Computational Photonics

Finite-Difference Time-Domain

Finite-Difference Time-Domain (FDTD)

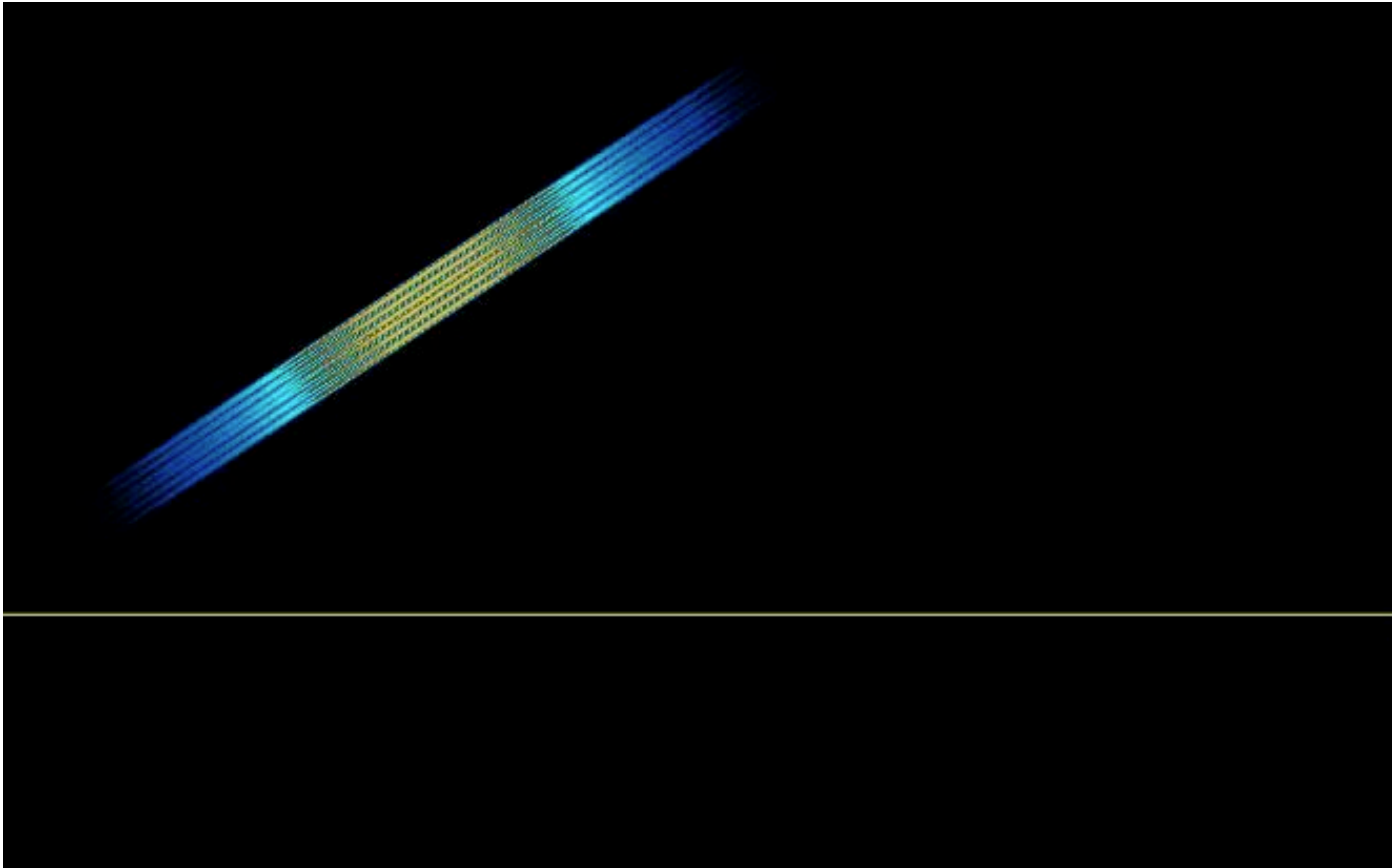
- *ab initio*: direct solution of Maxwell's equations

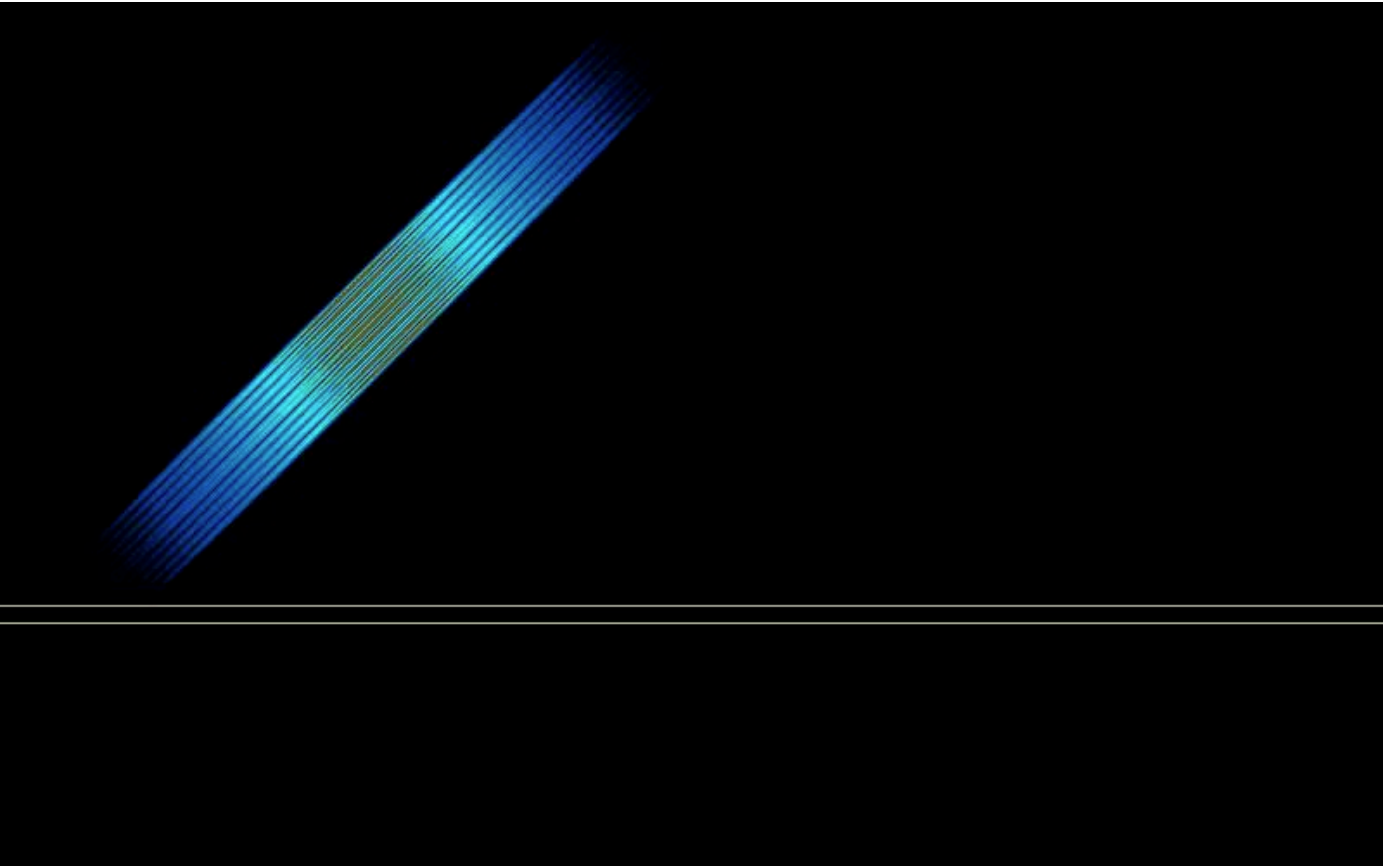
- probably the most often used numerical technique, since an implementation is straight forward
  (but cumbersome implementation of proper boundary conditions)

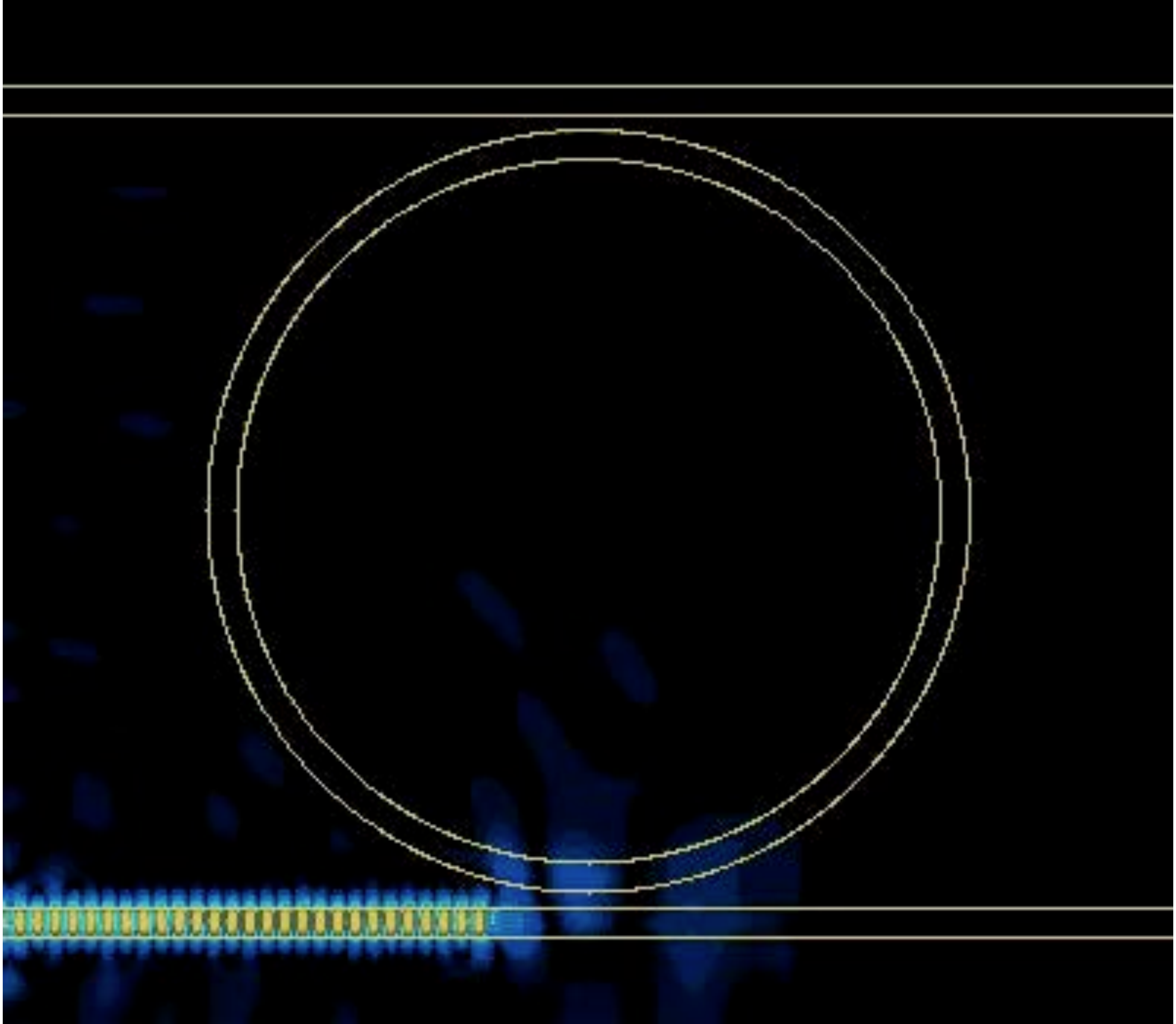
- requires excessive computational resources for reasonable problems in 3D *(implementation for clusters)*

