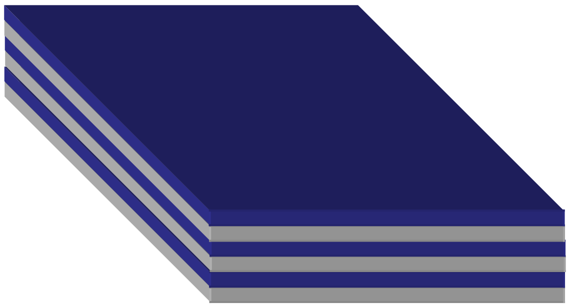
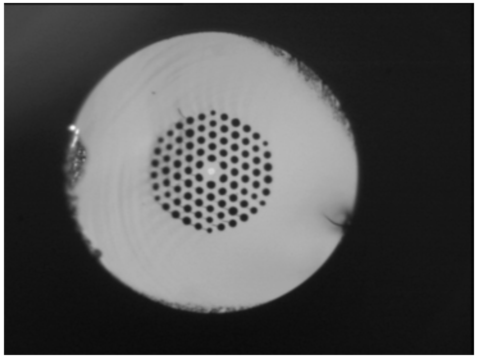
Computational Photonics

Guided modes in slab geometries

Calculating the modes of waveguides





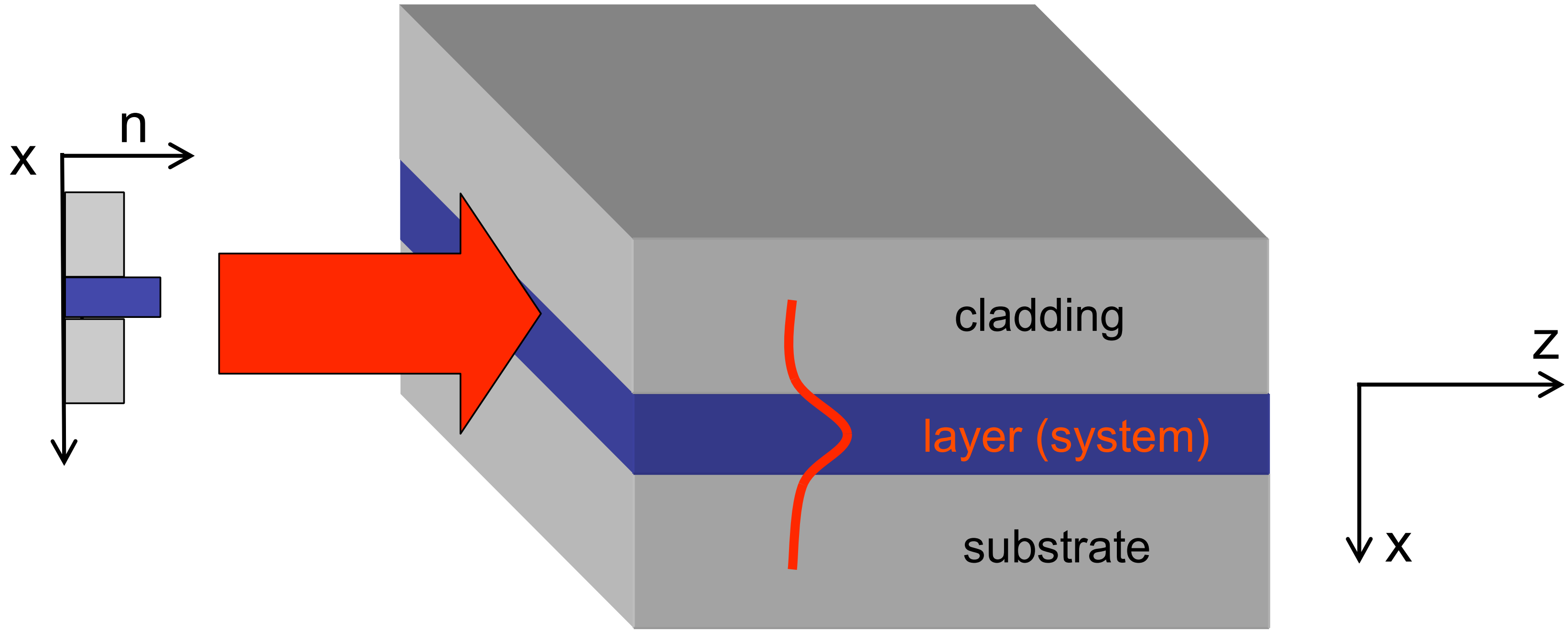
guided modes are self-consistent solutions to Maxwell's equations that propagate invariant (except a phase accumulation) in a principal propagation direction along which the geometry does not change

need to know dispersion relation (dependency of the propagation constant on the frequency) and field distributions

study those properties here for systems of increasing complexity

(film - scalar 2D - full vectorial 2D)

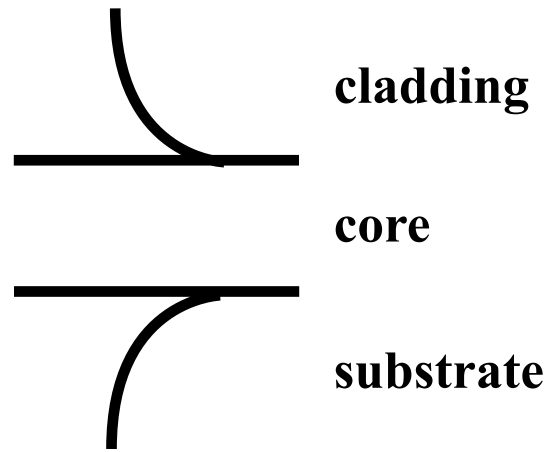
Modes in a layer system



- no y dependency, phase rotation in z-direction
  - ⇨ wave propagation without diffraction
  - ⇨ miniaturisation of optics
  - ⇨ optical signal communication

How can waves be bound?

