① $\boxed{1a}$ The ground state at $t=0$ is

$|N\,N \dots N \dots\rangle$, because that minimises the Coulomb energy which by definition is $\geqslant 0$

At $t=0$ the single-(anti) boson excitations are

$$|R\rangle = |NN\dots N \underset{\substack{\uparrow \\ R\text{-th site}}}{N+1} N\dots\rangle \quad \text{- extra boson}$$

$$|\tilde{R}\rangle = |NN\dots N \underset{\substack{\uparrow \\ R\text{-th site}}}{N-1} N\dots\rangle \quad \text{- „antiboson"}$$

Consider a reduced space of states when there is only one ~~boson~~ extra boson in the system. The other states are separated by the „single-boson" states under consideration by large gaps $\sim U$

Matrix element between states $|r\rangle$ and $|r'\rangle$ - $-t_{rr'}N$

$$H_{eff} = U - \sum_{\langle r,r'\rangle} Nt\,|r\rangle\langle r'| \quad \text{- tight-binding Ham-n}$$

$$|k\rangle = \sum_r e^{-ikr}\,|r\rangle$$

$$E_k = -2Nt\cos k_1 - \dots - 2Nt\cos k_d$$

$k_1, \dots, k_d$ are the momentum projections on different coordinate axes

$\boxed{1b}$ $-\cos k_i \approx -1 + \frac{k_i^2}{2}$ in the long-wavelength limit

$E(k) \approx$ ~~[illegible]~~ $\underbrace{U - 2Ntd}_{\text{Gap} = E_0} + Nt\,k^2$

$$Z_g = \prod_k \left(1 + e^{-\beta E_k}\right)$$

$$\Omega = -T \ln Z_g = -T \sum_k \ln\left(1 + e^{-\beta E_k}\right) \approx -T \sum_k e^{-\beta E_k}$$

$$C = -T \frac{\partial^2 \Omega}{\partial T^2} = \sum_k \frac{E_k^2}{T^2} e^{-\beta E_k} \approx \frac{E_0^2}{T^2} \sum_k e^{-\beta E_k} \qquad (*)$$

It is sufficient to differentiate the exponential only because it is a rapid function of the temperature

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There is an easier alternative way to see the capacitance

The total energy of the excitations $\mathcal{E} = \sum_k E_k e^{-\beta E_k}$

$$C = \frac{\partial \mathcal{E}}{\partial T} = \sum_k \frac{E_k^2}{T^2} e^{-\beta E_k}$$

$$(*) \rightarrow C = \frac{E_0^2}{T^2} N_0 \int \frac{dk}{(2\pi)^d} e^{-\beta E_0} e^{\beta N t k^2} =$$

Number of sites

$$= N_0 \frac{E_0^2}{T^2} e^{-\frac{E_0}{T}} \left(\int \frac{dk_i}{2\pi} e^{\beta N t k_i^2} \right)^d = \boxed{N_0 \frac{E_0^2}{T^2} e^{-\frac{E_0}{T}} (4\pi \beta N t)^{-\frac{d}{2}} = C}$$

$$C \sim T^{\frac{d}{2} - 2} e^{-\frac{E_0}{T}}$$

$\boxed{1c}$

$$b_r^+ b_{r'} = [\Delta_r^* + (b_r^+ - \Delta_r^*)][\cancel{b}\,\Delta_{r'} + (b_{r'} - \Delta_{r'})] \rightarrow$$

$$\xrightarrow{MF} \Delta_r^* \Delta_{r'} + (b_r^+ - \Delta_r^*)\Delta_{r'} + \Delta_r^*(b_{r'} - \Delta_{r'})$$

((Δ_r is site-independent, cf. the situation of the problem

$$|\Delta|^2 + (b_r^+ - \Delta^*)\Delta + \Delta^*(b_r - \Delta) =$$

$$= b_r^+ \Delta + \Delta^* b_r - |\Delta|^2$$

Let us write the hopping Ham-n as $-\sum_{r,r'} t_{rr'} b_r^+ b_{r'}$

Then in the MF Ham-n we have $-\sum_{r,r'} (b_r^+\Delta + \Delta^* b_r - |\Delta|^2)\, t_{rr'} =$

$$= -2td\Delta\, b_r^+ - 2td\,\Delta^* b_r + 2td\,|\Delta|^2$$

$$H_{MF} = \sum_r \left[U(\hat{n}_r - N)^2 - 2td\,\Delta b_r^+ - 2td\,\Delta^* b_r + 2td|\Delta|^2 \right]$$