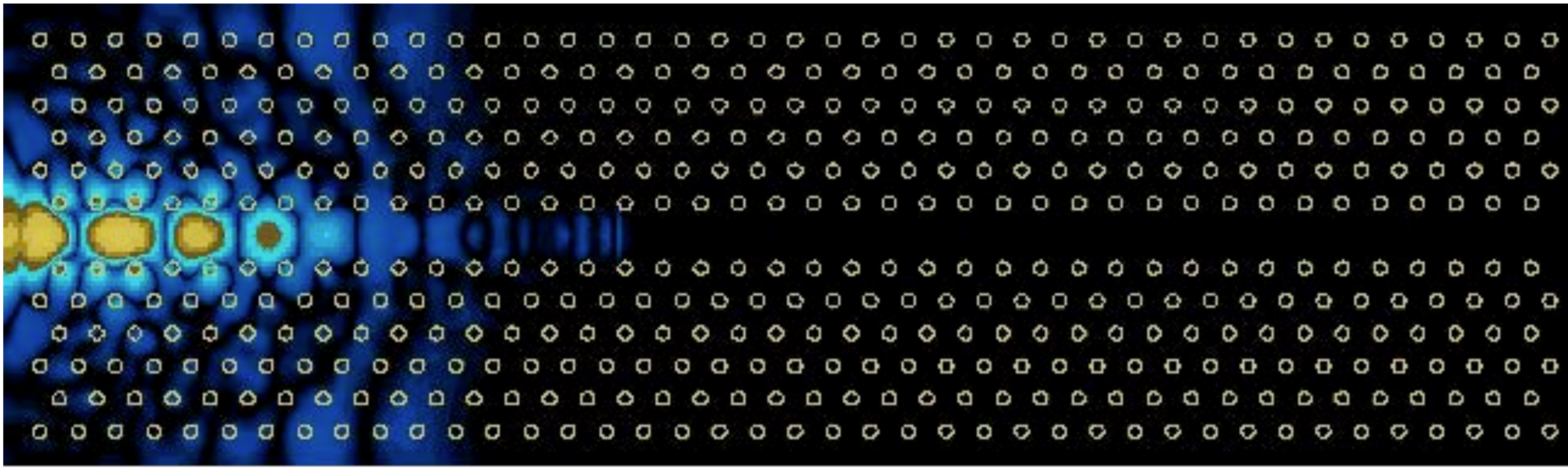
- implementation is absolutely general but often doesn't take explicit advantage of symmetries

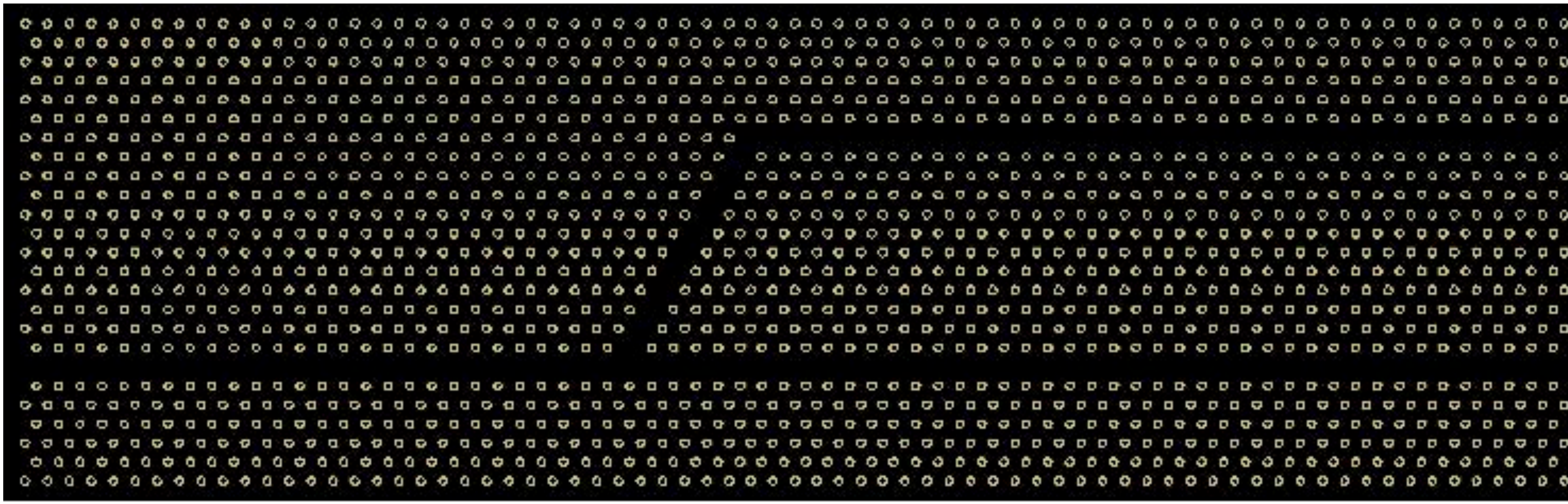
- principally all kinds of materials are treatable
  (dispersive or nonlinear materials)

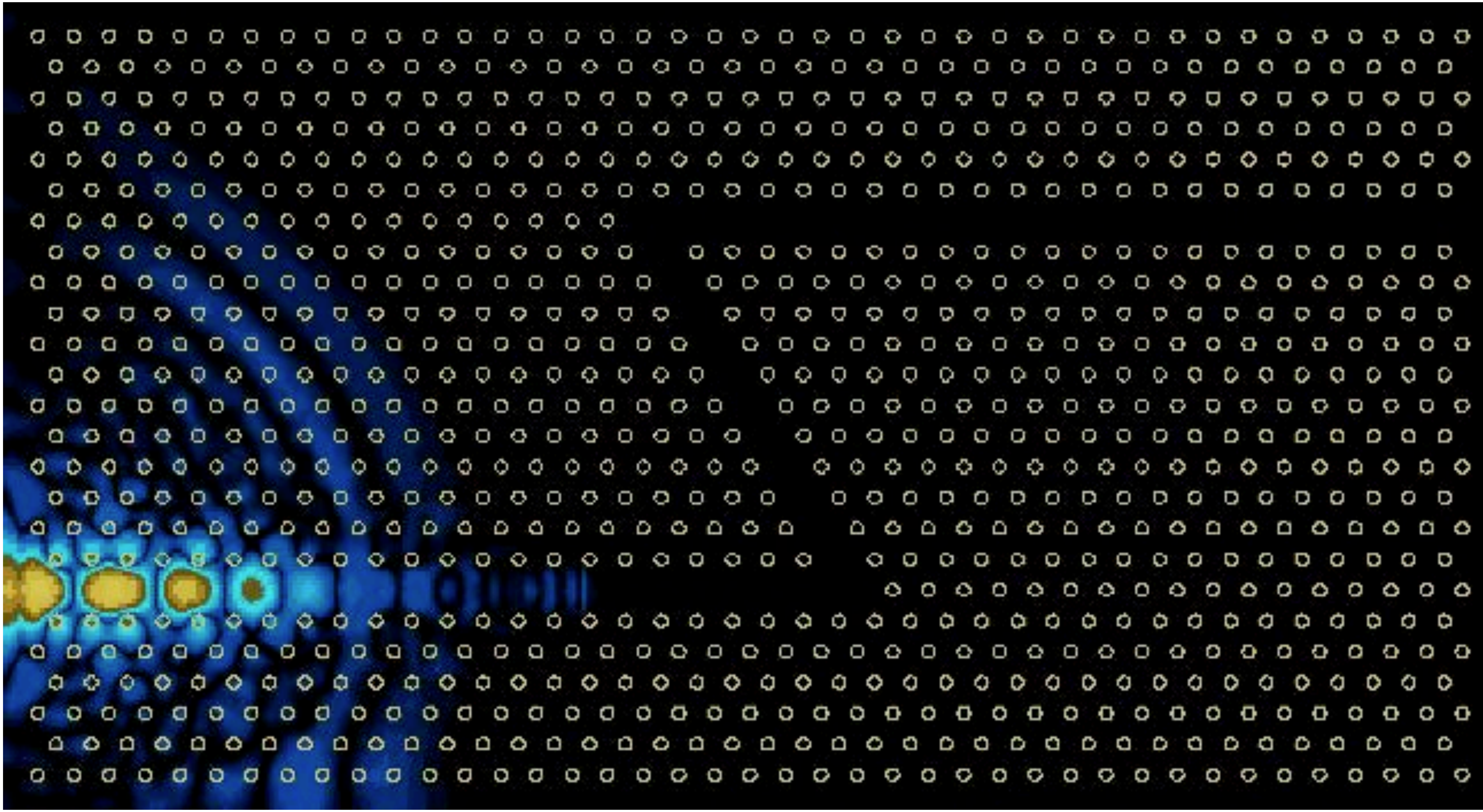












Finite-Difference Time-Domain (FDTD)

Maxwell's equations

$$\mathrm{rot}\,\mathbf{E}(\mathbf{r},t) = -\frac{\partial \mathbf{B}(\mathbf{r},t)}{\partial t} \qquad \mathrm{rot}\,\mathbf{H}(\mathbf{r},t) = \mathbf{j}_{\mathrm{makr}}(\mathbf{r},t) + \frac{\partial \mathbf{D}(\mathbf{r},t)}{\partial t}$$

$$\mathrm{div}\,\mathbf{D}(\mathbf{r},t) = \rho_{\mathrm{ext}}(\mathbf{r},t) \qquad \mathrm{div}\,\mathbf{B}(\mathbf{r},t) = 0$$

constitutive relations

$$\mathbf{D}(\mathbf{r},t) = \varepsilon_0 \mathbf{E}(\mathbf{r},t) + \mathbf{P}(\mathbf{r},t)$$

$$\mathbf{B}(\mathbf{r},t) = \mu_0 \mathbf{H}(\mathbf{r},t) + \mathbf{M}(\mathbf{r},t)$$

- $\mathbf{E}(\mathbf{r},t)$ electric field
- $\mathbf{H}(\mathbf{r},t)$ magnetic field
- $\mathbf{D}(\mathbf{r},t)$ dielectric flux density
- $\mathbf{B}(\mathbf{r},t)$ magnetic flux density
- $\mathbf{P}(\mathbf{r},t)$ dielectric polarisation
- $\mathbf{M}(\mathbf{r},t)$ magnetic polarisation
- $\rho_{\mathrm{ext}}(\mathbf{r},t)$ external charge density
- $\mathbf{j}_{\mathrm{makr}}(\mathbf{r},t)$ macroscopic current density

Finite-Difference Time-Domain (FDTD)

constitutive relations for a linear, isotropic, and *dispersionless* dielectric media

$$\mathbf{D}(\mathbf{r},t) = \varepsilon_0 \varepsilon(\mathbf{r})\mathbf{E}(\mathbf{r},t)$$

$$\mathbf{B}(\mathbf{r},t) = \mu_0 \mathbf{H}(\mathbf{r},t)$$

this is only an approximation that allows for convenient simulations; a dispersionless material does not exist

$$\text{rot}\,\mathbf{E}(\mathbf{r},t) = -\mu_0 \frac{\partial \mathbf{H}(\mathbf{r},t)}{\partial t} \qquad \text{rot}\,\mathbf{H}(\mathbf{r},t) = \varepsilon_0 \varepsilon(\mathbf{r}) \frac{\partial \mathbf{E}(\mathbf{r},t)}{\partial t}$$

$$\text{div}\left[\varepsilon_0 \varepsilon(\mathbf{r})\mathbf{E}(\mathbf{r},t)\right] = 0 \qquad \text{div}\,\mathbf{H}(\mathbf{r},t) = 0,$$