→ principal mechanism: total internal reflection



- plane wave in propagation direction: $$e^{ik_z z}$$
- evanescent wave in substrate and cladding: $$k_z^2 > \frac{\omega^2}{c^2} \max\{\epsilon_{\mathrm{s,c}}(\omega)\}$$
- oscillating solution in core: $$A \sin(k_{fx} x) + B \cos(k_{fx} x)$$

→ $$k_z^2 < \frac{\omega^2}{c^2} \max\{\epsilon_{\mathrm{i}}(\omega)\}$$

How to find their dispersion relation

$$\frac{\omega}{c}\max\{\mathrm{n}_{\mathrm{s,c}}(\omega)\} < k_z < \frac{\omega}{c}\max\{\mathrm{n}_{\mathrm{i}}(\omega)\}$$

guided waves are resonances of the system

⟹ singularities of R and T

$$R = \frac{F_{\mathbf{R}}}{F_{\mathbf{I}}} = \frac{\left(\alpha_{\mathbf{s}} k_{\mathbf{sx}} M_{22} - \alpha_{\mathbf{c}} k_{\mathbf{cx}} M_{11}\right) - \mathbf{i}\left(M_{21} + \alpha_{\mathbf{s}} k_{\mathbf{sx}} \alpha_{\mathbf{c}} k_{\mathbf{cx}} M_{12}\right)}{\left(\alpha_{\mathbf{s}} k_{\mathbf{sx}} M_{22} + \alpha_{\mathbf{c}} k_{\mathbf{cx}} M_{11}\right) + \mathbf{i}\left(M_{21} - \alpha_{\mathbf{s}} k_{\mathbf{sx}} \alpha_{\mathbf{c}} k_{\mathbf{cx}} M_{12}\right)}$$

singularities:

$$\left(\alpha_{\mathbf{s}} k_{\mathbf{sx}} M_{22} + \alpha_{\mathbf{c}} k_{\mathbf{cx}} M_{11}\right) + \mathbf{i}\left(M_{21} - \alpha_{\mathbf{s}} k_{\mathbf{sx}} \alpha_{\mathbf{c}} k_{\mathbf{cx}} M_{12}\right) = 0$$

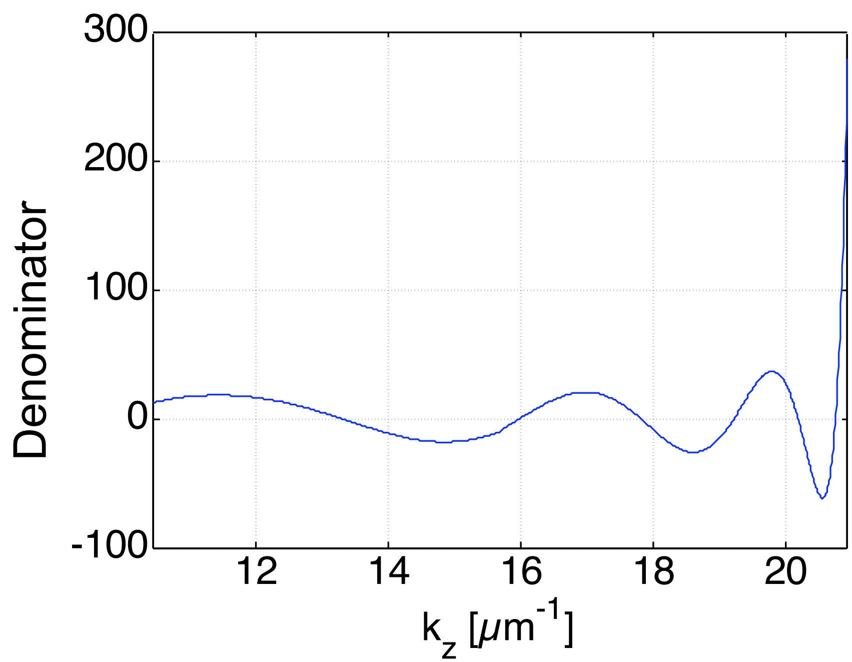
the problem of finding a guided mode is reduced to finding a root

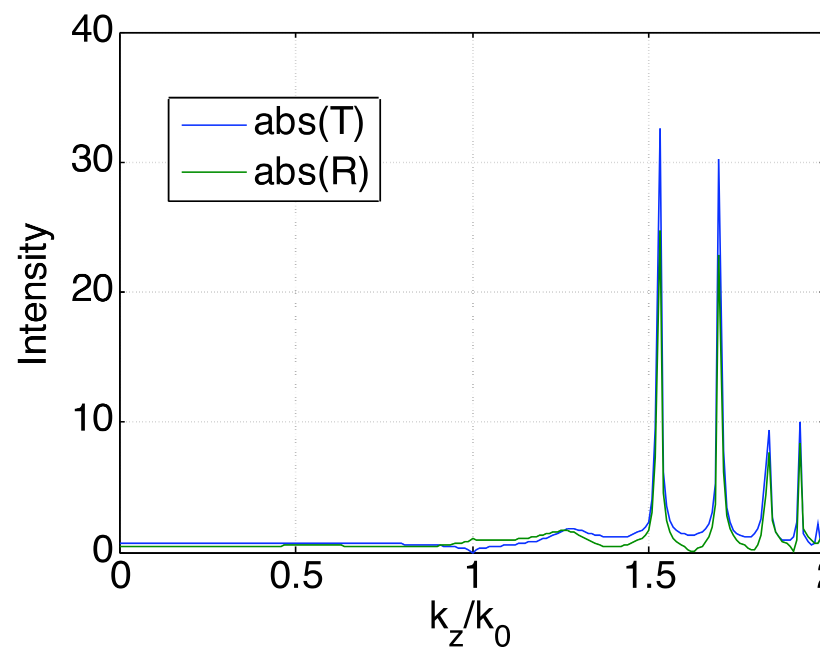
Reduced to the problem of finding a root

