Computational Photonics
Tutorial 1:
Transfer matrix method

3 May 2023

Dr. Markus Nyman, markus.nyman@kit.edu
M.Sc. Nigar Asadova, nigar.asadova@kit.edu

Exercises for everyone:

- Task I: Calculation of a transfer matrix
- Task II: Reflection and transmission coefficients

Extended exercises:

- Task III*: Field distribution
- Task IV*: Time animation of the field

Computational Photonics

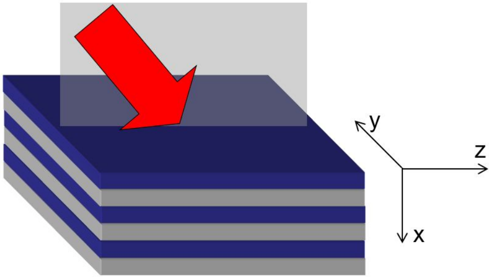
Seminar 03, SS 2020

Homework 0: Implementation of the Matrix Method

- calculation of the transfer matrix
- calculation of reflection and transmission characteristics of stratified media
- calculation of fields inside layers

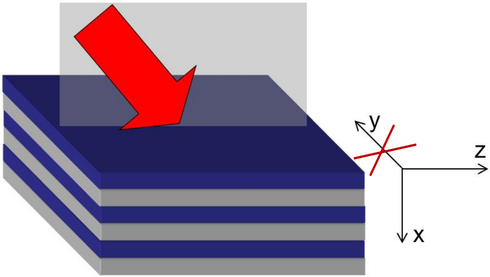
Abbe School of Photonics | JENA
Friedrich-Schiller-Universität

Optics in stratified media



- Bragg mirror
- mirror with chirp for compensating dispersion
- interferometer

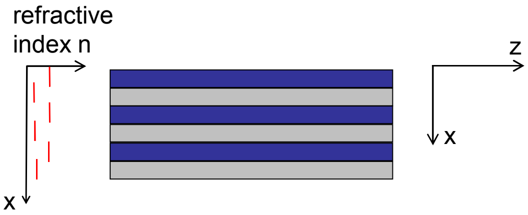
Optics in stratified media



Plane of incidence = x-z-plane

$\Rightarrow$ no y-dependency

A stratified (layered) medium

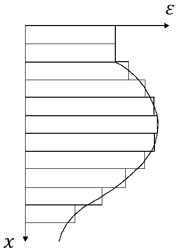


A stratified (layered) medium

each layer with index i is characterized by its thickness d_i and its dielectric constant $\varepsilon_i(\omega)$

an arbitrary continuous variation of the refractive index can be discretized with a sufficient large number of layers

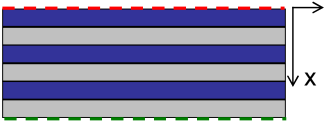
→ important for so called 'GRIN' – graded index waveguides



EM fields in the stratified media

Requirements:

- stationarity
- infinite extension of the layers in the z-y-plane
- illuminating field incident in the x-z-plane



Ansatz:
$$\mathbf{E}_{\mathbf{real}}(x,z,t)=\mathrm{Re}\left[\mathbf{E}(x)\exp\left(ik_z z-i\omega t\right)\right]$$
$$\mathbf{H}_{\mathbf{real}}(x,z,t)=\mathrm{Re}\left[\mathbf{H}(x)\exp\left(ik_z z-i\omega t\right)\right]$$

Separation into TE und TM polarization

TE: $\mathbf{E}_{\mathrm{TE}}=\begin{pmatrix}0\\E_y\\0\end{pmatrix},\quad \mathbf{H}_{\mathrm{TE}}=\begin{pmatrix}H_x\\0\\H_z\end{pmatrix}$

TM: $\mathbf{H}_{\mathrm{TM}}=\begin{pmatrix}0\\H_y\\0\end{pmatrix},\quad \mathbf{E}_{\mathrm{TM}}=\begin{pmatrix}E_x\\0\\E_z\end{pmatrix}$

Boundary conditions

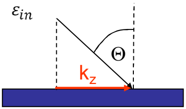
Fields: $\mathbf{E}_t$ and $\mathbf{H}_t$ continuous

TE: $E = E_y$ and H_z
TM: $H = H_y$ and E_z

➔ Performing all computations with the **tangential components**,
(if necessary the normal components can be derived)