Finite-Difference Time-Domain (FDTD)

Maxwell's equations can be written as a set of six coupled differential equations of the the first kind

$$\frac{\partial \mathbf{H}(\mathbf{r},t)}{\partial t} = -\frac{1}{\mu_0}\operatorname{rot}\mathbf{E}(\mathbf{r},t) \qquad \frac{\partial \mathbf{E}(\mathbf{r},t)}{\partial t} = \frac{1}{\varepsilon_0\varepsilon(\mathbf{r})}\operatorname{rot}\mathbf{H}(\mathbf{r},t)$$

$$\frac{\partial H_x}{\partial t} = \frac{1}{\mu_0}\left[\frac{\partial E_y}{\partial z} - \frac{\partial E_z}{\partial y}\right]$$

$$\frac{\partial H_y}{\partial t} = \frac{1}{\mu_0}\left[\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z}\right]$$

$$\frac{\partial H_z}{\partial t} = \frac{1}{\mu_0}\left[\frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x}\right]$$

$$\frac{\partial E_x}{\partial t} = \frac{1}{\epsilon_0\epsilon(\mathbf{r})}\left[\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right]$$

$$\frac{\partial E_y}{\partial t} = \frac{1}{\epsilon_0\epsilon(\mathbf{r})}\left[\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right]$$

$$\frac{\partial E_z}{\partial t} = \frac{1}{\epsilon_0\epsilon(\mathbf{r})}\left[\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right]$$

1D Problem

assume that there is no dependence in y and z ⟹ dynamics only in x

$$\frac{\partial H_x}{\partial t} = 0 \qquad \frac{\partial H_y}{\partial t} = \frac{1}{\mu_0}\frac{\partial E_z}{\partial x} \qquad \frac{\partial H_z}{\partial t} = -\frac{1}{\mu_0}\frac{\partial E_y}{\partial x}$$

$$\frac{\partial E_x}{\partial t} = 0 \qquad \frac{\partial E_y}{\partial t} = -\frac{1}{\epsilon_0\epsilon(\mathbf{r})}\frac{\partial H_z}{\partial x} \qquad \frac{\partial E_z}{\partial t} = \frac{1}{\varepsilon_0\varepsilon(\mathbf{r})}\frac{\partial H_y}{\partial x}$$

grouping together non-mixing transverse electromagnetic components (TEM)

z-polarised E-field ⟹

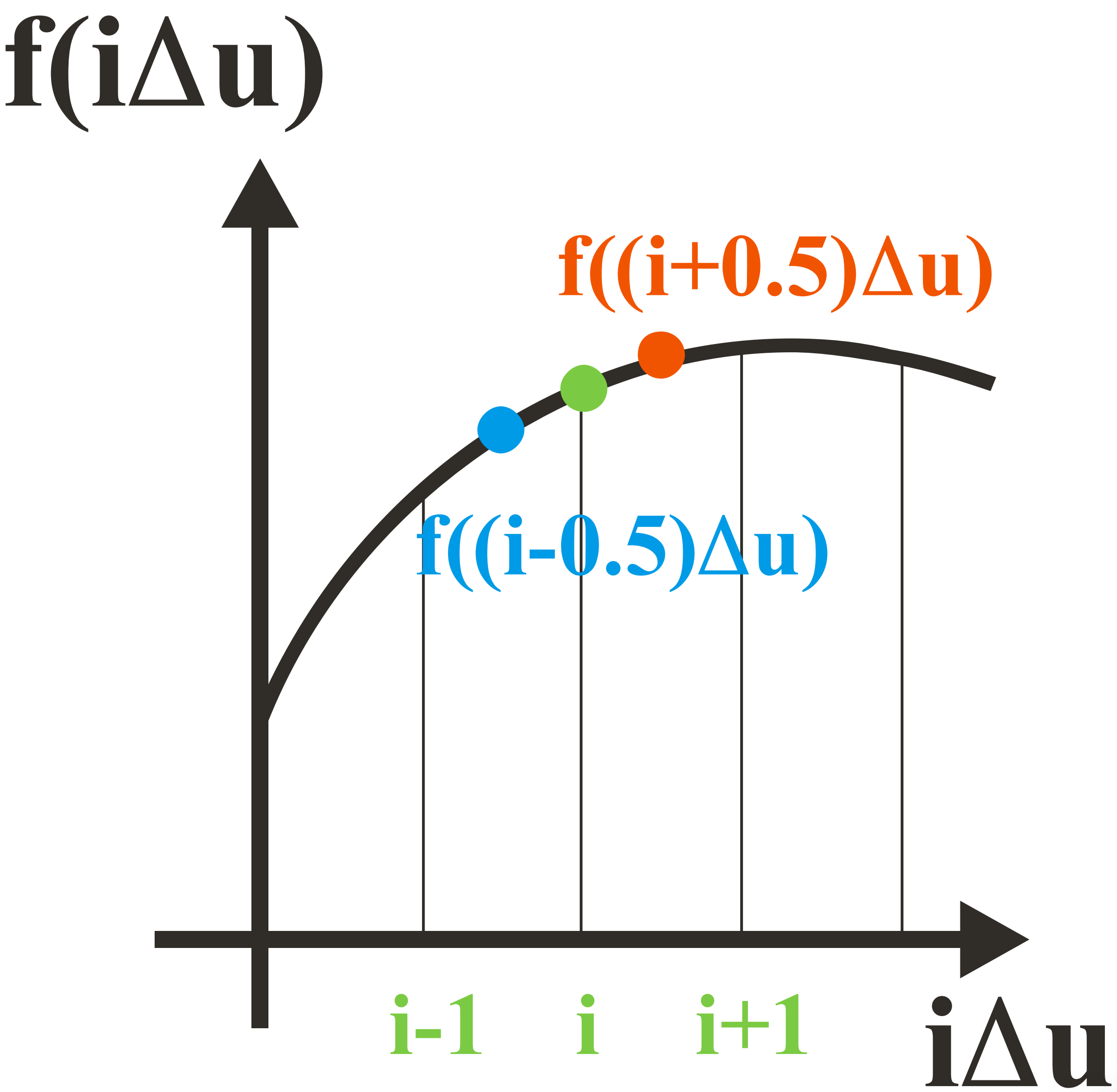
$$\frac{\partial E_z}{\partial t} = \frac{1}{\varepsilon_0\varepsilon(\mathbf{r})}\frac{\partial H_y}{\partial x} \qquad \frac{\partial H_y}{\partial t} = \frac{1}{\mu_0}\frac{\partial E_z}{\partial x}$$

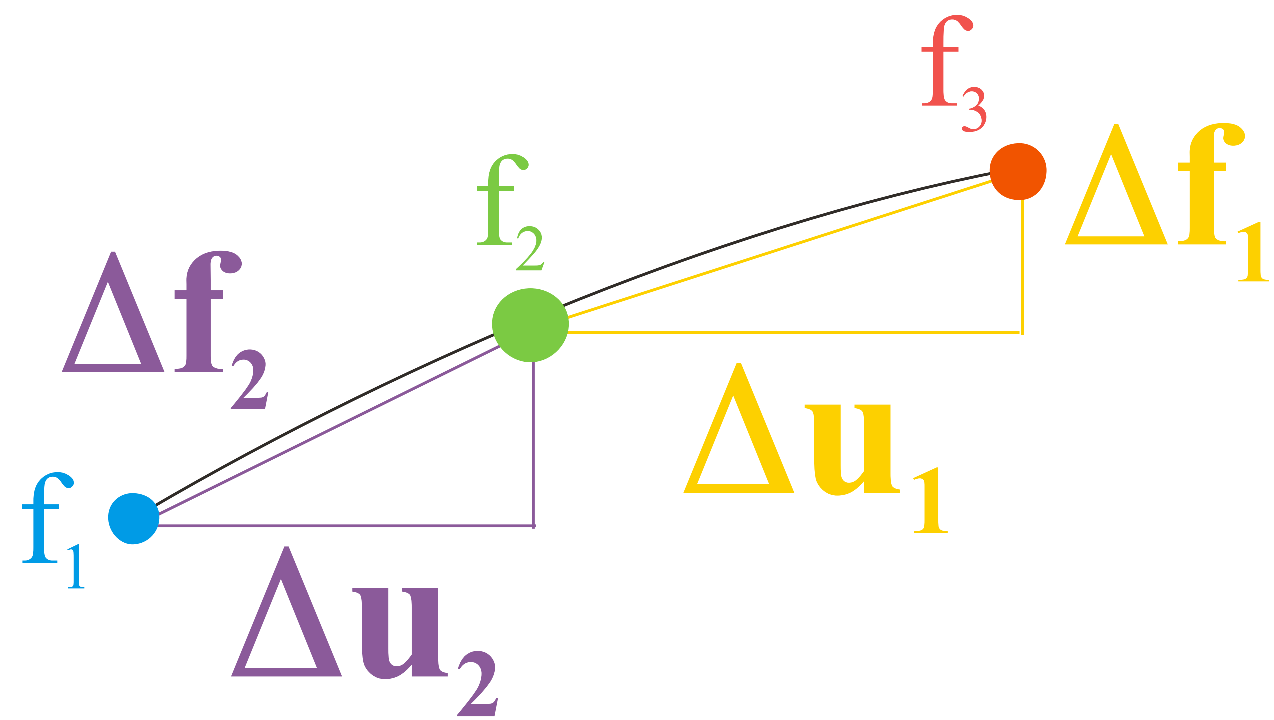
y-polarised E-field ⟹

$$\frac{\partial E_y}{\partial t} = -\frac{1}{\epsilon_0\epsilon(\mathbf{r})}\frac{\partial H_z}{\partial x} \qquad \frac{\partial H_z}{\partial t} = -\frac{1}{\mu_0}\frac{\partial E_y}{\partial x}$$