field in substrate/superstrate is evanescent

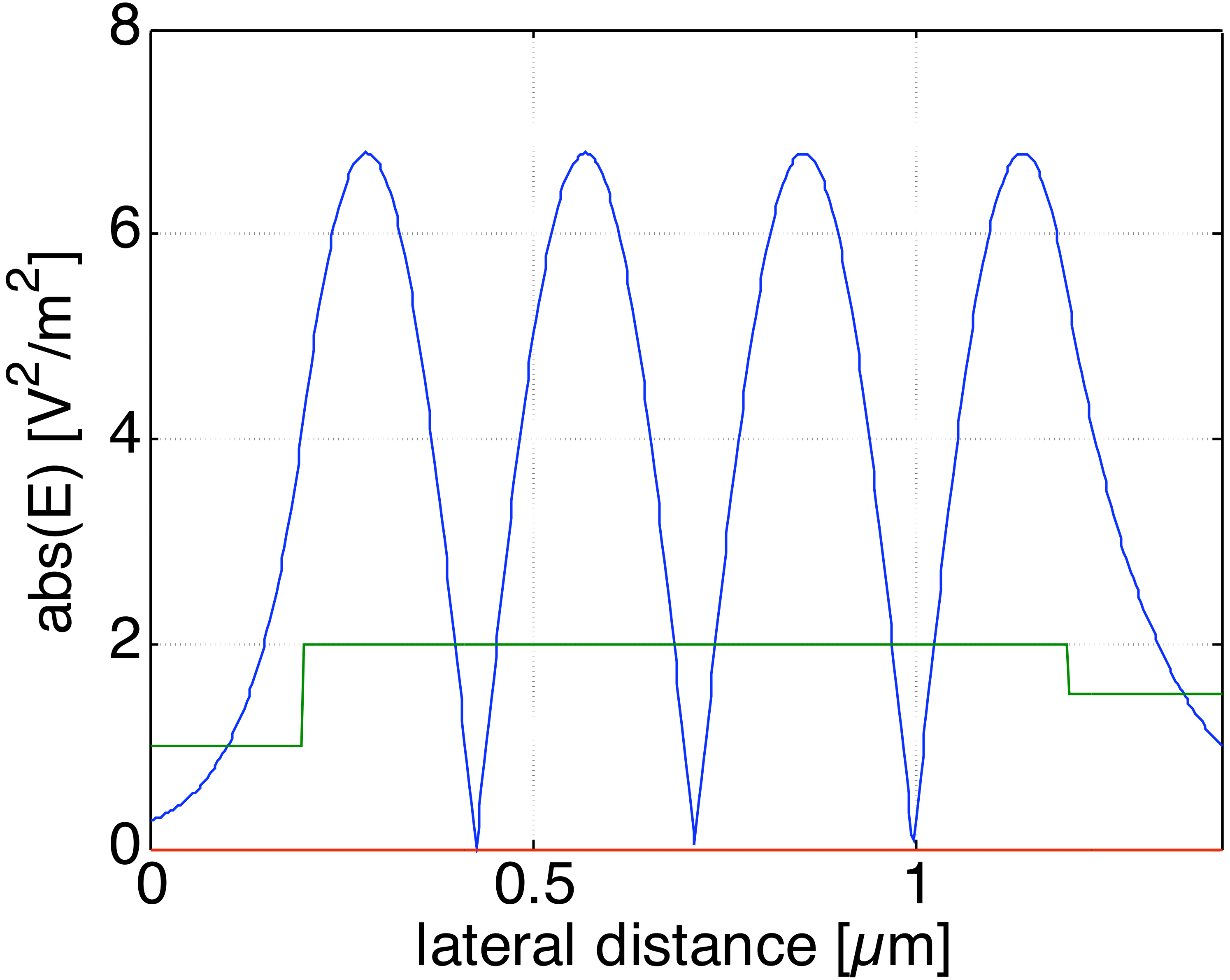
$$k_{sx} = i\mu_s, \qquad k_{cx} = i\mu_c, \qquad \mu_{s,c} = \sqrt{k_z^2 - \frac{\omega^2}{c^2}\varepsilon_{s,c}(\omega)} > 0$$

$$\alpha_s \mu_s M_{22}^{TE,TM} + \alpha_c \mu_c M_{11}^{TE,TM} + M_{21}^{TE,TM} + \alpha_c \mu_c \alpha_s \mu_s M_{12}^{TE,TM} = 0$$

