1

a) $\vec{\tau}_k = \psi_k^+ \vec{\tau} \psi_k$ with $\psi_k = \begin{pmatrix} c_{k\uparrow} \\ c^+_{-k\downarrow} \end{pmatrix}$, $\psi_k^+ = \left(c^+_{k\uparrow}, c_{-k\downarrow} \right)$, $\vec{\tau} = (\tau^x, \tau^y, \tau^z)$.

$$\Rightarrow \tau_k^x = c^+_{k\uparrow} c^+_{-k\downarrow} + c_{-k\downarrow} c_{k\uparrow}$$

$$\tau_k^y = -i\left(c^+_{k\uparrow} c^+_{-k\downarrow} - c_{-k\downarrow} c_{k\uparrow} \right)$$

$$\tau_k^z = c^+_{k\uparrow} c_{k\uparrow} - c_{-k\downarrow} c^+_{-k\downarrow} = n_{k\uparrow} + n_{-k\downarrow} - 1 \quad \text{with} \quad n_{k\sigma} = c^+_{k\sigma} c_{k\sigma}.$$

Show: $\left[\tau_k^\alpha, \tau_k^\beta \right] = 2i\, \varepsilon^{\alpha\beta\gamma} \tau_k^\gamma$.

i) Show $\left[\tau_k^z, \tau_k^x \right] = 2i\, \tau_k^y$ on next page. Use $\boxed{[AB, C] = A\{B, C\} - \{A, C\} B}$.

ii) $$\left[\tau_k^z, \tau_k^y \right] = \left[c^+_{k\uparrow} c_{k\uparrow} + c^+_{-k\downarrow} c_{-k\downarrow}, -i\left(c^+_{k\uparrow} c^+_{-k\downarrow} - c_{-k\downarrow} c_{k\uparrow} \right) \right] =$$

$$= -i\Big\{ c^+_{k\uparrow} \left[c_{k\uparrow}, c^+_{k\uparrow} c^+_{-k\downarrow} \right] - \left[c^+_{k\uparrow}, c_{-k\downarrow} c_{+k\uparrow} \right] c_{k\uparrow} + c^+_{-k\downarrow} \left[c_{-k\downarrow}, c^+_{+k\uparrow} c^+_{-k\downarrow} \right]$$

see next page

$$- \left[c^+_{-k\downarrow}, c_{-k\downarrow} c_{k\uparrow} \right] c_{-k\downarrow} \Big\} =$$

$$= -i\left\{ c^+_{k\uparrow} c^+_{-k\downarrow} + c_{-k\downarrow} c_{k\uparrow} - c^+_{-k\downarrow} c^+_{k\uparrow} - c_{k\uparrow} c_{-k\downarrow} \right\} = \underline{\underline{-2i\, \tau_k^x}}. \quad \square$$

Nambu spinor:
$$\Psi_k = \left(c_{k\uparrow},\ c^+_{-k\downarrow} \right)^T$$
$$\Psi^+_k = \left(c^+_{k\uparrow},\ c_{-k\downarrow} \right)$$

Isospin operators: $\vec{\tau}_k = \Psi^+_k \vec{\tau}\, \Psi_k \Rightarrow \tau^z_k = \left(c^+_{k\uparrow},\ c_{-k\downarrow} \right) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} c_{k\uparrow} \\ c^+_{-k\downarrow} \end{pmatrix} =$

$$\Leftrightarrow \boxed{\begin{aligned} \tau^z_k &= c^+_{k\uparrow} c_{k\uparrow} + c^+_{-k\downarrow} c_{-k\downarrow} - 1 \\ \tau^x_k &= c^+_{k\uparrow} c^+_{-k\downarrow} + c_{-k\downarrow} c_{k\uparrow} \\ \tau^y_k &= -i\left(c^+_{k\uparrow} c^+_{-k\downarrow} - c_{-k\downarrow} c_{k\uparrow} \right) \end{aligned}}$$

$$[\tau^z_k, \tau^x_k] =$$
$$= \left[c^+_{k\uparrow} c_{k\uparrow} + c^+_{-k\downarrow} c_{-k\downarrow},\ c^+_{k\uparrow} c^+_{-k\downarrow} + c_{-k\downarrow} c_{k\uparrow} \right] =$$
$$= c^+_{k\uparrow} \left[c_{k\uparrow}, c^+_{k\uparrow} c^+_{-k\downarrow} \right] + \left[c^+_{k\uparrow}, c_{-k\downarrow} c_{k\uparrow} \right] c_{k\uparrow} + c^+_{-k\downarrow} \left[c_{-k\downarrow}, c^+_{k\uparrow} c^+_{-k\downarrow} \right]$$
$$+ \left[c^+_{-k\downarrow}, c_{-k\downarrow} c_{k\uparrow} \right] c_{-k\downarrow} =$$