Discretising the equations

on a discrete mesh the differential operator is calculated as





$$\frac{\partial f}{\partial u} \approx \frac{\Delta f_1}{\Delta u_1} = \frac{f_3 - f_2}{\Delta u_1}$$

first order approximation

Discretising the equations

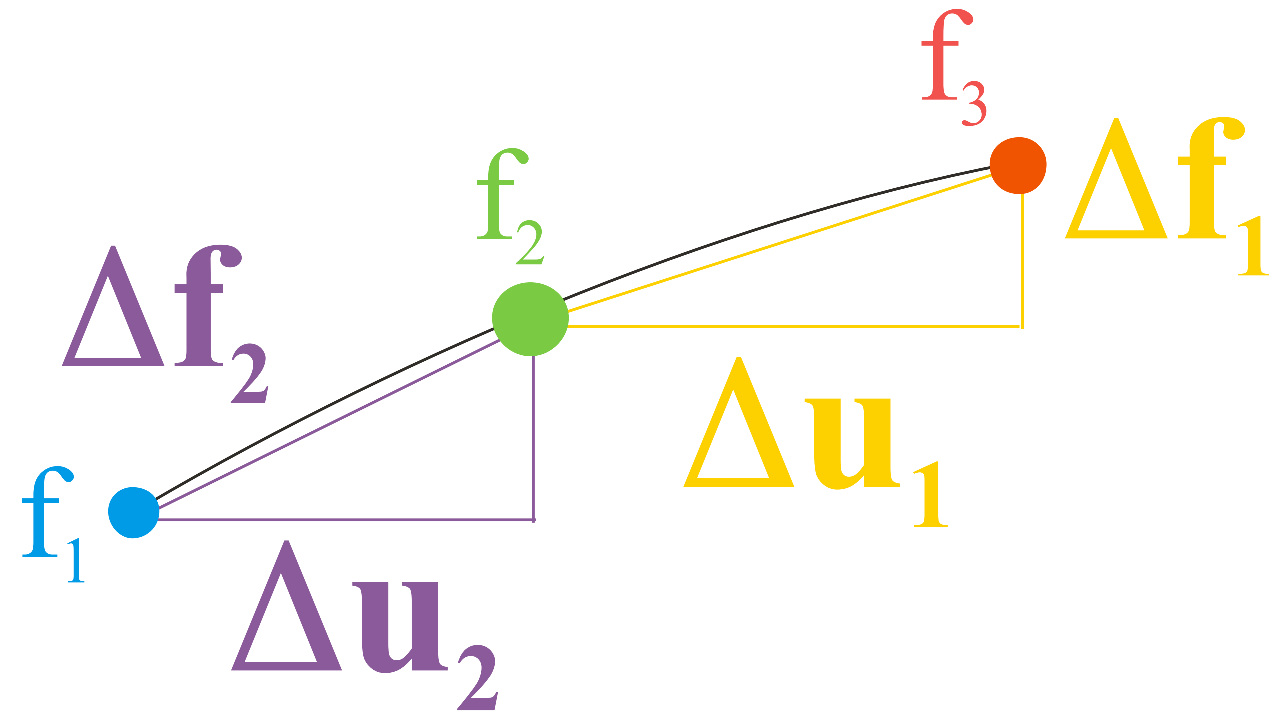
on a discrete mesh the differential operator is calculated as

$$\frac{\partial f}{\partial u} \approx \frac{\left(\frac{\Delta f_1}{\Delta u_1} + \frac{\Delta f_2}{\Delta u_2}\right)}{2}$$

$$\frac{\partial f}{\partial u} \approx \frac{f_3 - f_2 + f_2 - f_1}{2\Delta u}$$

$$\frac{\partial f}{\partial u} \approx \frac{f_3 - f_1}{2\Delta u}$$

second order approximation



$$\frac{\partial f}{\partial u} \approx \frac{\Delta f_1}{\Delta u_1} = \frac{f_3 - f_2}{\Delta u_1}$$

first order approximation

Discretising the equations

$$(i, n) = (i\Delta x, n\Delta t)$$

$$E_z(i\Delta x, n\Delta t) = E_i^n \Rightarrow \quad \frac{\partial E_i^n}{\partial x} = \frac{E_{i+1}^n - E_{i-1}^n}{2\Delta x} + O\left[(\Delta x)^2\right]$$

$$\frac{\partial E_i^n}{\partial t} = \frac{E_i^{n+1} - E_i^{n-1}}{2\Delta t} + O\left[(\Delta t)^2\right]$$

$$H_y(i\Delta x, n\Delta t) = H_i^n \Rightarrow \quad \frac{\partial H_i^n}{\partial x} = \frac{H_{i+1}^n - H_{i-1}^n}{2\Delta x} + O\left[(\Delta x)^2\right]$$

$$\frac{\partial H_i^n}{\partial t} = \frac{H_i^{n+1} - H_i^{n-1}}{2\Delta t} + O\left[(\Delta t)^2\right]$$

$$\varepsilon(i\Delta x) = \varepsilon_i$$

Discretising the equations

$$\frac{\partial E_z}{\partial t} = \frac{1}{\varepsilon_0 \varepsilon(\mathbf{r})} \frac{\partial H_y}{\partial x}$$

$$\Rightarrow \quad \frac{\partial E_i^n}{\partial t} = \frac{1}{\varepsilon_0 \varepsilon_i} \frac{\partial H_i^n}{\partial x}$$

$$\Rightarrow \quad \frac{E_i^{n+1} - E_i^{n-1}}{2\Delta t} \approx \frac{1}{\varepsilon_0 \varepsilon_i} \frac{H_{i+1}^n - H_{i-1}^n}{2\Delta x}$$

$$\Rightarrow \quad E_i^{n+1} \approx E_i^{n-1} + \frac{1}{\varepsilon_0 \varepsilon_i} \frac{\Delta t}{\Delta x} \left[H_{i+1}^n - H_{i-1}^n \right]$$

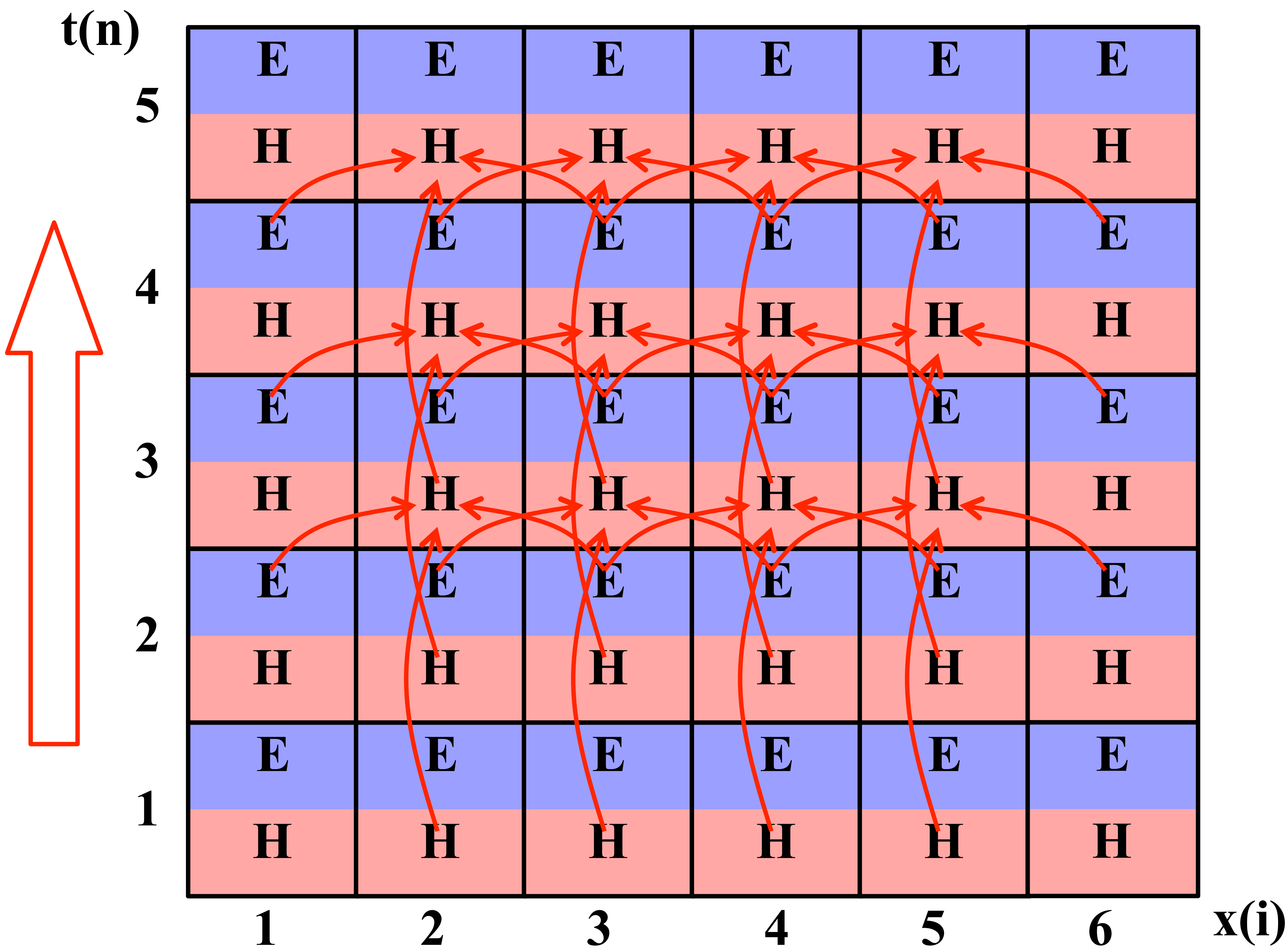
$$\frac{\partial H_y}{\partial t} = \frac{1}{\mu_0} \frac{\partial E_z}{\partial x}$$

$$\Rightarrow \quad \frac{\partial H_i^n}{\partial t} = \frac{1}{\mu_0} \frac{\partial E_i^n}{\partial x}$$

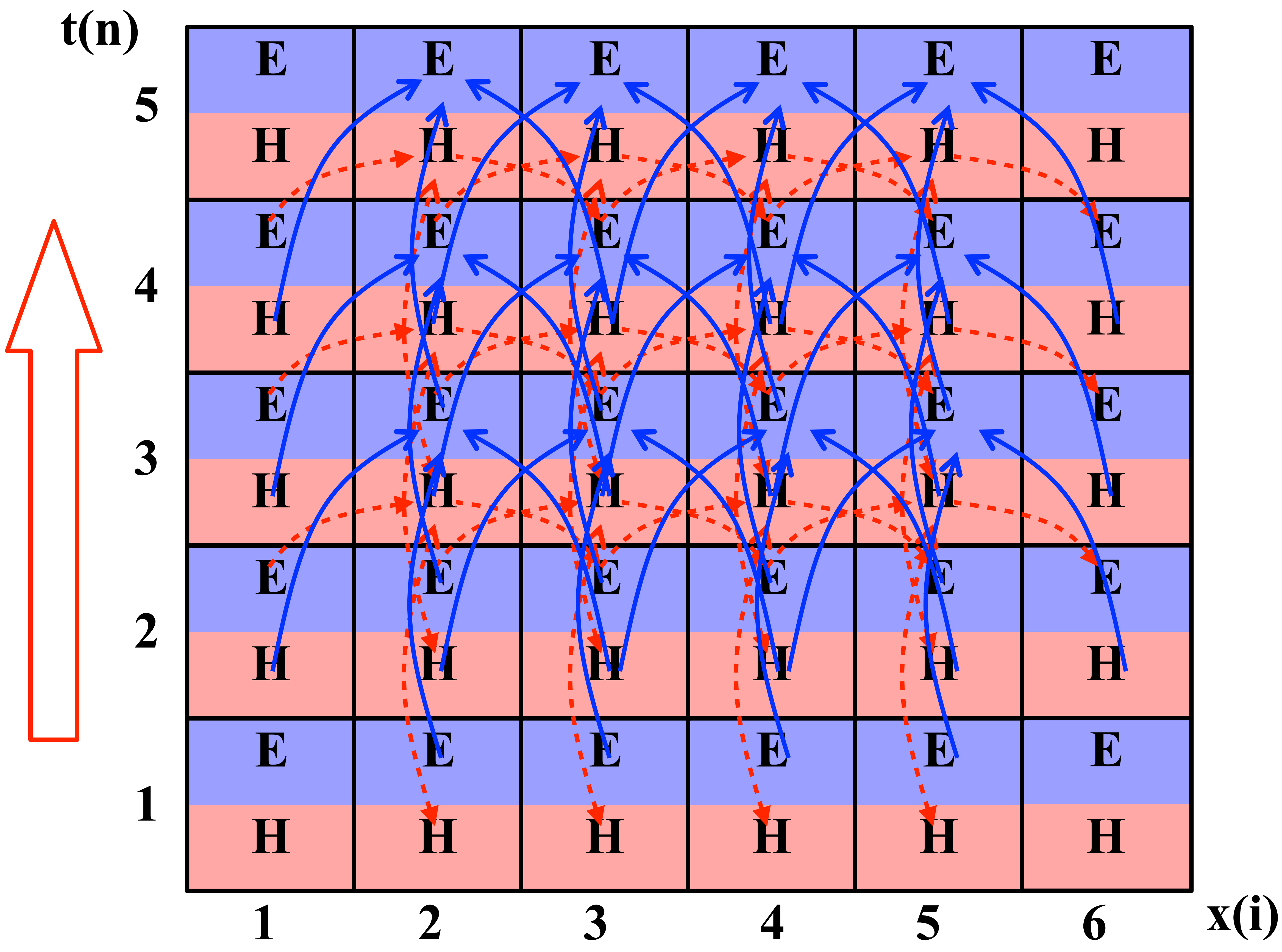
$$\Rightarrow \quad \frac{H_i^{n+1} - H_i^{n-1}}{2\Delta t} \approx \frac{1}{\mu_0} \frac{E_{i+1}^n - E_{i-1}^n}{2\Delta x}$$

$$\Rightarrow \quad H_i^{n+1} \approx H_i^{n-1} + \frac{1}{\mu_0} \frac{\Delta t}{\Delta x} \left[E_{i+1}^n - E_{i-1}^n \right]$$