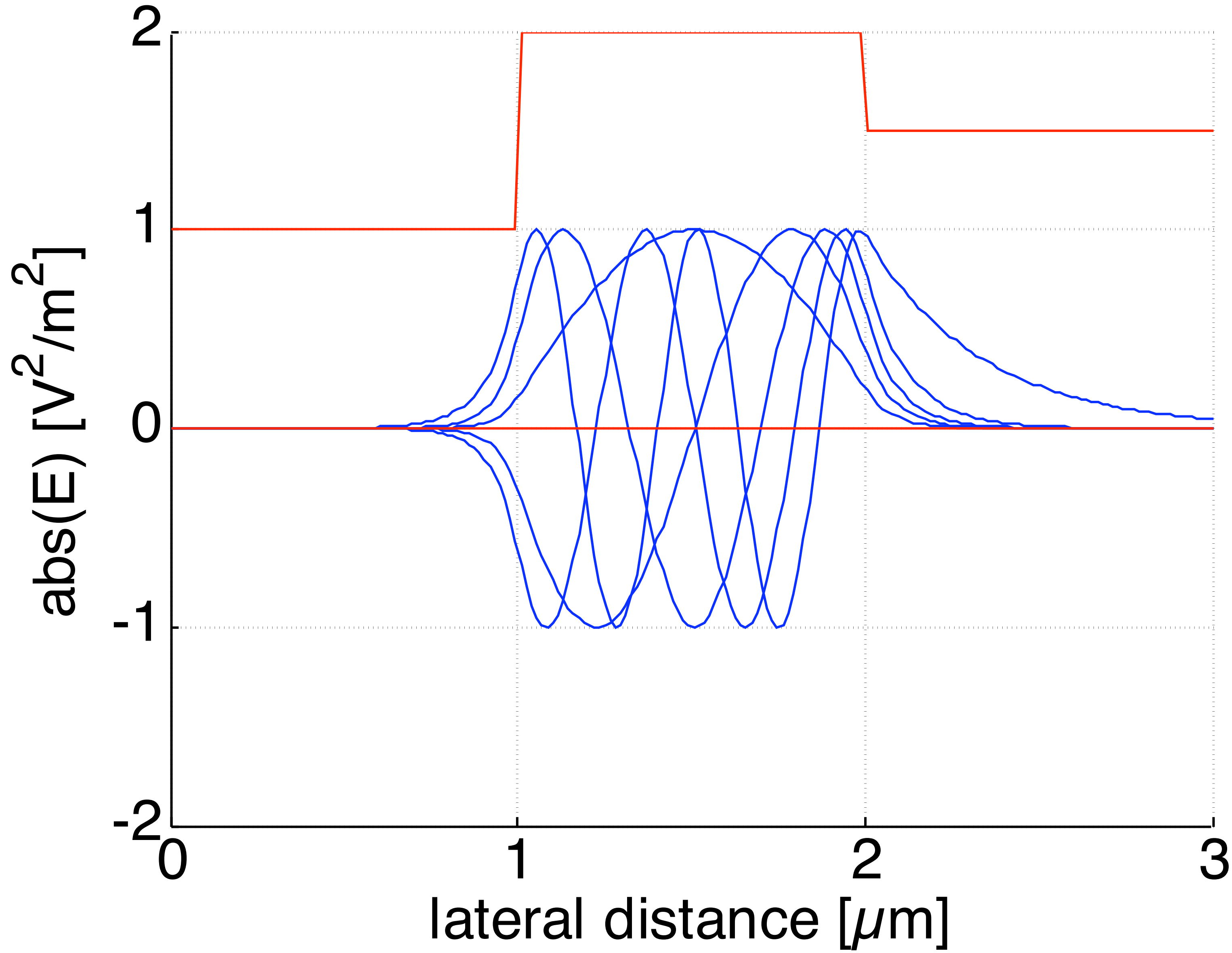




How do the modes look like



How do the modes look like



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General properties of guided modes

Characteristics of the mode solution

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right]u(x,y,\omega)+\left[k^2(x,y,\omega)-\beta^2(\omega)\right]u(x,y,\omega)=0$$

- u and u' converge to zero, physical fields have finite energy
- solutions for u and u' must be continuous and bound; second derivative of u must be finite due to the finite refractive index discontinuities

⇨ these properties determine the eigenvalue problem

⇨ discrete solutions are called modes

Orthogonality and normalisation

consider two solutions

$$\left[\triangle_\perp+k^2(x,y,\omega)-\beta_a^2\right]u_a(x,y,\omega)=0$$

$$\left[\triangle_\perp+k^2(x,y,\omega)-\beta_b^2\right]u_b(x,y,\omega)=0$$

Orthogonality and normalisation

multiplying each equation with the opposite mode and subtracting both equations

$$\left(\beta_a^2 - \beta_b^2\right) u_a u_b = u_b \triangle_\perp u_a - u_a \triangle_\perp u_b$$

integration over the mode area

$$\left(\beta_a^2 - \beta_b^2\right) \int_{\mathrm{A}\infty} u_a u_b dA = \int_{\mathrm{A}\infty} \left(u_b \triangle_\perp u_a - u_a \triangle_\perp u_b\right) dA$$

rhs is transformed using Green's second identity to a line integral that actually vanishes since the fields at infinity are supposed to be decayed gives

$$\left(\beta_a^2 - \beta_b^2\right) \int_{\mathrm{A}\infty} u_a u_b dA = 0$$

Characteristics of the mode solution

fulfil this condition for different propagation constants we require

$$\int_{\mathrm{A}\infty} u_a u_b dA = \delta_{a,b}$$

- modes are orthogonal: $$\int_A u_a u_b dA = \delta_{ab}$$

- modes are normalised: $$\int_A u^2 dA = 1$$

Derived quantities of modes

- phase velocity:
  (velocity of the phase front)

$$v_{\mathrm{P}} = \frac{\omega}{\beta} = \frac{2\pi c}{\lambda\beta}$$

- group velocity:
  (velocity of the energy)

$$v_{\mathrm{g}} = \frac{\partial\omega}{\partial\beta} = -\frac{2\pi c}{\lambda^2}\frac{\partial\lambda}{\partial\beta}$$

can conveniently be calculated with the following method
(using wavelengths as arguments and dropping space coordinates)

calculation at two different wavelengths

$$\left[\triangle_\perp + k_0^2(\lambda)\epsilon(\lambda) - \beta^2(\lambda)\right] u(\lambda) = 0$$

$$\left[\triangle_\perp + k_0^2(\lambda')\epsilon(\lambda') - \beta^2(\lambda')\right] u(\lambda') = 0$$

Derived quantities of modes

same as before; multiplication with the other mode and subtracting

$$\left\{\left[\beta^2(\lambda)-\beta^2(\lambda')\right]-\left[k_0^2(\lambda)\epsilon(\lambda)-k_0^2(\lambda')\epsilon(\lambda')\right]\right\}u(\lambda)u(\lambda')=u(\lambda')\triangle_\perp u(\lambda)-u(\lambda)\triangle_\perp u(\lambda')$$

again, integrating over the cross section and using Green's second identity to show that the rhs is zero gives

$$\frac{\left[\beta^2(\lambda)-\beta^2(\lambda')\right]}{\lambda-\lambda'}\int_{\mathrm{A}\infty}u(\lambda)u(\lambda')dA=\frac{4\pi^2}{\lambda-\lambda'}\int_{\mathrm{A}\infty}\left(\frac{\epsilon(\lambda)}{\lambda^2}-\frac{\epsilon(\lambda')}{\lambda'^2}\right)u(\lambda)u(\lambda')dA$$

considering now the limiting case of $\lambda\to\lambda'$

$$\beta(\lambda)\,\frac{\partial\beta(\lambda)}{\partial\lambda}\int_{\mathrm{A}\infty}u^2(\lambda)dA=4\pi^2\int_{\mathrm{A}\infty}u^2(\lambda)\frac{\partial}{\partial\lambda}\left(\frac{\epsilon(\lambda)}{\lambda^2}\right)dA$$

this expression for $\partial\lambda/\partial\beta$ can be inserted into the original equation