$$[AB, C] = A\{B, C\} - \{A, C\}B$$
$$= ABC - CAB \quad = ABC + \cancel{AC}B - \cancel{AC}B - CAB =$$
$$= ABC - CAB \checkmark$$

$$\cancel{[A, BC] = B\{A, C\}}$$
$$\cancel{= ABC - BCA = BAC + BCA}$$

$$-c^+_{k\uparrow}(-1)c^+_{-k\downarrow} - c_{-k\downarrow}c_{k\uparrow} - c^+_{-k\downarrow}c^+_{k\uparrow} - (-1)c_{k\uparrow}c_{-k\downarrow} =$$

$$= c^+_{k\uparrow}c^+_{-k\downarrow} + c^+_{k\uparrow}c^+_{-k\downarrow} - c_{-k\downarrow}c_{k\uparrow} - c_{-k\downarrow}c_{k\uparrow} =$$

$$= 2\left(c^+_{k\uparrow}c^+_{-k\downarrow} - c_{-k\downarrow}c_{k\uparrow}\right) = \underline{\underline{2i\tau^y_k}} = [\tau^z_k, \tau^x_k].$$

$$\boxed{[\tau^\alpha_k, \tau^\beta_k] = 2i\,\varepsilon^{\alpha\beta\gamma}\,\tau^\gamma_k}$$

(see p. 4. as well for $[\tau^x_k, \tau^y_k] = 2i\tau^z_k$.

(b) BCS-Ham.:

$$H_{BCS} = \sum_{k,\sigma} \varepsilon_k c^+_{k\sigma} c_{k\sigma} + \sum_k \left[\Delta^* c_{-k\downarrow} c_{k\uparrow} + c^+_{k\uparrow} c^+_{-k\downarrow}\Delta\right]$$

$$+ \frac{V}{g_0}|\Delta|^2 = \sum_k \psi^+_k (\vec{h}_k \cdot \vec{\tau})\psi_k + \frac{V}{g_0}|\Delta|^2 + \sum_k \varepsilon_k$$

$$= -\sum_k \vec{B}_k \cdot \vec{\tau}_k + \text{const.} \; ; \; \vec{B}_k = -\vec{h}_k = -(\Delta_1, \Delta_2, \varepsilon_k) = |B_k|\,\hat{m}_k. \quad |B_k| = \sqrt{\Delta_1^2 + \Delta_2^2 + \varepsilon_k^2} = E_k.$$

iii) $[\tau^x_k, \tau^y_k] = [c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow} + c_{-k\downarrow} c_{k\uparrow}, -i(c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow} - c_{-k\downarrow} c_{k\uparrow})] =$

$$= -i\{ -c^\dagger_{k\uparrow}[c^\dagger_{-k\downarrow}, c_{-k\downarrow} c_{k\uparrow}] - [c^\dagger_{k\uparrow}, c_{-k\downarrow} c_{k\uparrow}] c^\dagger_{-k\downarrow}$$

$$+ c_{-k\downarrow}[c_{k\uparrow}, c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow}] + [c_{-k\downarrow}, c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow}] c_{k\uparrow} \} =$$

$$= -i\{ -c^\dagger_{k\uparrow} c_{k\uparrow} + c_{-k\downarrow} c^\dagger_{-k\downarrow} + c_{-k\downarrow} c^\dagger_{-k\downarrow} - c^\dagger_{k\uparrow} c_{k\uparrow} \} =$$

$$= i\{ 2n_{k\uparrow} + 2n_{-k\downarrow} - 2 \} = \underline{\underline{2i\tau^z_k}} . \ \square$$

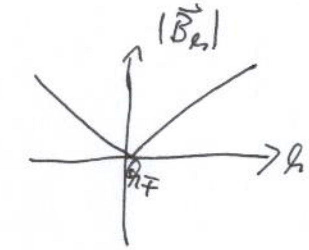
Magnitude: $|\vec{B}_k| = E_k = \sqrt{\Delta_1^2 + \Delta_2^2 + \varepsilon_k^2}$

Direction: $$\hat{n}_k = \frac{\vec{B}_k}{|\vec{B}_k|} = -\left(\frac{\Delta_1}{E_k},\ \frac{\Delta_2}{E_k},\ \frac{\varepsilon_k}{E_k} \right).$$

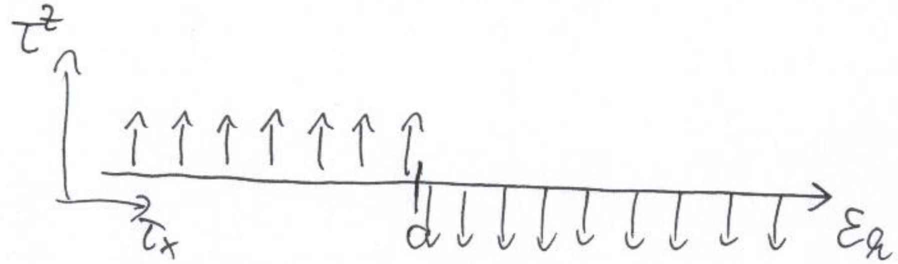
(c)

Assume that gap is real $\Rightarrow \Delta_2 = 0$.