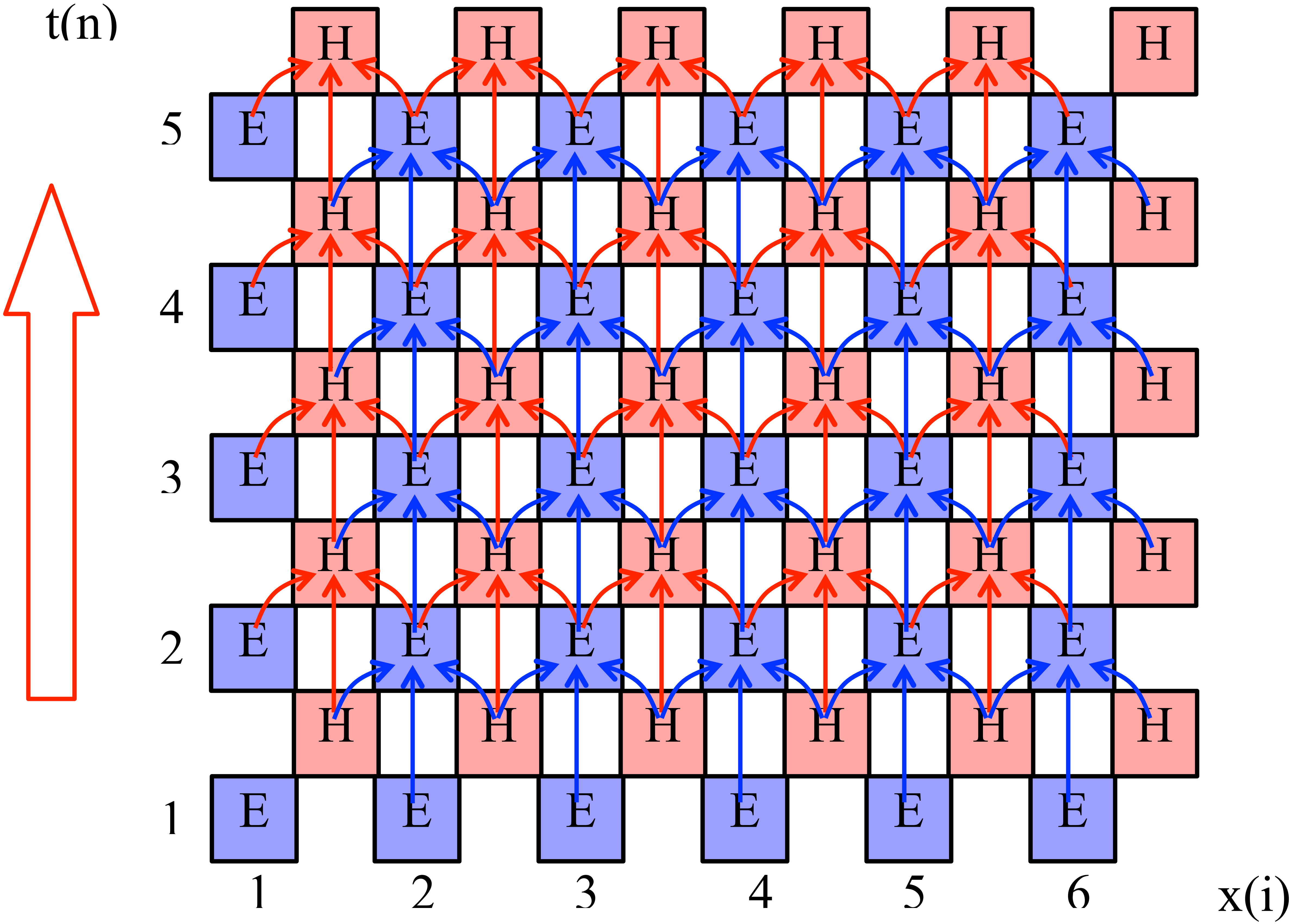
FDTD discretisation for the H field



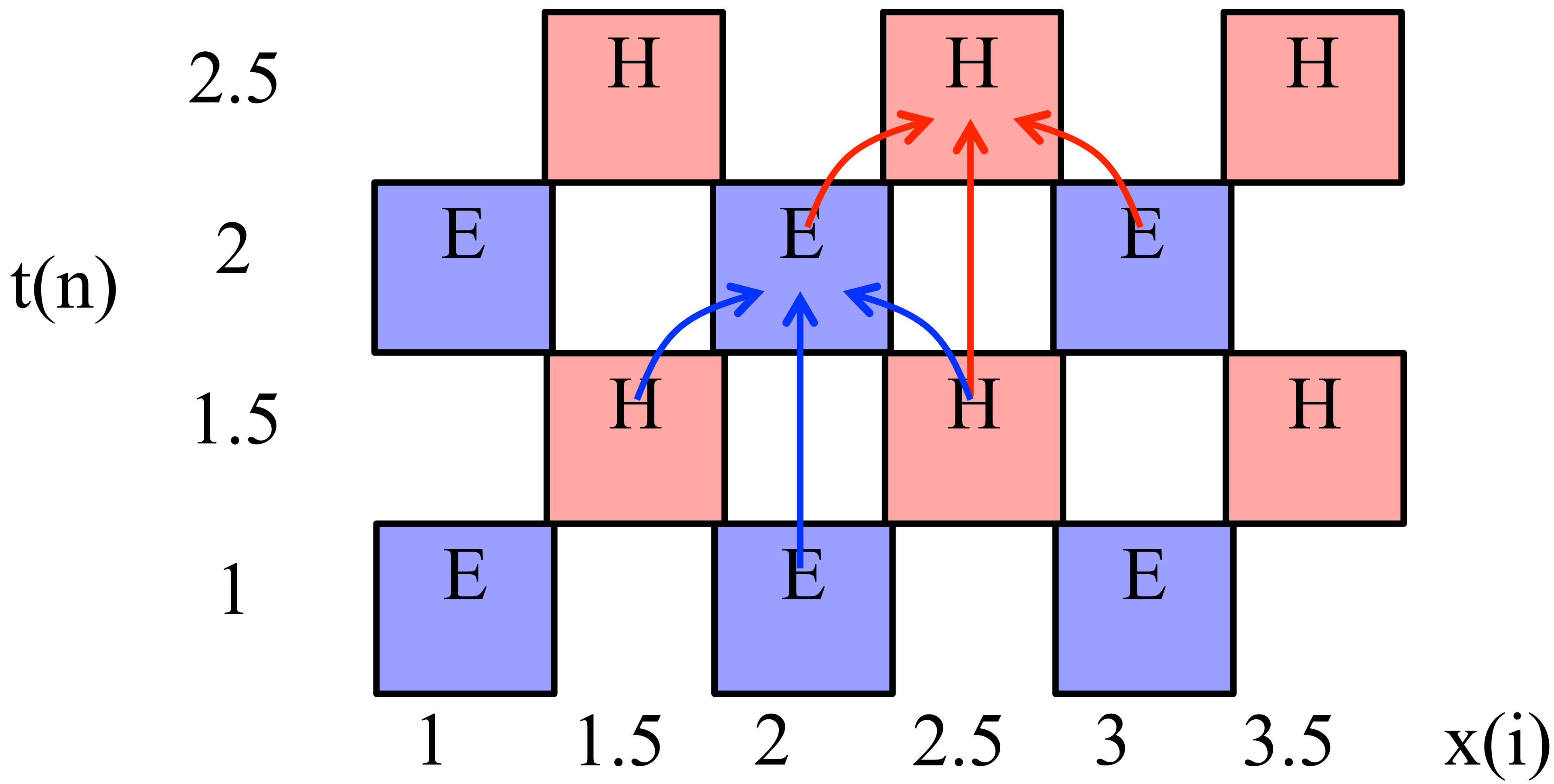
FDTD discretisation for the H field



Yee grid



Yee grid and leap-frog time steps



$$E_i^{n+1} \approx E_i^n + \frac{1}{\varepsilon_0 \varepsilon_i} \frac{\Delta t}{\Delta x} \left[H_{i+0.5}^{n+0.5} - H_{i-0.5}^{n+0.5} \right]$$

$$H_{i+0.5}^{n+0.5} \approx H_{i+0.5}^{n-0.5} + \frac{1}{\mu_0} \frac{\Delta t}{\Delta x} \left[E_{i+1}^n - E_i^n \right]$$

Properties: divergence, resolution

a vanishing divergence of the fields is always fulfilled in 1D since fields are always transversally polarised to the direction of field change

resolution of discretisation Δx and Δt from physical arguments:

spatial grid resolution Δx must be fine enough to resolve the finest structures of the material distribution and the fields ⟹ rule of thumb $\Delta x \leq \lambda/(20 n_{max})$, with n_{max} being the highest refractive index in the simulation domain

temporal step size Δt is limited by the speed of light, i.e. the interaction in space can reach only up to the next neighbour ⟹ sets upper limit to the phase velocity ⟹ $\Delta t \leq \Delta x / c$

in higher dimensions $$\Delta t \leq \frac{1}{c}\left(\frac{1}{\Delta x^2}+\frac{1}{\Delta y^2}+\frac{1}{\Delta z^2}\right)^{-\frac{1}{2}}$$

Courant criteria

FDTD in 3D

- Cartesian coordinate system (x, y, z) for the field and the material
- fields depend additionally on time t

discretisation in 3D

$$\mathbf{E}(x, y, z, t) = \mathbf{E}(i\Delta x, j\Delta y, k\Delta z, n\Delta t) = \mathbf{E}^n_{ijk}$$

$$\mathbf{E}(x, y, z, t) = \begin{Bmatrix} E^n_x(i, j, k) \\ E^n_y(i, j, k) \\ E^n_z(i, j, k) \end{Bmatrix}$$

field consist of three components; each is written separately

FDTD in 3D

same discrete approximation of differential operators

$$\frac{\partial f(i\Delta x, j\Delta y, k\Delta z, n\Delta t)}{\partial x} = \frac{f^n_{i+\frac{1}{2},j,k} - f^n_{i-\frac{1}{2},j,k}}{\Delta x} + O\left[(\Delta x)^2\right]$$

$$\frac{\partial f(i\Delta x, j\Delta y, k\Delta z, n\Delta t)}{\partial t} = \frac{f^{n+\frac{1}{2}}_{i,j,k} - f^{n-\frac{1}{2}}_{i,j,k}}{\Delta t} + O\left[(\Delta t)^2\right]$$

approach used for all differential equations, e.g.

