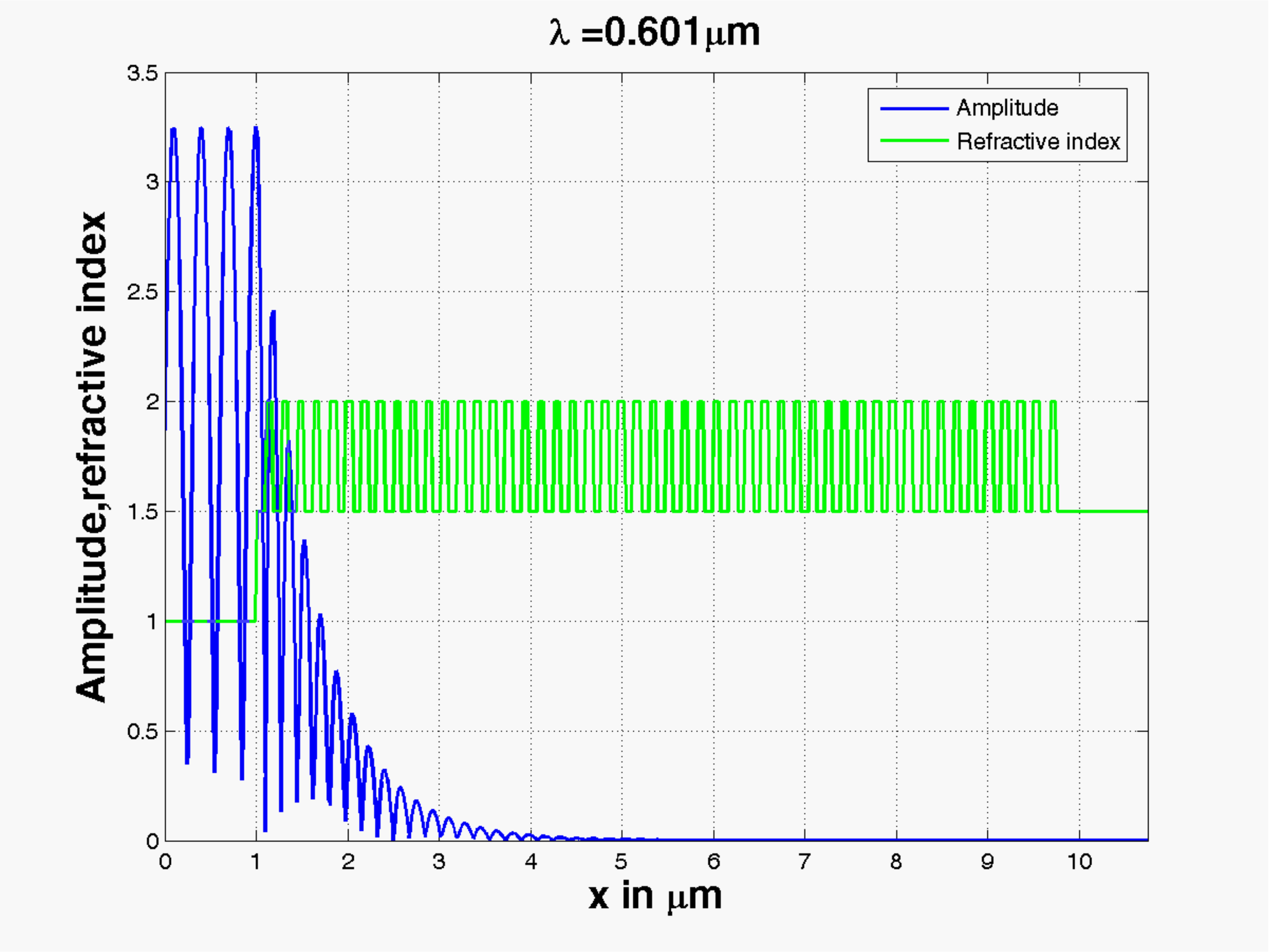
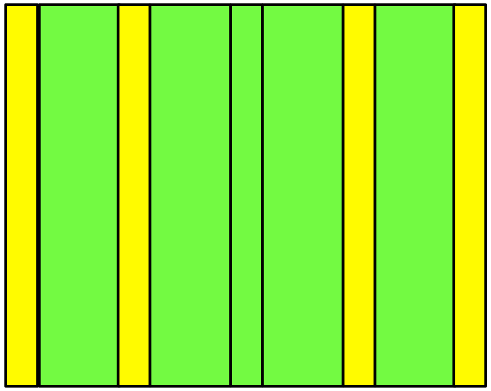