Derived quantities of modes

- group velocity:
  (velocity of the energy)

$$v_{\mathrm{g}} = -\frac{c\beta(\lambda)}{\lambda^2 2\pi} \frac{\int_{A\infty} u^2(\lambda) dA}{\int_{A\infty} u^2(\lambda) \frac{\partial}{\partial\lambda}\left(\frac{\epsilon(\lambda)}{\lambda^2}\right) dA}$$

- group velocity dispersion:
  (measure for the spread of a pulse)

$$D = \frac{\partial \frac{1}{v_g}}{\partial\lambda}$$

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General properties of guided modes

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Finite difference to solve guided eigenmodes in scalar approximation

Finite-difference method for waveguide modes

starting from the wave equation

$$\nabla \times \nabla \times \mathbf{E}(\mathbf{r}, \omega) = \frac{\omega^2}{c^2} \epsilon(\mathbf{r}, \omega) \mathbf{E}(\mathbf{r}, \omega)$$

neglecting the divergence of the electric field

$$\nabla \cdot \mathbf{D}(\mathbf{r}, \omega) = 0 \qquad \epsilon_0 \nabla \cdot [\epsilon(\mathbf{r}, \omega) \mathbf{E}(\mathbf{r}, \omega)] \approx 0$$

we obtain the Helmholtz equation

$$\triangle \mathbf{E}(\mathbf{r}, \omega) + \frac{\omega^2}{c^2} \epsilon(\mathbf{r}, \omega) \mathbf{E}(\mathbf{r}, \omega) = 0$$

neglecting the vectorial properties $\Rightarrow$ scalar Helmholtz equation

$$\triangle v(\mathbf{r}, \omega) + k^2(\mathbf{r}, \omega) v(\mathbf{r}, \omega) = 0 \quad \text{with} \quad k^2(\mathbf{r}, \omega) = \frac{\omega^2}{c^2} \epsilon(\mathbf{r}, \omega)$$

Stationary solutions of the scalar Helmholtz equation

search for the stationary states (modes) of the problem with $\epsilon(\mathbf{r}, \omega) = \epsilon(x, y, \omega)$

$$v(\mathbf{r}, \omega) = u(x, y, \omega) e^{i\beta(\omega) z}$$

eigenvalue equation for the propagation constant $\beta(\omega)$

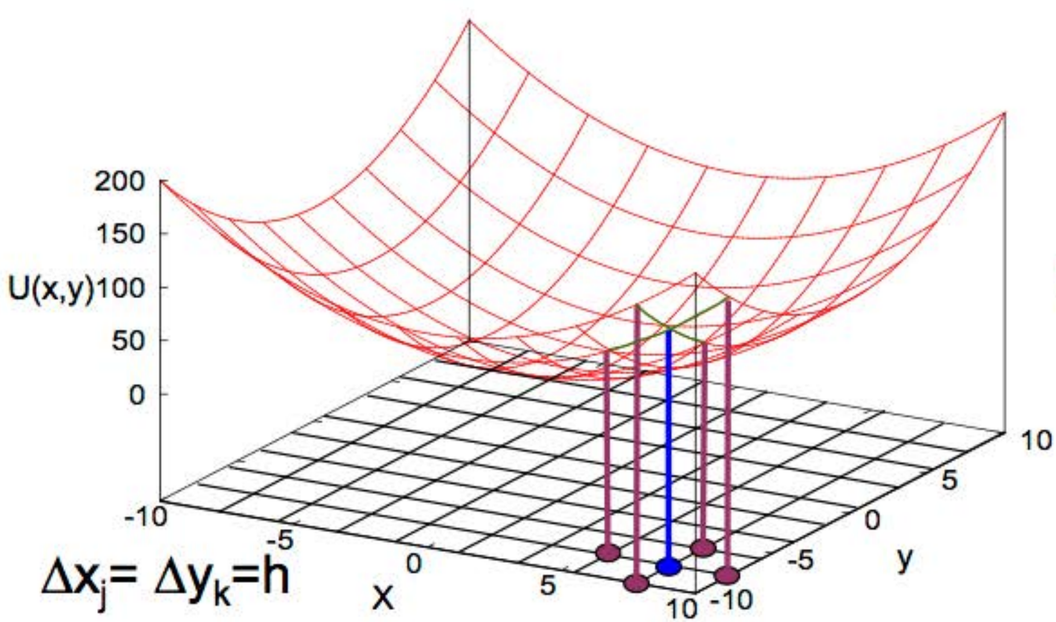
$$\triangle_{\perp} u(x, y, \omega) + \left[k^2(x, y, \omega) - \beta^2(\omega)\right] u(x, y, \omega) = 0$$

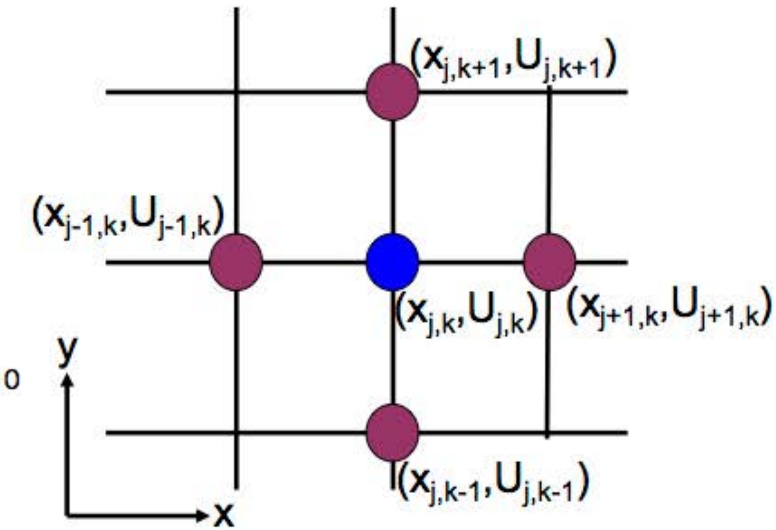
$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right] u(x, y, \omega) + \left[k^2(x, y, \omega) - \beta^2(\omega)\right] u(x, y, \omega) = 0$$

eigenvalue solved by a finite difference scheme for the transversal Laplace operator

$$\frac{\partial^2 u(x, y, \omega)}{\partial x^2} + \frac{\partial^2 u(x, y, \omega)}{\partial y^2}$$