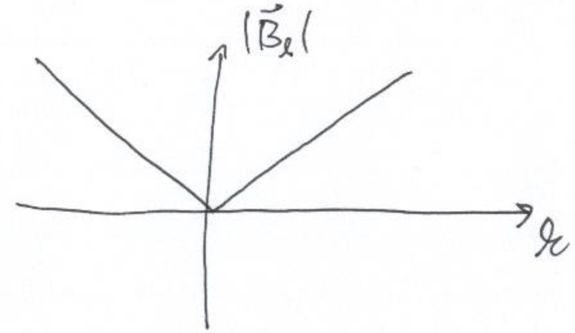
$$\Rightarrow \hat{n}_k = -\left(\frac{\Delta_1}{E_k},\ 0,\ \frac{\varepsilon_k}{E_k} \right) = -\left(\sin\theta_k,\ 0,\ \cos\theta_k \right).$$



i) Spin configuration in normal state: (not asked for): $\Delta_1 = 0 \Rightarrow E_k = |\varepsilon_k| \Rightarrow$



$\varepsilon_F = v_F(k - k_F)$ (linearized around FS).

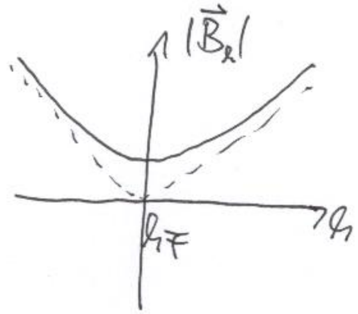


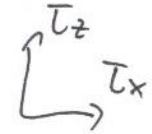
$$H = -\sum_k \vec{B}_k \cdot \vec{\tau}_k = -\sum_k |\vec{B}_k|\, \hat{n}_k \cdot \vec{\tau}_k .$$

$\vec{\tau}_k$ points along $\hat{n}_k$ in GS.

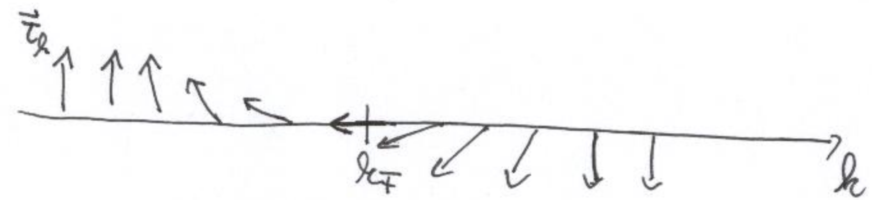
ii) Spin configuration in <u>sc. state</u> :

$$\hat{m}_k = -\left(\frac{\Delta_1}{E_k},\ 0,\ \frac{\varepsilon_k}{E_k}\right) = \frac{-1}{E_k}\left(\Delta_1,\ 0,\ v_F(k-k_F)\right) = -\left(\sin\theta_k,\ 0,\ \cos\theta_k\right).$$





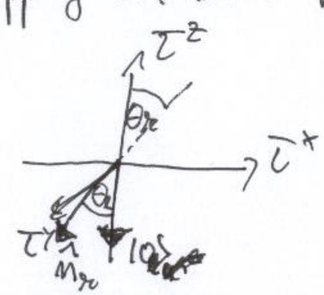
$$\sin\theta_k = \frac{\Delta_1}{E_k}\ ; \quad \cos\theta_k = \frac{\varepsilon_k}{E_k}.$$



$$H = -\sum_k \vec{B}_k \cdot \vec{\tau}_k .$$

(d) Isospin state that corresponds to electron vacuum reads : $|\downarrow\downarrow\downarrow\downarrow\ldots\rangle$ = spins pointing down $\forall k$ (i.e. along $-z$).

Now apply rotation operator $\exp\left[-\frac{i}{2}\theta_k \tau_k^y\right]|0\rangle = |\theta_k\rangle$



that rotates each spin around τ_k^y axis to be in the configuration to be aligned with direction of $\vec{B}_k$ (or $\hat{m}_k$).