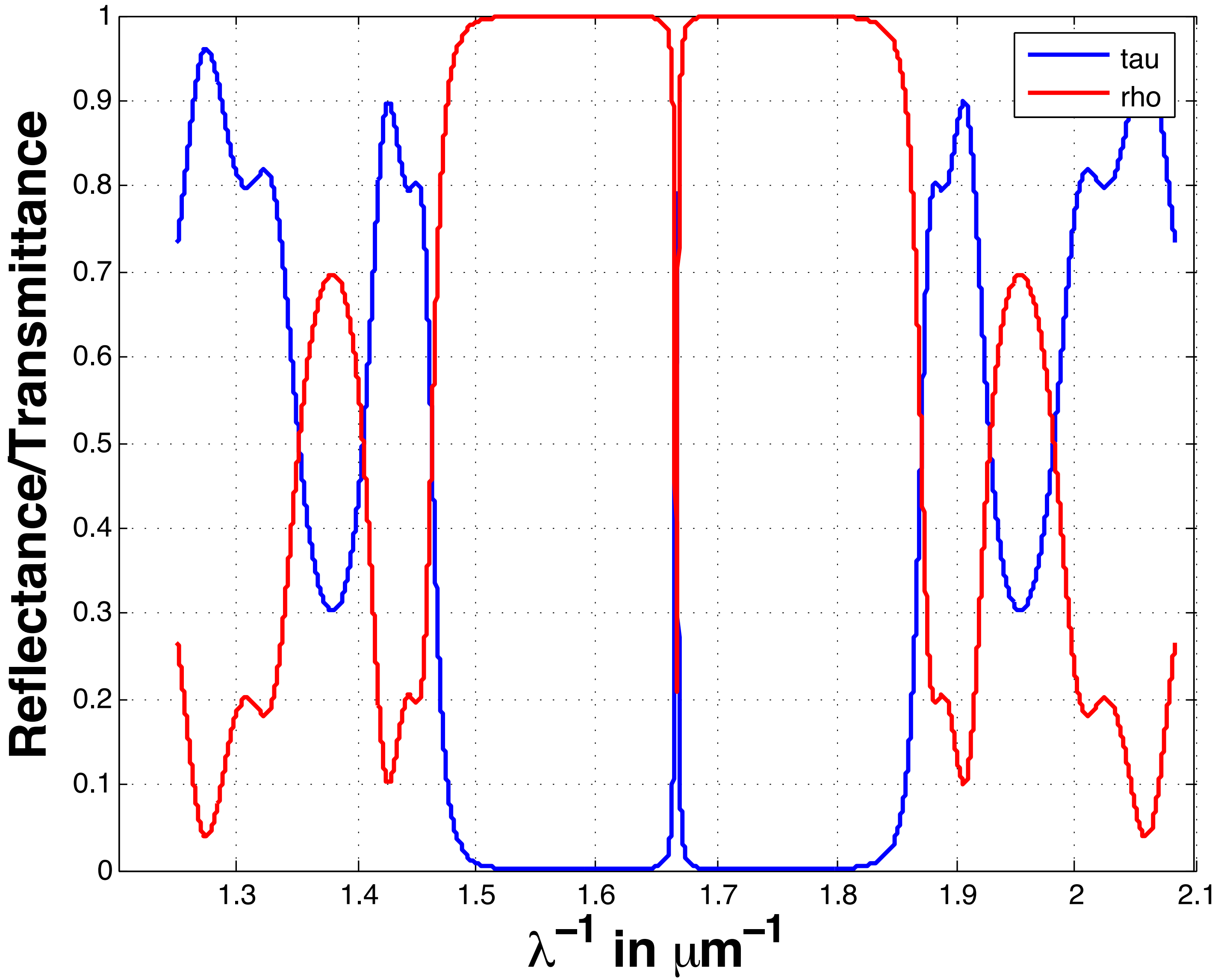
Example: Bragg-mirror with defect

→ series of alternating dielectric layers with a chosen thickness, such that reflected light interferes constructively