Discretisation of the Laplace operator





$$\left.\frac{\partial^2 u(x,y,\omega)}{\partial x^2}\right|_{x_j,y_k} \approx \frac{u(x_{j+1},y_k,\omega)-2u(x_j,y_k,\omega)+u(x_{j-1},y_k,\omega)}{h^2}$$

$$\left.\frac{\partial^2 u(x,y,\omega)}{\partial y^2}\right|_{x_j,y_k} \approx \frac{u(x_{j,y_{k+1}},\omega)-2u(x_j,y_k,\omega)+u(x_{j,y_{k-1}},\omega)}{h^2}$$

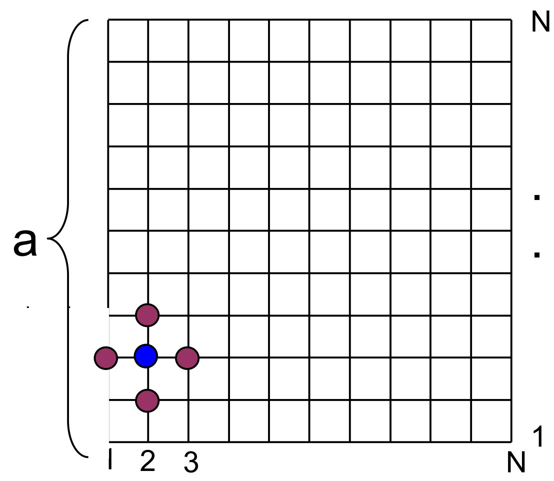
Discretisation of the Laplace operator

linear equation for each variable $u(x_j, y_k, \omega) = u_{j,k}$

$$\triangle_{(x,y)} u \rightarrow (\triangle u)_{j,k}$$

assuming: ⇨ quadratic area of size a x a

⇨ equidistant discretisation of the area with N x N points

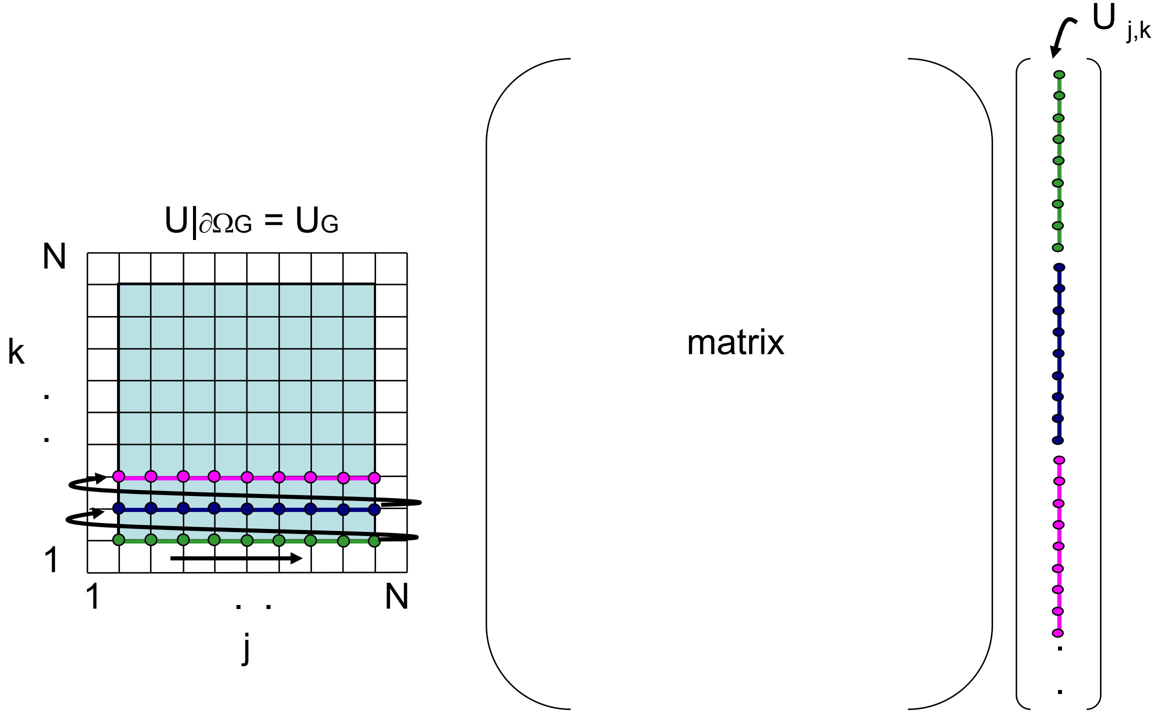


e.g. $$(\triangle u)_{2,3} = \frac{-4u_{2,3} + u_{3,3} + u_{1,3} + u_{2,4} + u_{2,2}}{h^2}$$

Matrix notation of the eigenvalue equation

$u_{j,k}$ is originally a 2D variable depending on x-direction (j) and y-direction (k)