$$\frac{\partial E_x}{\partial t} = \frac{1}{\varepsilon_0 \varepsilon(\mathbf{r})}\left[\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} - j_x\right]$$

⟹ Interleaved grid in space and time for E and H field

(Yee-cell)

K.S. Yee, IEEE Trans. Antennas Propagat., AP-14, 302, 1966

FDTD in 3D

same discrete approximation of differential operators

$$\frac{\partial f(i\Delta x, j\Delta y, k\Delta z, n\Delta t)}{\partial x} = \frac{f^n_{i+\frac{1}{2},j,k} - f^n_{i-\frac{1}{2},j,k}}{\Delta x} + O\left[(\Delta x)^2\right]$$

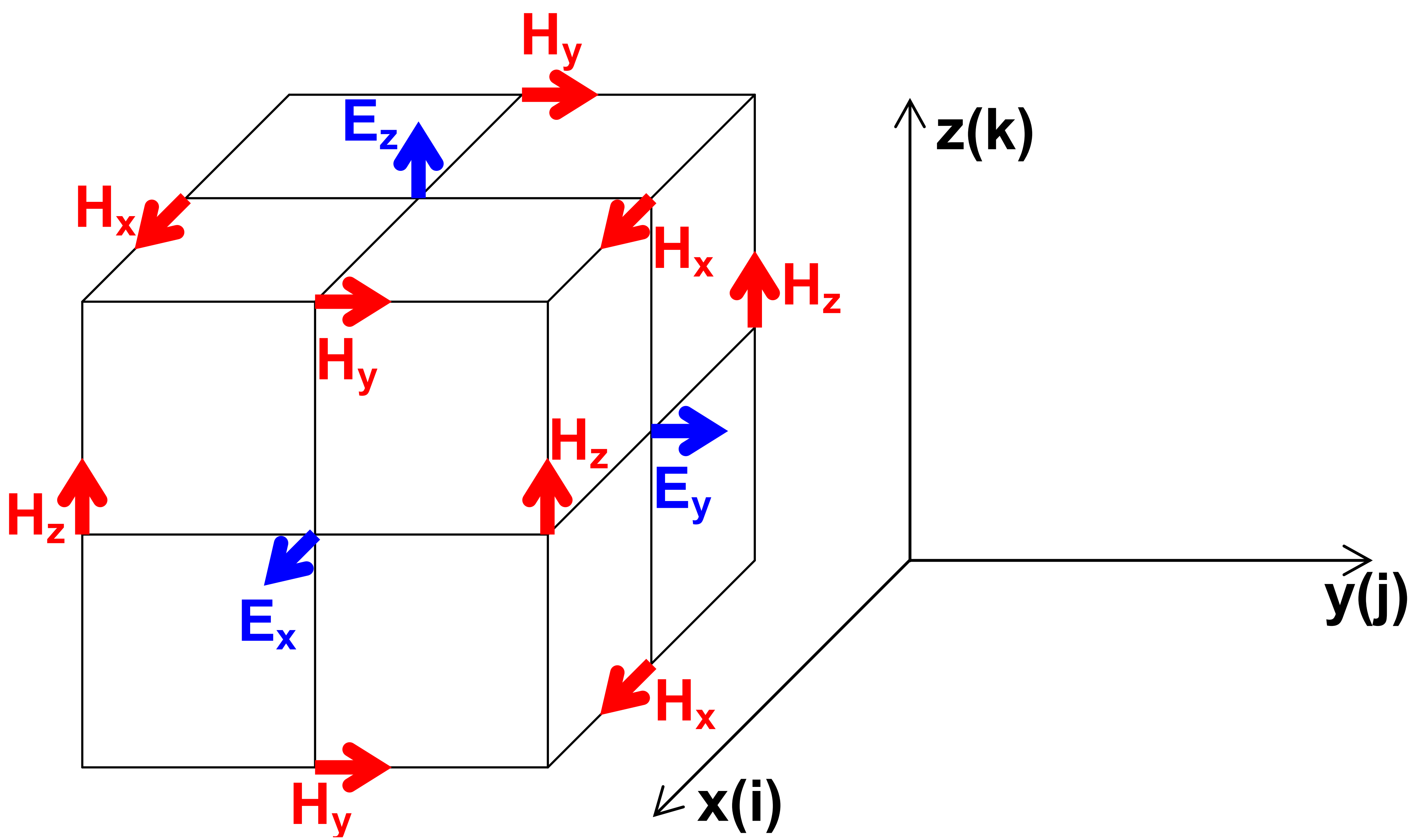
$$\frac{\partial f(i\Delta x, j\Delta y, k\Delta z, n\Delta t)}{\partial t} = \frac{f^{n+\frac{1}{2}}_{i,j,k} - f^{n-\frac{1}{2}}_{i,j,k}}{\Delta t} + O\left[(\Delta t)^2\right]$$

approach used for all differential equations, e.g.

$$\frac{\partial E_x}{\partial t} = \frac{1}{\varepsilon_0 \varepsilon(\mathbf{r})}\left[\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} - j_x\right]$$

E and H field are evaluated a half-step (in space and time) apart

FDTD in 3D



Properties: no divergence

Leapfrog time steps

central differencing ⇒ second order explicit method

FDTD in 3D

discretising MWEQ

$$\frac{\partial E_x}{\partial t} = \frac{1}{\varepsilon_0 \varepsilon(\mathbf{r})}\left[\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} - j_x\right]$$

$$E_x\Big|_{i,j+\frac{1}{2},k+\frac{1}{2}}^{n+\frac{1}{2}} = E_x\Big|_{i,j+\frac{1}{2},k+\frac{1}{2}}^{n-\frac{1}{2}} + \frac{\Delta t}{\varepsilon_0 \varepsilon_{i,j+\frac{1}{2},k+\frac{1}{2}}}\left(\frac{H_z\Big|_{i,j+1,k+\frac{1}{2}}^{n} - H_z\Big|_{i,j,k+\frac{1}{2}}^{n}}{\Delta y} - \frac{H_y\Big|_{i,j+\frac{1}{2},k+1}^{n} - H_y\Big|_{i,j+\frac{1}{2},k}^{n}}{\Delta z} - j_x\Big|_{i,j+\frac{1}{2},k+\frac{1}{2}}^{n}\right)$$

Other equations for FDTD in 3D

$$E_x\Big|_{i,j+\frac{1}{2},k+\frac{1}{2}}^{n+\frac{1}{2}} = E_x\Big|_{i,j+\frac{1}{2},k+\frac{1}{2}}^{n-\frac{1}{2}} + \frac{\Delta t}{\varepsilon_0 \varepsilon_{i,j+\frac{1}{2},k+\frac{1}{2}}}\left(\frac{H_z\Big|_{i,j+1,k+\frac{1}{2}}^{n} - H_z\Big|_{i,j,k+\frac{1}{2}}^{n}}{\Delta y} - \frac{H_y\Big|_{i,j+\frac{1}{2},k+1}^{n} - H_y\Big|_{i,j+\frac{1}{2},k}^{n}}{\Delta z} - j_x\Big|_{i,j+\frac{1}{2},k+\frac{1}{2}}^{n} \right)$$

$$E_y\Big|_{i-\frac{1}{2},j+1,k+\frac{1}{2}}^{n+\frac{1}{2}} = E_y\Big|_{i-\frac{1}{2},j+1,k+\frac{1}{2}}^{n-\frac{1}{2}} + \frac{\Delta t}{\varepsilon_0 \varepsilon_{i-\frac{1}{2},j+1,k+\frac{1}{2}}}\left(\frac{H_x\Big|_{i-\frac{1}{2},j+1,k+1}^{n} - H_x\Big|_{i-\frac{1}{2},j+1,k}^{n}}{\Delta z} - \frac{H_z\Big|_{i,j+1,k+\frac{1}{2}}^{n} - H_z\Big|_{i-1,j+1,k+\frac{1}{2}}^{n}}{\Delta x} - j_y\Big|_{i-\frac{1}{2},j+1,k+\frac{1}{2}}^{n} \right)$$

$$E_z\Big|_{i-\frac{1}{2},j+\frac{1}{2},k+1}^{n+\frac{1}{2}} = E_z\Big|_{i-\frac{1}{2},j+\frac{1}{2},k+1}^{n-\frac{1}{2}} + \frac{\Delta t}{\varepsilon_0 \varepsilon_{i-\frac{1}{2},j+\frac{1}{2},k+1}}\left(\frac{H_y\Big|_{i,j+\frac{1}{2},k+1}^{n} - H_y\Big|_{i-1,j+\frac{1}{2},k+1}^{n}}{\Delta x} - \frac{H_x\Big|_{i-\frac{1}{2},j+1,k+1}^{n} - H_x\Big|_{i-\frac{1}{2},j,k}^{n}}{\Delta z} - j_z\Big|_{i-\frac{1}{2},j+\frac{1}{2},k+1}^{n} \right)$$

Other equations for FDTD in 3D

$$H_x\Big|_{i-\frac{1}{2},j+1,k+1}^{n+1} = H_x\Big|_{i-\frac{1}{2},j+1,k+1}^{n} + \frac{\Delta t}{\mu_0}\left(\frac{E_y\Big|_{i-\frac{1}{2},j+1,k+\frac{3}{2}}^{n+\frac{1}{2}} - E_y\Big|_{i-\frac{1}{2},j+1,k+\frac{1}{2}}^{n+\frac{1}{2}}}{\Delta z} - \frac{E_z\Big|_{i-\frac{1}{2},j+\frac{3}{2},k+1}^{n+\frac{1}{2}} - E_z\Big|_{i-\frac{1}{2},j+\frac{1}{2},k+1}^{n+\frac{1}{2}}}{\Delta y} \right)$$

$$H_y\Big|_{i,j+\frac{1}{2},k+1}^{n+1} = H_y\Big|_{i,j+\frac{1}{2},k+1}^{n} + \frac{\Delta t}{\mu_0}\left(\frac{E_z\Big|_{i+\frac{1}{2},j+\frac{1}{2},k+1}^{n+\frac{1}{2}} - E_z\Big|_{i-\frac{1}{2},j+\frac{1}{2},k+1}^{n+\frac{1}{2}}}{\Delta x} - \frac{E_x\Big|_{i,j+\frac{1}{2},k+\frac{3}{2}}^{n+\frac{1}{2}} - E_x\Big|_{i,j+\frac{1}{2},k+\frac{1}{2}}^{n+\frac{1}{2}}}{\Delta z} \right)$$

$$H_z\Big|_{i,j+1,k+\frac{1}{2}}^{n+1} = H_z\Big|_{i,j+1,k+\frac{1}{2}}^{n} + \frac{\Delta t}{\mu_0}\left(\frac{E_x\Big|_{i,j+\frac{3}{2},k+\frac{1}{2}}^{n+\frac{1}{2}} - E_x\Big|_{i,j+\frac{1}{2},k+\frac{1}{2}}^{n+\frac{1}{2}}}{\Delta y} - \frac{E_y\Big|_{i+\frac{1}{2},j+1,k+\frac{1}{2}}^{n+\frac{1}{2}} - E_y\Big|_{i-\frac{1}{2},j+1,k+\frac{1}{2}}^{n+\frac{1}{2}}}{\Delta x} \right)$$