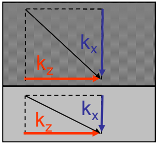
transversal wavevector is constant in the stack and is determined by the angle of incidence:

$$\Rightarrow k_z = \frac{\omega}{c}\sqrt{\varepsilon_{in}}\sin\theta = \frac{2\pi}{\lambda_0}\sqrt{\varepsilon_{in}}\sin\theta$$



normal component varies in the stack:
$\Rightarrow k_x$ depends on the permittivity of each layer
$\Rightarrow$ **dispersion relation**

$$k_x{}^2 = \frac{\omega^2}{c^2}\varepsilon(\omega) - k_z^2$$



Computing the fields by continuous components (TE)

Helmholtz-equation:

$$\Bigg[\frac{\partial^2}{\partial x^2}+\underbrace{\frac{\omega^2}{c^2}\varepsilon(\omega)-k_z^2}_{k_x^2}\Bigg]E_y(x)=0 \qquad i\omega\mu_0 H_z(x)=\frac{\partial}{\partial x}E_y(x)$$

Solution: $$E_y(x)=C_1\cos(k_x x)+C_2\sin(k_x x)$$

$$i\omega\mu_0 H_z(x)=\frac{\partial}{\partial x}E_y(x)=k_x\left[-C_1\sin(k_x x)+C_2\cos(k_x x)\right]$$

Determination of C_1, C_2:

$$E_y(0)=C_1 \qquad \left.\frac{\partial}{\partial x}E_y\right|_0=k_x C_2$$

Solution

TE:

$$E_y(x) = \cos(k_x x) E_y(0) + \frac{1}{k_x}\sin(k_x x)\frac{\partial}{\partial x}E_y\Big|_0$$

$$\frac{\partial}{\partial x}E_y = -k_x \sin(k_x x) E_y(0) + \cos(k_x x)\frac{\partial}{\partial x}E_x\Big|_0$$

TM:

$$H_y(x) = \cos(k_x x) H_y(0) + \frac{\varepsilon}{k_x}\sin(k_x x)\frac{1}{\varepsilon}\frac{\partial}{\partial x}H_y\Big|_0$$

$$\frac{1}{\varepsilon}\frac{\partial}{\partial x}H_y = -\frac{k_x}{\varepsilon}\sin(k_x x) H_y(0) + \cos(k_x x)\frac{1}{\varepsilon}\frac{\partial}{\partial x}H_y\Big|_0$$

TE/TM:

$$F(x) = \cos(k_x x) F(0) + \frac{1}{qk_x}\sin(k_x x) G(0)$$

$$G(x) = -qk_x \sin(k_x x) F(0) + \cos(k_x x) G(0)$$

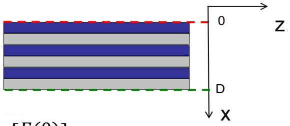
TE: $F = E_y,\quad G = i\omega\mu_0 H_z = \frac{\partial}{\partial x}E_y,\quad q = 1$

TM: $F = H_y,\quad G = -i\omega\varepsilon_0 E_z = q\frac{\partial}{\partial x}H_y,\quad q = 1/\varepsilon$

Summary: Matrix method

Need to know: $F(0)$, $G(0)$, $k_x^{(i)}$, ε_i, d_i

We want to calculate the fields $F(D)$, $G(D)$



$$\begin{bmatrix} F(D) \\ G(D) \end{bmatrix} = \prod_i \hat{m}_i \begin{bmatrix} F(0) \\ G(0) \end{bmatrix} = \hat{M} \begin{bmatrix} F(0) \\ G(0) \end{bmatrix}$$

$$\hat{m}_i = \begin{bmatrix} \cos\left(k_x^{(i)} d_i\right) & \frac{1}{q_i k_x^{(i)}} \sin\left(k_x^{(i)} d_i\right) \\ -q_i k_x^{(i)} \sin\left(k_x^{(i)} d_i\right) & \cos\left(k_x^{(i)} d_i\right) \end{bmatrix}$$

TE: $F = E_y, \quad G = \frac{\partial}{\partial x} E_y, \quad q_i = 1$

TM: $F = H_y, \quad G = q_i \frac{\partial}{\partial x} H_y, \quad q_i = 1/\varepsilon_i$

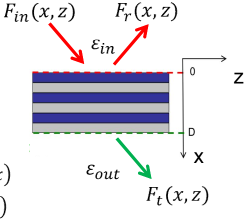
with

$$\left[k_x^{(i)}\right]^2 = \left(\frac{2\pi}{\lambda_0}\right)^2 \varepsilon_i(\omega) - k_z^2$$