At the transition point, if it exists, Δ vanishes

Treat the Δ-dependent part as a perturbation

~~The ground state to the first order in Δ:~~

~~$|g\rangle \approx |g\rangle - \sum_R \frac{2td\sqrt{N}}{U} |R\rangle_1$~~

~~Ground state for t=0, cf. 1a~~

] $|g\rangle$ – the ground state ~~t~~ at $t=0$, cf. 1a

$|R\rangle$ – the state of a single boson on site R

$|\tilde{R}\rangle$ – antiboson on site R

$$\langle g| \Delta^* b_r |R\rangle = \Delta^* \sqrt{N}$$

$$\langle g| \Delta\, b_r^+ |\tilde{R}\rangle = \Delta\sqrt{N}$$

The modified by the tunnelling ground state up to the first order in Δ:

$$|g\rangle' \approx |g\rangle - \sum_R \frac{2td\sqrt{N}\Delta}{U}|R\rangle - \sum_R \frac{2td\sqrt{N}\,\Delta^*}{U}|\tilde{R}\rangle$$

The modified ground-state energy

$$E' \approx \underbrace{-2\sum_R \frac{|2td\sqrt{N}\Delta|^2}{U}}_{\substack{\text{2-nd order}\\ \text{quantum mechanical}\\ \text{perturbation theory}}} + \sum_R 2td\,|\Delta|^2$$

GL expansion per grain (free energy at $T=0$ = simply energy):

$$F \approx \left(-\frac{8t^2d^2N}{U} + 2td\right)|\Delta|^2 + \gamma|\Delta|^4 + \dots$$

γ ← one can check this

The phase ~~at~~ with $|\Delta| \neq 0$ is energetically favourable if the term in the round brackets is <0

$$-\frac{4tdN}{U} + 1 = 0 \rightarrow$$

$$\rightarrow \boxed{\left(\frac{U}{t}\right)_c = 4Nd}$$

(5.2)

$$H = \sum_{j=-\infty}^{\infty} \left[J_1 (a_j^+ a_{j+1} + a_{j+1}^+ a_j) + J_2 (a_j a_{j+1} + a_{j+1}^+ a_j^+) - 2B a_j^+ a_j \right].$$

a) Write H in momentum space: $a_n = \int_{-\pi}^{\pi} \frac{dk}{2\pi} a_k e^{ikn}$

$$\rightsquigarrow H = \sum_j \int_{k,q} \left[J_1 (a_k^+ a_q e^{iq} + a_k^+ a_q e^{-ik})\overset{e^{i(q-k)j}}{!} + J_2 (a_k a_q \overset{e^{i(q+k)j}}{e^{+iq}} + a_k^+ a_q^+ e^{-ik})\, e^{-i(q+k)j} - 2B a_k^+ a_q \right\} e^{i(q-k)j} \Big] =$$

$$\Rightarrow \boxed{H = \int_{-\pi}^{\pi} \frac{dk}{2\pi} \left[J_1 a_k^+ a_k \, 2\cos k + J_2 (a_k a_{-k} e^{-ik} + a_k^+ a_{-k}^+ e^{-ik}) - 2B a_k^+ a_k \right].}$$

$$\sum_j e^{i(k-q)j} = \delta(k-q) 2\pi$$

$$\int_k a_k a_{-k} e^{-ik} \underset{\substack{\text{anti-commute} \\ \{a_k, a_{-k}\}=0}}{=} - \int_k a_{-k} a_k e^{-ik} \underset{k'=-k}{=} - \int_{k'} a_{k'}$$

$$\Rightarrow \boxed{H = \int_{-\pi}^{\pi} \frac{dk}{2\pi} \left[2(J_1 \cos k - B) a_k^{+} a_k - i J_2 \sin k \left(a_k a_{-k} + \underbrace{a_k^{+} a_{-k}^{+}}_{= - a_{-k}^{+} a_k^{+}} \right) \right].}$$

ⓑ Diagonalize H & find quasiparticle spectrum E_k.