Divergence free nature of the Yee-discretization

$$\operatorname{rot}\mathbf{E}(\mathbf{r},t) = -\mu_0 \frac{\partial \mathbf{H}(\mathbf{r},t)}{\partial t} \qquad \operatorname{rot}\mathbf{H}(\mathbf{r},t) = \varepsilon_0\varepsilon(\mathbf{r})\frac{\partial \mathbf{E}(\mathbf{r},t)}{\partial t} + \mathbf{j}_{\text{makr}}$$

$$\operatorname{div}\left[\varepsilon_0\varepsilon(\mathbf{r})\mathbf{E}(\mathbf{r},t)\right] = 0 \qquad \operatorname{div}\mathbf{H}(\mathbf{r},t) = 0$$

$$\frac{\partial}{\partial t}\operatorname{div}(\varepsilon_0\mathbf{E}) = ? \qquad \frac{\partial}{\partial t}\oiiint_{\text{Yee cell}} \operatorname{div}(\varepsilon_0\mathbf{E})\,dV = \frac{\partial}{\partial t}\oiint_{\text{Yee cell}} \varepsilon_0\mathbf{E}\,d\mathbf{f}$$

$$\frac{\partial}{\partial t}\oiint_{\text{Yee cell}} \varepsilon_0\mathbf{E}\,d\mathbf{f} = \underbrace{\varepsilon_0\frac{\partial}{\partial t}\left(E_x\Big|_{i,j+\frac{1}{2},k+\frac{1}{2}} - E_x\Big|_{i-1,j+\frac{1}{2},k+\frac{1}{2}}\right)\Delta y\Delta z}_{\text{Term 1}}$$

$$+\underbrace{\varepsilon_0\frac{\partial}{\partial t}\left(E_y\Big|_{i-\frac{1}{2},j+1,k+\frac{1}{2}} - E_y\Big|_{i-\frac{1}{2},j,k+\frac{1}{2}}\right)\Delta x\Delta z}_{\text{Term 2}}$$

$$+\underbrace{\varepsilon_0\frac{\partial}{\partial t}\left(E_z\Big|_{i-\frac{1}{2},j+\frac{1}{2},k+1} - E_z\Big|_{i-\frac{1}{2},j+\frac{1}{2},k}\right)\Delta x\Delta y}_{\text{Term 3}}$$

Divergence free nature of the Yee-discretisation

Term 1 with rot equation $\frac{\partial E_x}{\partial t} = \frac{1}{\varepsilon_0 \varepsilon(\mathbf{r})}\left[\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right]$

$$\text{Term 1} = \left(\frac{H_z\big|_{i,j+1,k+\frac{1}{2}} - H_z\big|_{i,j,k+\frac{1}{2}}}{\Delta y} - \frac{H_y\big|_{i,j+\frac{1}{2},k+1} - H_y\big|_{i,j+\frac{1}{2},k}}{\Delta z} \right)$$

$$- \left(\frac{H_z\big|_{i-1,j+1,k+\frac{1}{2}} - H_z\big|_{i-1,j,k+\frac{1}{2}}}{\Delta y} - \frac{H_y\big|_{i-1,j+\frac{1}{2},k+1} - H_y\big|_{i-1,j+\frac{1}{2},k}}{\Delta z} \right)$$

$$\frac{\partial}{\partial t} \oint\!\!\oint_{\text{Yee cell}} \varepsilon_0 \mathbf{E} d\mathbf{f} = (\text{Term 1})\Delta y \Delta z + (\text{Term 2})\Delta x \Delta z + (\text{Term 3})\Delta x \Delta y = 0$$

$$\oint\!\!\oint\!\!\oint_{\text{Yee cell}} \operatorname{div}[\varepsilon_0 \mathbf{E}(t=0)] dV = 0 \quad \& \quad \frac{\partial}{\partial t} \oint\!\!\oint\!\!\oint_{\text{Yee cell}} \operatorname{div}(\varepsilon_0 \mathbf{E}) dV = 0 \Rightarrow \operatorname{div}[\varepsilon_0 \mathbf{E}] = 0$$

Computational procedure

- using the spatial differences of the E Field that are known for the time step n Δt to calculate the H field at the time step (n+1/2) Δt

- using the spatial differences of the H Field that are known for the time step (n+1/2) Δt to calculate the E field at the time step (n+1) Δt

- using the spatial differences of the E Field that are known for the time step (n+1) Δt to calculate the H field at the time step (n+3/2) Δt

⋮

⟹ leap-frog algorithm *(discretisation applies to all components)*

⟹ close to the physical world as the spatial and temporal propagation is exactly simulated

Simplification in 2D

- problems are often invariant in one spatial direction, taking z (e.g. grating, cylindrical objects)

- derivations of the field along this directions are zero

$$\frac{\partial H_x}{\partial t} = \frac{1}{\mu}\left[\cancel{\frac{\partial E_y}{\partial z}} - \frac{\partial E_z}{\partial y}\right] \qquad \frac{\partial E_x}{\partial t} = \frac{1}{\epsilon}\left[\frac{\partial H_z}{\partial y} - \cancel{\frac{\partial H_y}{\partial z}} - \sigma E_x\right]$$

$$\frac{\partial H_y}{\partial t} = \frac{1}{\mu}\left[\frac{\partial E_z}{\partial x} - \cancel{\frac{\partial E_x}{\partial z}}\right] \qquad \frac{\partial E_y}{\partial t} = \frac{1}{\epsilon}\left[\cancel{\frac{\partial H_x}{\partial z}} - \frac{\partial H_z}{\partial x} - \sigma E_y\right]$$

$$\frac{\partial H_z}{\partial t} = \frac{1}{\mu}\left[\frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x}\right] \qquad \frac{\partial E_z}{\partial t} = \frac{1}{\epsilon}\left[\frac{\partial H_y}{\partial x} - \frac{\partial H x}{\partial y} - \sigma E_z\right]$$

⇨ Maxwell can be decoupled into 2 sets of each 3 differential equations

Simplification in 2D

- problems are often invariant in one spatial direction, taking z (e.g. grating, cylindrical objects)
- derivations of the field along this directions are zero

$$\frac{\partial H_z}{\partial t} = \frac{1}{\mu}\left[\frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x}\right]$$

$$\frac{\partial E_x}{\partial t} = \frac{1}{\epsilon}\left[\frac{\partial H_z}{\partial y} - \sigma E_x\right]$$

$$\frac{\partial E_y}{\partial t} = \frac{1}{\epsilon}\left[-\frac{\partial H_z}{\partial x} - \sigma E_y\right]$$

TM polarisation

$$\frac{\partial E_z}{\partial t} = \frac{1}{\epsilon}\left[\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} - \sigma E_z\right]$$

$$\frac{\partial H_x}{\partial t} = \frac{1}{\mu}\left[-\frac{\partial E_z}{\partial y}\right]$$

$$\frac{\partial H_y}{\partial t} = \frac{1}{\mu}\left[\frac{\partial E_z}{\partial x}\right]$$

TE polarisation

Introducing a light source

- sources modelled by adding their field to field in computational domain
- a physical model for sources is a macroscopic current density

$$\mathrm{rot}\,\mathbf{H}(\mathbf{r},t) = \mathbf{j}_{\mathrm{makr}}(\mathbf{r},t) + \frac{\partial \mathbf{D}(\mathbf{r},t)}{\partial t}$$

Examples for a temporal variation of the light source:

$$E_x^n(i,j,k) = E_x^n(i,j,k) + \sin(n\Delta t\omega)$$

(x-polarised plane wave)

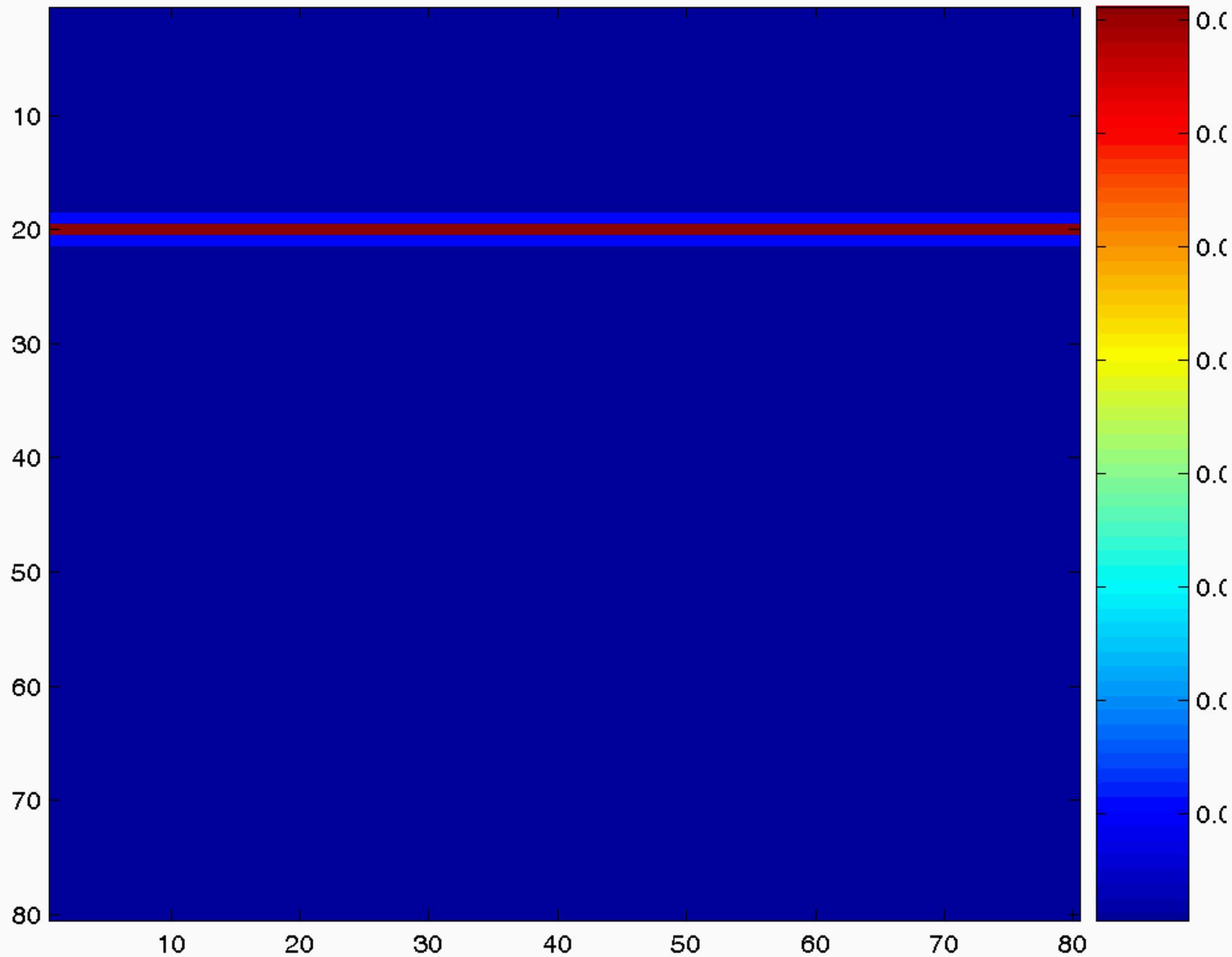
$$E_x^n(i,j,k) = E_x^n(i,j,k) + \delta_{n,n'}$$

(x-polarised impulse)

$$E_x^n(i,j,k_s) = E_x^n(i,j,k_s) + e^{-\left(\frac{i\Delta x}{\sigma_x}\right)^2} e^{-\left(\frac{j\Delta y}{\sigma_y}\right)^2} \sin(n\Delta t\omega)$$