introducing a defect will couple light evanescently through the structure

Example: Bragg-mirror with defect



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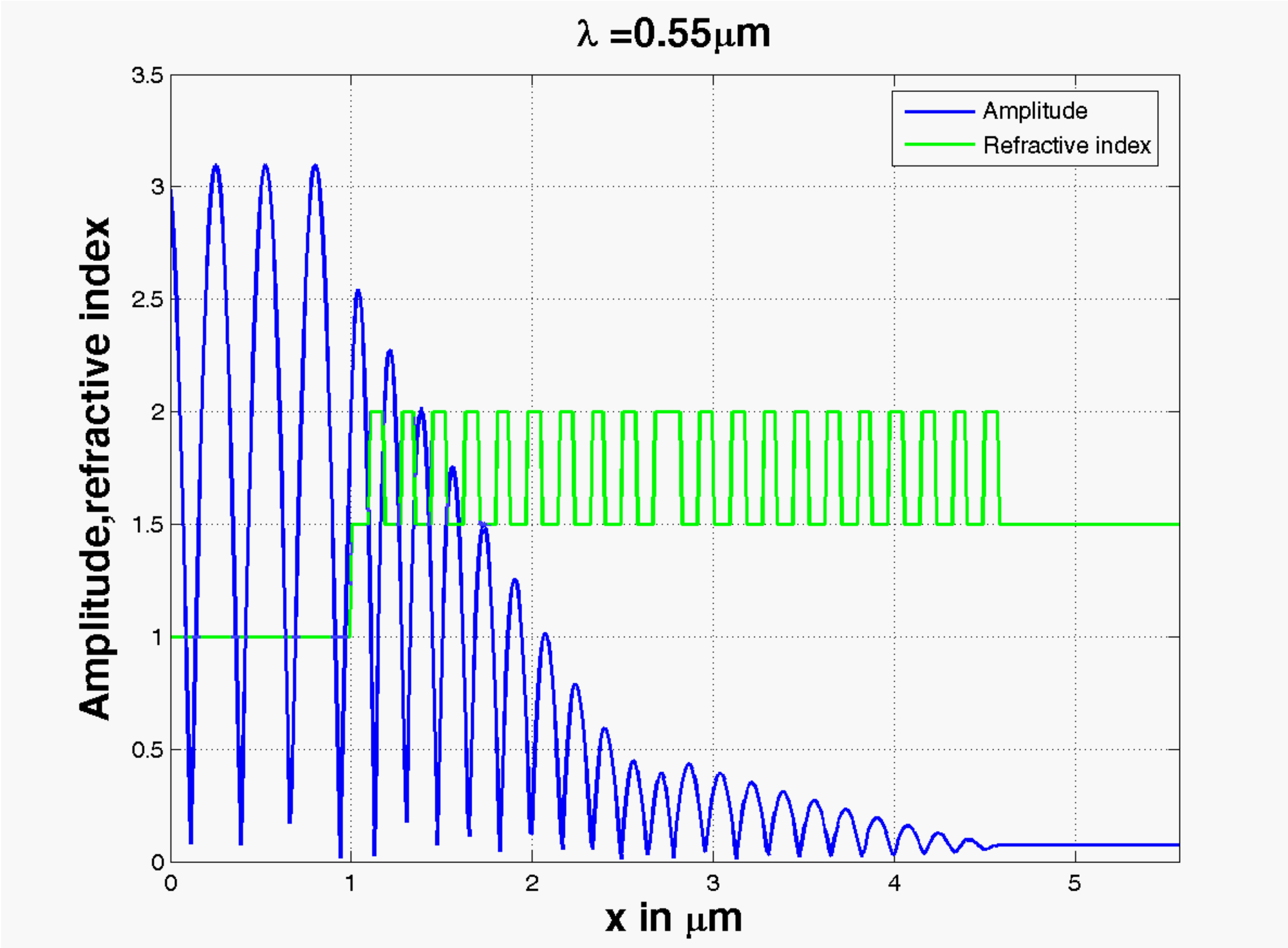

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