$$e^{-i\frac{\pi}{4}} a_k = u_k c_k + v_k c_{-k}^{+} \qquad e^{i\frac{\pi}{4}} a_k^{+} = u_k^{*} c_k^{+} + v_k^{*} c_{-k}$$

$$e^{-i\frac{\pi}{4}} a_{-k} = - v_k c_k^{+} + u_k c_{-k} \quad \Rightarrow \quad e^{i\frac{\pi}{4}} a_{-k}^{+} = - v_k^{*} c_k + u_k^{*} c_{-k}^{+}$$

$$H = \int_{-\pi}^{\pi} \frac{dk}{2\pi} \left[(a_k^{+}, a_{-k}) \underbrace{\begin{pmatrix} J_1 \cos k - B & -i \sin k J_2 \\ i \sin k J_2 & -(J_1 \cos k - B) \end{pmatrix}} \begin{pmatrix} a_k \\ a_{-k}^{+} \end{pmatrix} + (J_1 \cos k - B) \right]$$

$$\Rightarrow \text{diagonalize} \Rightarrow \text{eigenvalues: } \lambda^2 + \left(-(J_1 \cos k - B)^2 - J_2^2 \sin^2 k \right) = 0$$

$$\Rightarrow \lambda_{\pm} = \pm \sqrt{(J_1 \cos k - B)^2 + J_2^2 \sin^2 k} =: E_{\pm}(k)$$

$a_k = e^{i\frac{\pi}{4}} b_k \; , \; a_k^{+} = e^{-i\frac{\pi}{4}} b_k^{+}$ $\qquad e^{\pm i\frac{\pi}{2}} = \pm i$

$a_{-k} = e^{i\frac{\pi}{4}} b_k \; , \; a_{-k}^{+} = e^{-i\frac{\pi}{4}} b_{-k}^{T}$

$$\Rightarrow H = \int_k \left[(J_1 \cos k - B)\left(b_k^{+} b_k + b_{-k}^{+} b_{-k}\right) - i J_2 \sin k \left(i\, b_k b_{-k} + i\, b_{-k}^{+} b_k^{+} \right) \right] =$$

$$= \int_k \left[(J_1 \cos k - B)\left(b_k^{+} b_k + b_{-k}^{+} b_{-k}\right) + J_2 \sin k \left(b_k b_{-k} + b_{-k}^{+} b_k^{+} \right) \right] .$$

Put particle-hole i one <u>Nambu-~~Ob~~ Gorkov spinor:</u>

$$\vec{b} = \left(b_k \, , \, b_{-k}^{+}\right)^T \; ; \; \vec{b}^{+} = \left(b_k^{+} \, , \, b_{-k}\right) \qquad U = \begin{pmatrix} | & | \\ e_{v_1} & e_{v_2} \\ | & | \end{pmatrix} .$$

$$\Rightarrow H = \int_k \left(b_k^{+} , b_{-k}\right) \overset{UU^{+}}{\underset{!}{}} \underbrace{\begin{pmatrix} J_1 \cos k - B , & -J_2 \sin k \\ -J_2 \sin k , & -(J_1 \cos k - B) \end{pmatrix}}_{\Rightarrow \lambda^2 + \left[-(J_1 \cos k - B)^2 - J_2^2 \sin^2 k\right] = 0 \;\Rightarrow\; \overset{E_\pm(k)}{\text{as before}}} \overset{UU^{+}}{\underset{!}{}} \begin{pmatrix} b_k \\ b_{-k}^{+} \end{pmatrix} + J_1 \cos k - B .$$

$$(J_1 \cos k - B)\, \underbrace{b_k^{+} b_{-k}}_{= 1 - b_{-k} b_{-k}^{+}} \rightarrow$$

Bog. Al.: $b_k = u_k c_k + v_k c_{-k}^+$ $\quad b_k^+ = \bar{u}_k c_k^+ + \bar{v}_k c_{-k}$

$b_{-k} = -v_k c_k^+ + u_k c_{-k}$ $\Rightarrow$ $b_{-k}^+ = -\bar{v}_k c_k + \bar{u}_k c_{-k}^+$.

$\rightsquigarrow$

$$\begin{pmatrix} b_k \\ b_{-k}^+ \end{pmatrix} = \underbrace{\begin{pmatrix} u_k & v_k \\ -v_k^* & u_k^* \end{pmatrix}}_{=\,U} \begin{pmatrix} c_k \\ c_{-k}^+ \end{pmatrix}$$

(choose: $u_k = \cos\theta_k$, $v_k = \sin\theta_k$ $\rightarrow$ $u_k^2 + v_k^2 = 1$.) $\quad U^+ = \begin{pmatrix} u_k^* & -v_k \\ v_k^* & u_k \end{pmatrix} = \begin{pmatrix} \cos\theta_k & -\sin\theta_k \\ \sin\theta_k & \cos\theta_k \end{pmatrix}$. as well.

$$\Rightarrow \begin{pmatrix} u_k^* & -v_k \\ v_k^* & u_k \end{pmatrix} \begin{pmatrix} \partial_1 \cos k - B\,, & -\partial_2 \sin k \\ -\partial_2 \sin k\,, & -(\partial_1 \cos k - B) \end{pmatrix} \begin{pmatrix} u_k & v_k \\ -v_k & u_k^+ \end{pmatrix} = \begin{pmatrix} E_+(k) & 0 \\ 0 & E_-(k) \end{pmatrix}.$$

Choose: $u_k = \cos\theta_k$, $v_k = \sin\theta_k$, which fulfills $u_k^2 + v_k^2 = 1$.