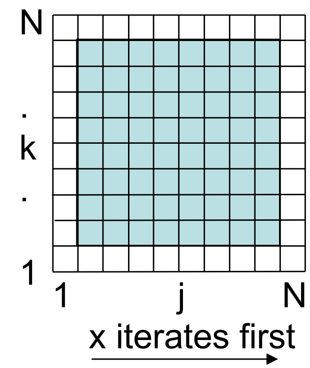
- unfolding of $u_{j,k}$ into a 1D vector
- each vector component $u_{j,k}$ results in an individual linear equation
- matrix dimension: number variables in x times number variables in y



Matrix notation of the eigenvalue equation

$$
\begin{pmatrix}
-4 & 1 & 0 & 0 & .. & 0 & 1 & 0 & 0 & 0 & 0 & & & & & \\
1 & -4 & 1 & 0 & .. & 0 & 0 & 1 & 0 & 0 & 0 & & & & & \\
 & & & & & & & & & & & & & & & \\
1 & 0 & 0 & 0 & .. & 0 & -4 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & .. & 0 & 1 & -4 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & .. & 0 & 0 & 1 & -4 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
 & & & & & & & & & & & & & & &
\end{pmatrix}
\begin{pmatrix}
U_{2,2} \\ U_{3,2} \\ \vdots \\ U_{N-2,2} \\ U_{N-1,2} \\ U_{2,3} \\ U_{3,3} \\ U_{4,3} \\ \vdots \\ U_{N-2,3} \\ U_{N-1,3} \\ U_{2,4} \\ U_{3,4} \\ U_{4,4}
\end{pmatrix}
$$

discrete Laplace operator



• • • • • • accounts for the 1D – problem (tridiagonal matrix)

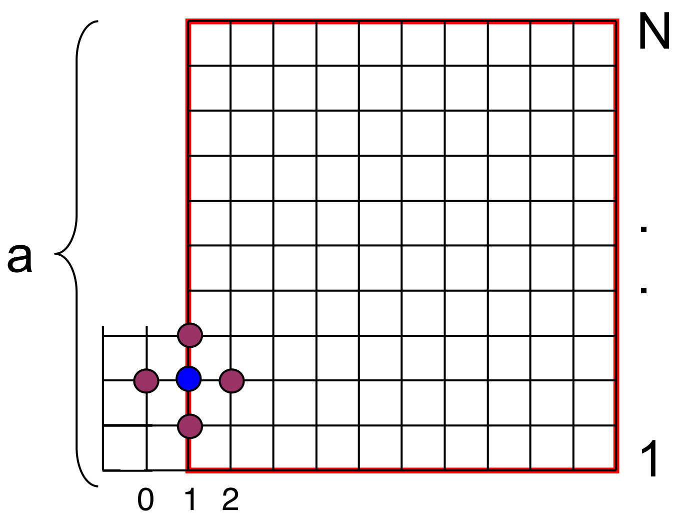
• • • • • • occurs for 2D problems

matrix: small number of non-zero values – ‚sparse matrix'

Boundary conditions

finite spatial grid to reflect an infinitely extended space

- suitable truncation necessary
- boundary conditions less critical since guided modes shall have a sufficiently decayed field



Boundaries of the grid?
Necessary to compute, e.g.

$$(\triangle u)_{1,3} = \frac{-4u_{1,3} + u_{2,3} + \boxed{u_{0,3}} + u_{1,4} + u_{1,2}}{h^2}$$

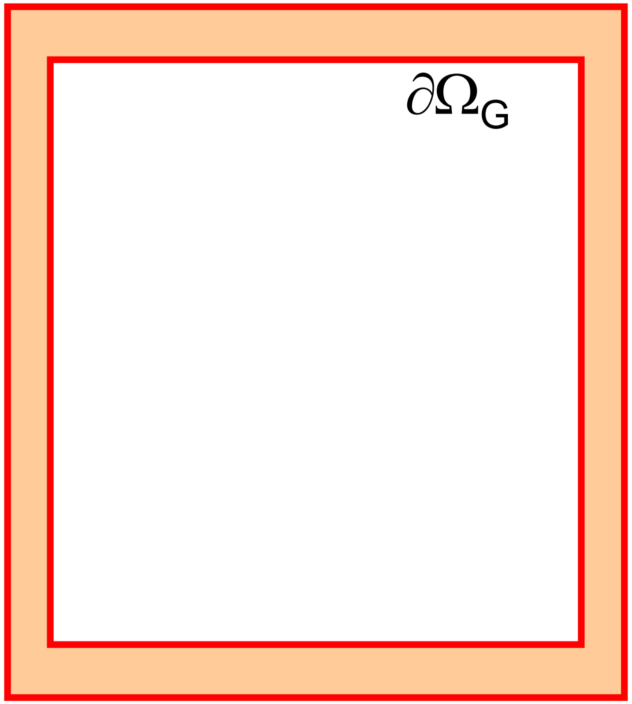
outside the grid

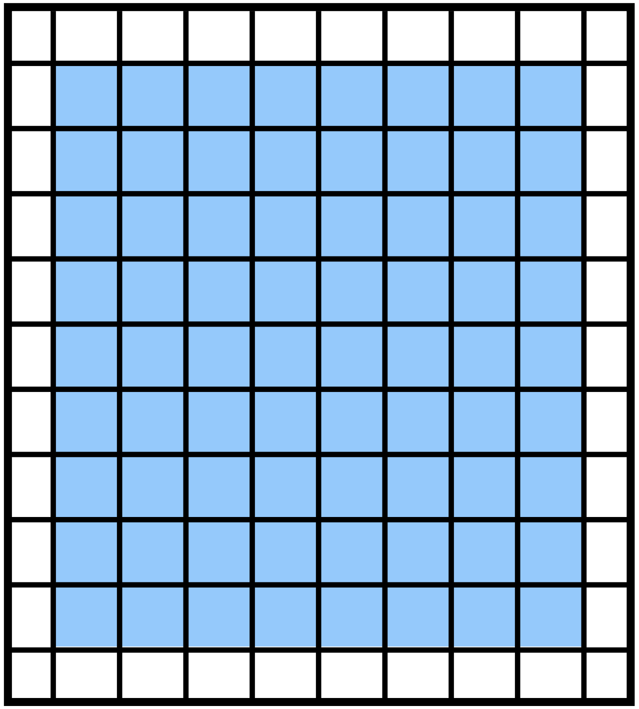
- boundary conditions

(compare with theory of partial differential equation)