Reflection and transmission coefficients of the fields

transmission coefficient $t = \frac{F_T}{F_{\text{in}}}$

reflection coefficient $r = \frac{F_R}{F_{\text{in}}}$



fields at ε_{in}:

$$F_{in}(x,z) = F_{in}(x)\exp(ik_z z) \qquad F_{in}(x) = F_{in}\exp\left(ik_x^{in}x\right)$$

$$F_r(x,z) = F_r(x)\exp(ik_z z) \qquad F_r(x) = F_r\exp\left(-ik_x^{in}x\right)$$

field at ε_{out}:

$$F_t(x,z) = F_t(x)\exp(ik_z z) \qquad F_t(x) = F_t\exp(ik_x^{out}(x-D))$$

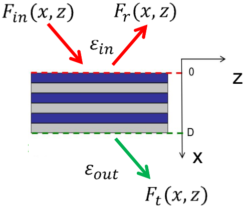
connection of fields at $x = 0$ and $x = D$ by transfer matrix

$$\begin{bmatrix} F(D) \\ G(D) \end{bmatrix} = \hat{M}\begin{bmatrix} F(0) \\ G(0) \end{bmatrix} \longrightarrow \begin{bmatrix} F_t \\ iq_{out}k_x^{out}F_t \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}\begin{bmatrix} F_{in} + F_r \\ iq_{in}k_x^{in}(F_{in} - F_r) \end{bmatrix}$$

Reflection and transmission coefficients of the fields

transmission coefficient $t = \dfrac{F_T}{F_{\text{in}}}$

reflection coefficient $r = \dfrac{F_R}{F_{\text{in}}}$

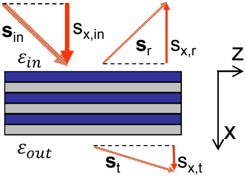


$$r = \frac{q_{in}k_x^{in}M_{22} - q_{out}k_x^{out}M_{11} - i\left(M_{21} + q_{in}k_x^{in}q_{out}k_x^{out}M_{12}\right)}{q_{in}k_x^{in}M_{22} + q_{out}k_x^{out}M_{11} + i\left(M_{21} - q_{in}k_x^{in}q_{out}k_x^{out}M_{12}\right)}$$

$$t = \frac{2q_{in}k_x^{in}}{q_{in}k_x^{in}M_{22} + q_{out}k_x^{out}M_{11} + i\left(M_{21} - q_{in}k_x^{in}q_{out}k_x^{out}M_{12}\right)}$$

Energy flux

defined via the normal component of the Poynting vector **S**



- Reflectivity:

$$R = \frac{S_{x,r}}{S_{x,in}} \qquad R = |r|^2$$

- Transmissivity:

$$T = \frac{S_{x,\mathrm{t}}}{S_{x,in}} \qquad T = \frac{q_{out}\,\mathrm{Re}\left(k_x^{out}\right)}{q_{in}\,\mathrm{Re}\left(k_x^{in}\right)}|t|^2$$

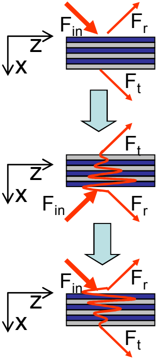
Field distribution

Goal: Computation of $F(x)$ inside the entire structure,
(the absolute values can be scaled)

Initial point: Take the known entries of the transmitted amplitude

$$\begin{bmatrix} F(D) \\ G(D) \end{bmatrix} = \begin{bmatrix} F_t \\ q_{out} \dfrac{\partial F_t}{\partial x} \end{bmatrix} = F_t \begin{bmatrix} 1 \\ i q_{out} k_x^{out} \end{bmatrix}$$

Now: $F_t = 1$



Approach:

1. Reverse the structure (incident vector becomes $(1, -i q_{out} k_x^{out})$
2. Calculate the field vector up to the next interface
3. From there, calculate the field to the next x-point of interest
4. Save the first value of the vector for this x-point
5. Iterate until all x-values are calculated and reverse the structure and the field

The real field

The observable (real) field

$$\mathbf{E_r}(x,z,t) = \mathrm{Re}\left[\mathbf{E}(x)\exp\left(ik_z z - i\omega t\right)\right]$$
$$\mathbf{H_r}(x,z,t) = \mathrm{Re}\left[\mathbf{H}(x)\exp\left(ik_z z - i\omega t\right)\right]$$

What you have actually calculated is the complex value of a certain component:

TE: $\mathbf{E}(x) = F(x)\,\mathbf{e_y}$

TM: $\mathbf{H}(x) = F(x)\,\mathbf{e_y}$

Task I : Transfer matrix

Goal : calculation of $\widehat{M}$

Python

```
import numpy as np
from matplotlib import pyplot as plt

def transfermatrix(thickness, epsilon, polarisation, wavelength, kz):
    '''Computes the transfer matrix for a given stratified medium.

    Parameters
    ----------
    thickness : 1d-array
        Thicknesses of the layers in µm.
    epsilon : 1d-array
        Relative dielectric permittivity of the layers.
    polarisation : str
        Polarisation of the computed field, either 'TE' or 'TM'.
    wavelength : float
        The wavelength of the incident light in µm.
    kz : float
        Transverse wavevector in 1/µm.

    Returns
    -------
    M : 2d-array
        The transfer matrix of the medium.
    '''
    pass
```

Task II: Reflection and transmission coefficients (1/2)

Goal: computation of r, t, R, t as a function of the wavelength

Python

```
def spectrum(thickness, epsilon, polarisation, wavelength, angle_inc, n_in, n_out):

    '''Computes the reflection and transmission of a stratified medium.

    Parameters
    ----------
    thickness : 1d-array
        Thicknesses of the layers in µm.
    epsilon : 1d-array
        Relative dielectric permittivity of the layers.
    polarisation : str
        Polarisation of the computed field, either 'TE' or 'TM'.
    wavelength : 1d-array
        The wavelength of the incident light in µm.
    angle_inc : float
        The angle of incidence in degree (not radian!).
    n_in, n_out : float
        The refractive indices of the input and output layers.
```

Task II: Reflection and transmission coefficients (2/2)

Goal: computation of r, t, R, t as a function of the wavelength

Python

```
    Returns
    -------
    t : 1d-array
        Transmitted amplitude
    r : 1d-array
        Reflected amplitude
    T : 1d-array
        Transmitted energy
    R : 1d-array
        Reflected energy
    '''
    pass
```

Task III*: Field distribution (1/2)

Goal: Computation of the complex field f at predefined values of x

Python

```
def field(thickness, epsilon, polarisation, wavelength, kz, n_in, n_out, Nx, l_in, l_out):
    '''Computes the field inside a stratified medium.

    The medium starts at x = 0 on the entrance side. The transmitted field
    has a magnitude of unity.

    Parameters
    ----------
    thickness : 1d-array
        Thicknesses of the layers in µm.
    epsilon : 1d-array
        Relative dielectric permittivity of the layers.
    polarisation : str
        Polarisation of the computed field, either 'TE' or 'TM'.
    wavelength : float
        The wavelength of the incident light in µm.
    kz : float
        Transverse wavevector in 1/µm.
```

Task III*: Field distribution (2/2)

Goal: Computation of the complex field f at predefined values of x

Python

```
    n_in, n_out : float
        The refractive indices of the input and output layers.
    Nx : int
        Number of points where the field will be computed.
    l_in, l_out : float
        Additional thickness of the input and output layers where the field will
be computed.

    Returns
    -------
    f : 1d-array
        Field structure
    index : 1d-array
        Refractive index distribution
    x : 1d-array
        Spatial coordinates
    '''
    pass
```

Task IV*: Time animation of the field

Goal: Visualization of the temporal evolution of the field

Python

```
def timeanimation(x, f, index, steps, periods):
    ''' Animation of a quasi-stationary field.

    Parameters
    ----------
    x : 1d-array
        Spatial coordinates
    f : 1d-array
        Field
    index : 1d-array
        Refractive index
    steps : int
        Total number of time points
    periods : int
        Number of the oscillation periods.

    '''
    pass
```

Example parameters

Define a Bragg mirror at 780nm:

```
>> eps1 = 2.25;
>> eps2 = 15.21;
>> d1 = 0.13;    %[µm]
>> d2 = 0.05;    %[µm]
>> N  = 5;
>> polarisation ='TE';
>> angle_inc = 0.0;
>> n_in = 1.0;
>> n_out = 1.5;
```

Create the arrays

```
>> epsilon = zeros(1, 2*N);
>> epsilon(1:2:2*N)  = eps1;
>> epsilon(2:2:2*N)  = eps2;
>> thickness = zeros(1, 2*N);
>> thickness(1:2:2*N)= d1;
>> thickness(2:2:2*N) = d2;
>> lambda = linspace(0.5, 1.5, 100); %[µm]
```

Now, e.g. calculate the transmission/reflection spectrum:

```
>> [t, r, T, R] = spectrum(thickness, epsilon,
                           lambda_vector, angl
```