(still fulfills fermionic comm. relations for c_k, c_k^+.)

Go to p. 9.

$u_k = \cancel{\cos}\,\cos\theta_k$; $v_k = \sin\theta_k$ fulfills $u_k^2 + v_k^2 = 1$ ✓

$\Rightarrow$ (1,2) - element.

$$\boxed{\begin{aligned} &2\sin\theta_k\cos\theta_k(\partial_1\cos k - B) = \partial_2 \sin k(\cos^2\theta_k - \sin^2\theta_k) \\ \Leftrightarrow\ &2\sin(2\theta_k)(\partial_1\cos k - B) = \partial_2\sin k \cdot 2\cos(2\theta_k)\end{aligned}}$$

$\Rightarrow$

$$\boxed{\tan 2\theta_k = \frac{\partial_2 \sin(k)}{\partial_1\cos k - B}.}$$

$$\sin 2\theta_k \overset{!!}{=} \pm\frac{\partial_2 \sin k}{\sqrt{(\partial_1\cos k - B)^2 + \partial_2 \sin k^2}}$$

$$\cos 2\theta_k = \frac{\partial_1\cos k - B}{\sqrt{(\partial_1\cos k - B)^2 + \partial_2\sin k^2}}$$

$$2u_k v_k(\partial_1\cos k - B) + \cancel{2}\,\partial_2 \sin k(v^2 - u^2) = 0$$

$$\Leftrightarrow\ 2\sin\theta_k\cos\theta_k(\partial_1\cos k - B) + \partial_2\sin k(\sin^2\theta_k - \cos^2\theta_k) = 0$$

$$\Leftrightarrow\ 2\tan\theta_k(\partial_1\cos k - B) + \partial_2\sin k(\tan^2\theta_k - 1) = 0.$$

$\cos\theta_k \neq 0$, i.e. $\theta_k \neq \frac{\pi}{2}, \frac{3}{2}\pi, \ldots$

~~$\Rightarrow \tan\theta_k =$~~

$$\Rightarrow\ \tan^2\theta_k + 2\tan\theta_k \cdot \frac{\partial_1\cos k - B}{\partial_2 \sin k} \mp 1 = 0$$

$$\Rightarrow\ \boxed{\tan\theta_k = -\frac{\partial_1\cos k - B}{\partial_2\sin k} \pm\sqrt{\frac{(\partial_1\cos k - B)^2}{\partial_2^2\sin^2 k} + 1}\ .}$$

$$\Rightarrow \quad H = \int_k \left[(c_k^+ , c_{-k}) \begin{pmatrix} \lambda_+(k) & 0 \\ 0 & -\lambda_+(k) \end{pmatrix} \begin{pmatrix} c_k \\ c_{-k}^+ \end{pmatrix} + \partial_1 \cos k - B \right] =$$

$$= \int_k \left[\lambda_+(k) \; (c_k^+ c_k) \overset{+(-\lambda_+)}{\cancel{*}} \underbrace{c_{-k} c_{-k}^+}_{= 1 - c_{-k}^+ c_{-k}} \;) + \partial_1 \cos k - B \right] =$$

Energy E_k shall be positive (~~as~~ excitation energy).

with $\lambda_\pm = \sqrt{(\partial_1 \cos k - B)^2 + \partial_2^2 \sin^2 k}$

$\Rightarrow$ def. $E = 2\lambda_+$

$$= \int_k \frac{E(k)}{2} \left(c_k^+ c_k + c_{-k}^+ c_{-k} \right) - \lambda_+(k) + \partial_1 \cos k - B \Big] =$$

$$\boxed{\begin{aligned} H &= \int_k \left[E_k \left(c_k^+ c_k \right) - \frac{E_k}{2} + \partial_1 \cos k - B \right] \\ &\text{with } E_k = 2\lambda_+(k) = 2\sqrt{(\partial_1 \cos(k) - B)^2 + \partial_2^2 \sin^2 k} . \end{aligned}}$$