Boundary conditions

example: Metal boundaries (Metal tube with boundaries $\partial\Omega_i$)





field inside a perfectly conducting metal vanishes

reduced number of unknowns and equations

$$(N-2)\times(N-2)$$

Final solution

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2(x, y, \omega)\right] u(x, y, \omega) = \beta(\omega)^2 u(x, y, \omega)$$

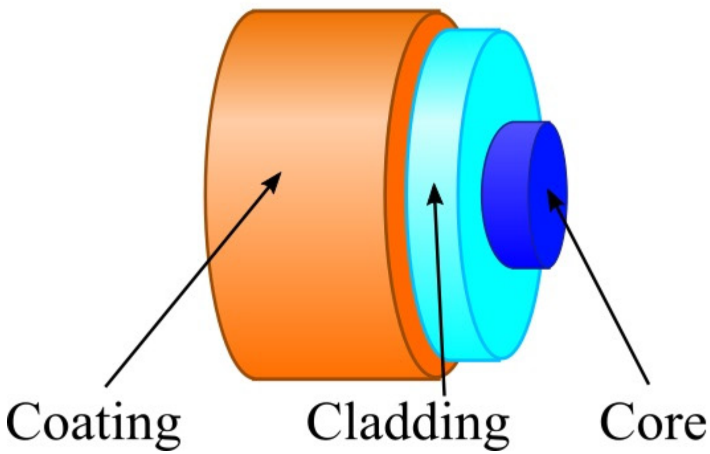
$$\left[(\triangle u)_{j,k} + k_{j,k}^2\right] u_{j,k} = \beta^2 u_{j,k}$$

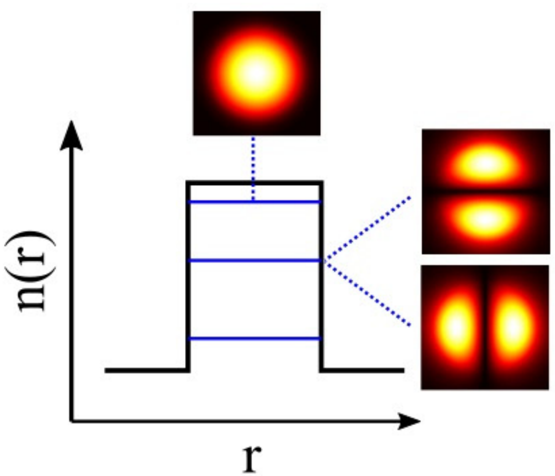
$$Au = \beta^2 u$$

$\mathbf{A}$ is the sum of a matrix encoding the Laplace operator and a diagonal matrix containing information on the material at size j,k

solvable with eigenvalue solver of preference

Example





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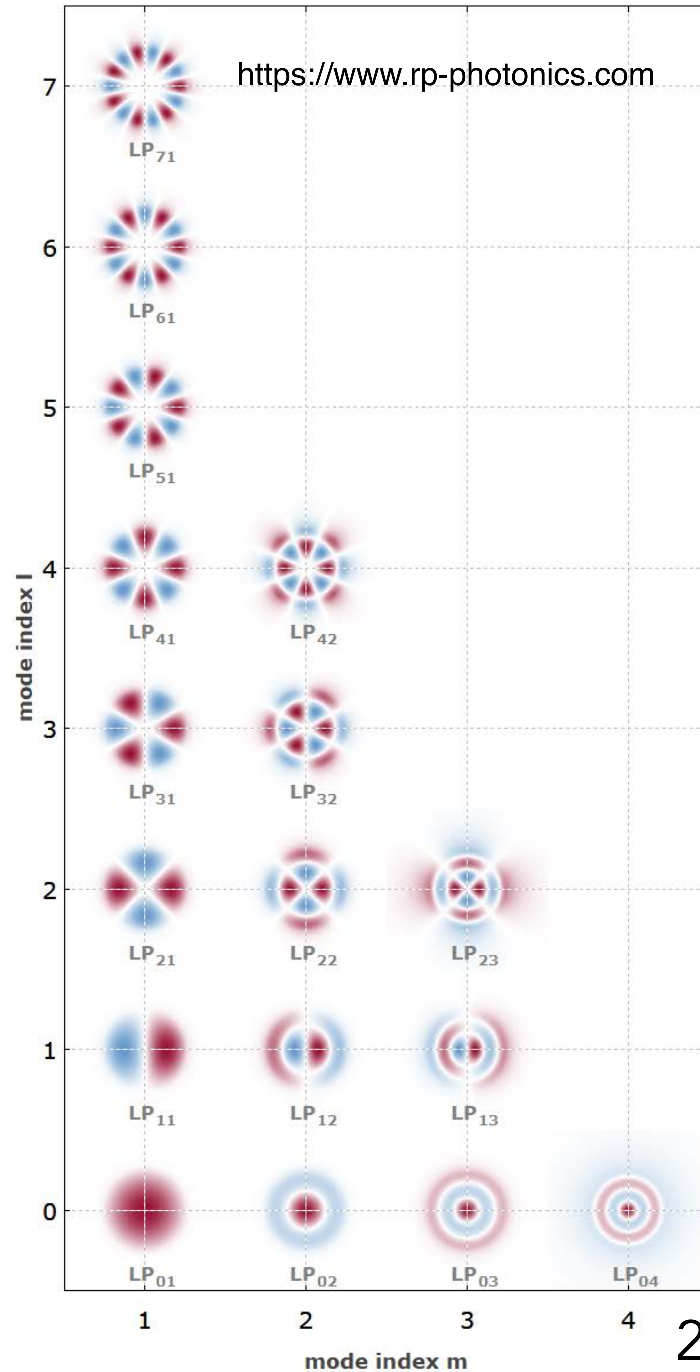

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