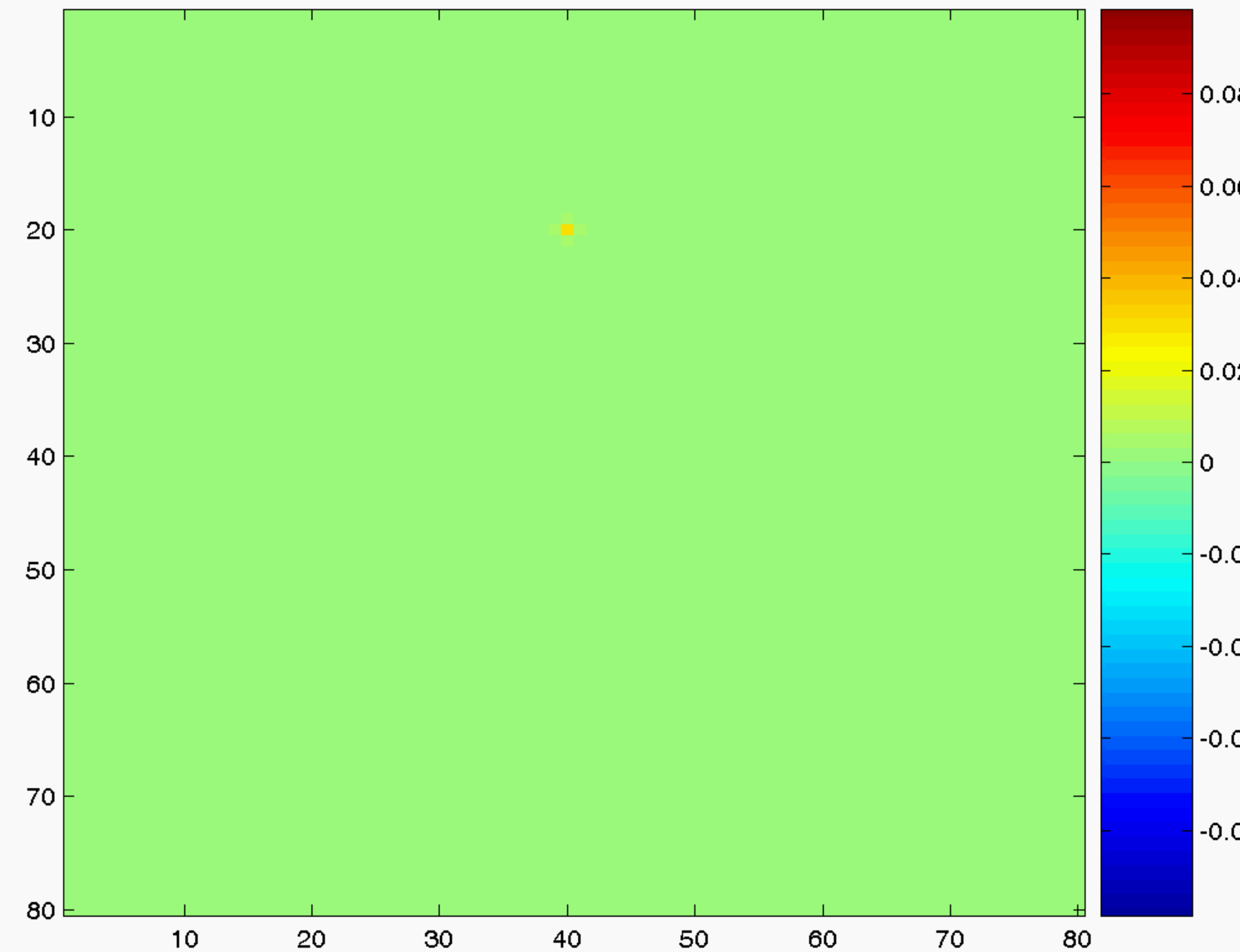
(x-polarised Gaussian wave with the waist in k_s)

Introducing a light source



example of a plane wave

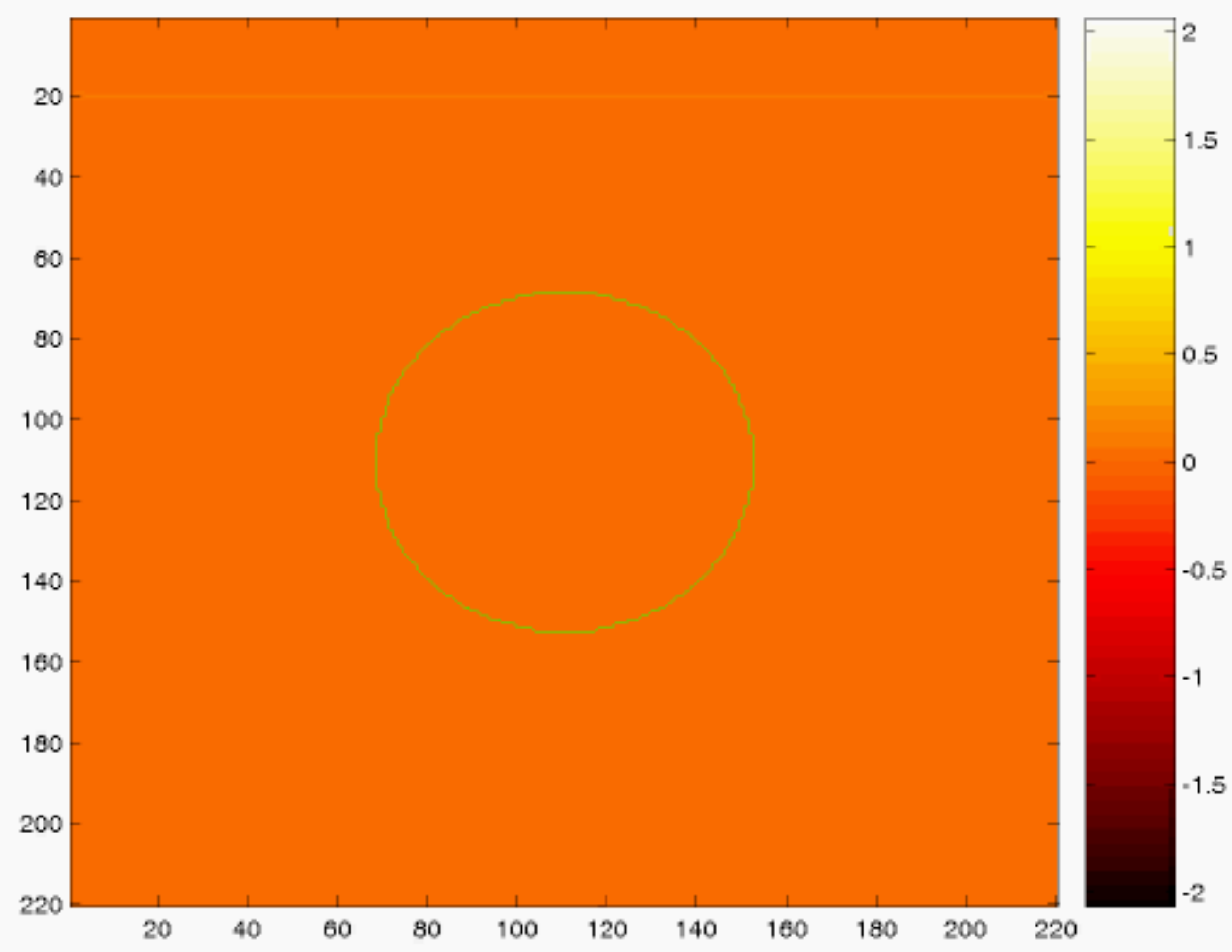
(2D-configuration, TM, Hy)



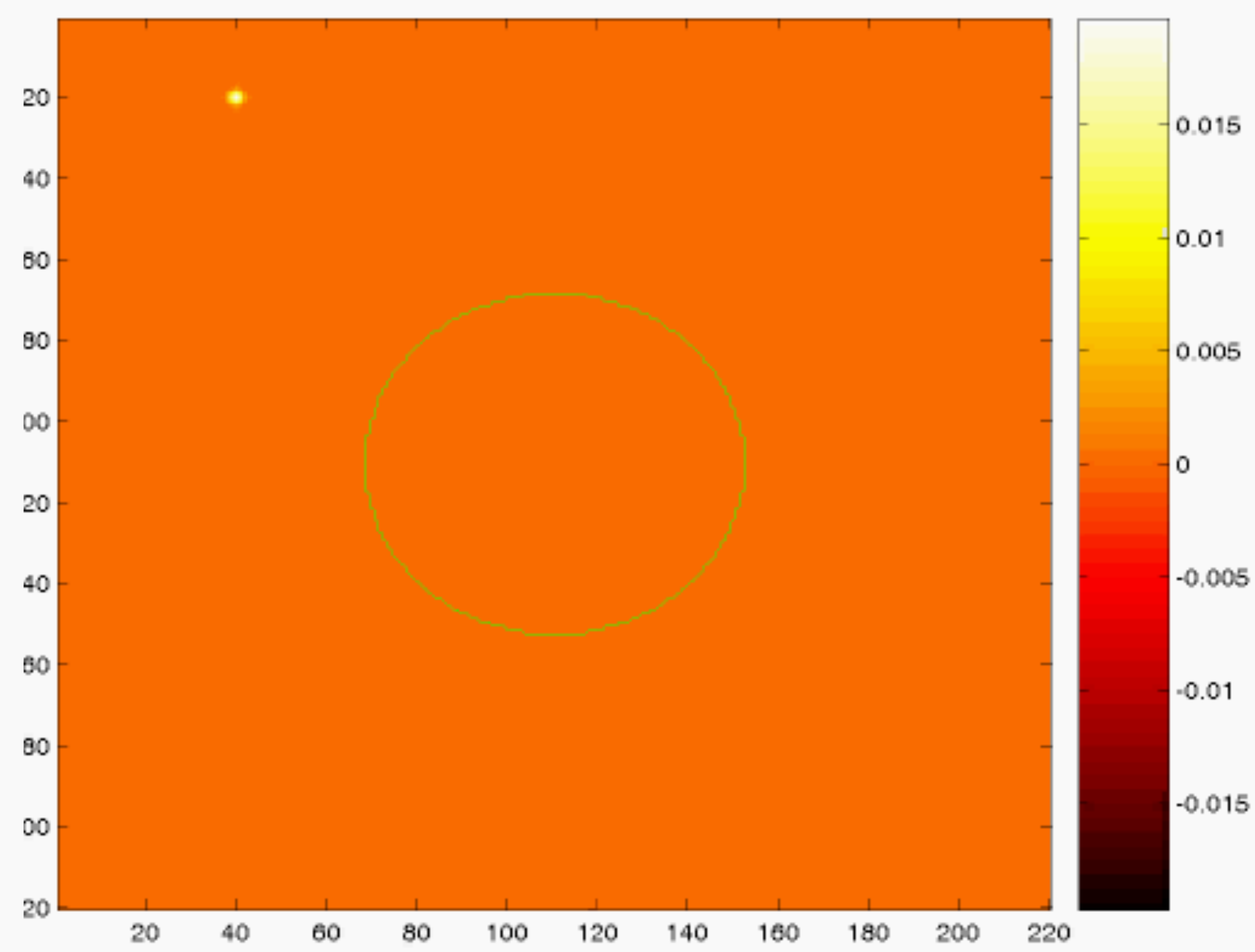
example of a point source

(2D-configuration, TM, Hy)

Introducing a light source



example of a plane wave
(2D-configuration, TM, Hy)



example of a point source
(2D-configuration, TM, Hy)

(cylinder with n=2 and D=2 λ)

Finite-Difference Time-Domain