This is the BCS ground state.

$$|\theta_k\rangle = \exp\left[-\frac{i}{2}\theta_k \psi_k^\dagger \tau^y \psi_k\right] |\downarrow_k\rangle = \left(\cos\left(\frac{\theta_k}{2}\right) - i \sin\left(\frac{\theta_k}{2}\right) \psi_k^\dagger \tau^y \psi_k\right) |\downarrow_k\rangle =$$

$$= \cos\left(\frac{\theta_k}{2}\right) |\downarrow_k\rangle - \sin\left(\frac{\theta_k}{2}\right) \underbrace{c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow} |\downarrow_k\rangle}_{= |\uparrow_k\rangle} .$$

BCS ground state.

$$|BCS\rangle = \prod_k |\theta_k\rangle = \prod_k \left(\underbrace{\cos\left(\frac{\theta_k}{2}\right)}_{=u_k} + \underbrace{\sin\left(\frac{\theta_k}{2}\right)}_{=v_k} c^\dagger_{-k\downarrow} c^\dagger_{k\uparrow} \right) |0\rangle .$$

$$u_k = \cos\frac{\theta_k}{2} = \sqrt{\frac{1}{2}(1+\cos\theta_k)} = \sqrt{\frac{1}{2}\left(1+\frac{\varepsilon_k}{E_k}\right)} .$$

$$v_k = \sin\frac{\theta_k}{2} = \sqrt{\frac{1}{2}(1-\cos\theta_k)} = \sqrt{\frac{1}{2}\left(1-\frac{\varepsilon_k}{E_k}\right)} .$$

(1) $\cos\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} =$

$= \cos^2\frac{\theta}{2} - 1 + \cos^2\frac{\theta}{2} =$

$= -1 + 2\cos^2\frac{\theta}{2}$

$\Leftrightarrow \cos^2\frac{\theta}{2} = \frac{1}{2}(1+\cos\theta)$.

(2) $\cos\theta = 1 - 2\sin^2\frac{\theta}{2}$

$\Leftrightarrow \sin^2\frac{\theta}{2} = \frac{1}{2}(1-\cos\theta)$.

(e) Find two eigenvectors with isospin parallel & anti-parallel to $\vec{B}_r$.

i) $$\hat{n}_r \cdot \tau \underbrace{\begin{pmatrix} u_r \\ v_r \end{pmatrix}}_{=\vec{v}_1} = \begin{pmatrix} -\cos\theta_r & -\sin\theta_r \\ -\sin\theta_r & \cos\theta_r \end{pmatrix} \begin{pmatrix} \cos\frac{\theta_r}{2} \\ \sin\frac{\theta_r}{2} \end{pmatrix} = \begin{pmatrix} -\cos\theta_r \cos\frac{\theta_r}{2} - \sin\theta_r \sin\frac{\theta_r}{2} \\ -\sin\theta_r \cos\frac{\theta_r}{2} + \cos\theta_r \sin\frac{\theta_r}{2} \end{pmatrix} =$$

$$\hat{n}_r = -(\sin\theta_r, 0, \cos\theta_r) \qquad = - \begin{pmatrix} \cos\frac{\theta_r}{2} \\ \sin\frac{\theta_r}{2} \end{pmatrix}.$$

$$\Rightarrow H = -\sum_r \vec{B}_r \cdot \vec{\tau}_r = -\sum_r E_r \hat{n}_r \cdot \tau_r$$

The state $\begin{pmatrix} u_r \\ v_r \end{pmatrix}$ is aligned ~~against~~ anti-parallel to the field $\vec{B}_r \sim \hat{n}_r$. It has energy $+E_r$.

~~$\vec{B}_r$~~

$\vec{B}_r$!!

$$-\vec{B}_r \cdot \vec{\tau}_r \begin{pmatrix} u_r \\ v_r \end{pmatrix} = +E_r \begin{pmatrix} u_r \\ v_r \end{pmatrix}$$

① $$\left(-\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}\right)\cos\frac{\theta}{2} - 2\sin^2\frac{\theta}{2}\cos\frac{\theta}{2} = -\underbrace{\left(\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}\right)}_{=1}\cos\frac{\theta}{2}$$

② $$\left(-2\sin\frac{\theta}{2}\cos^2\frac{\theta}{2} + \cos^2\frac{\theta}{2}\sin\frac{\theta}{2} - \sin^2\frac{\theta}{2}\sin\frac{\theta}{2}\right) = -\underbrace{\left(\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}\right)}_{=1}\sin\frac{\theta}{2}$$

ii) the orthonormal state $\begin{pmatrix} -v_r^* \\ u_r^* \end{pmatrix} = \vec{v}_2 \Rightarrow (-v_r, u_r)\begin{pmatrix} u_r \\ v_r \end{pmatrix} = 0$.

(sesquilinear product $\vec{v}_2^{\dagger} \cdot \vec{v}_1 = 0$.)

Can also be explicitly checked: $\hat{n}_r \cdot \vec{\tau} \begin{pmatrix} -v_r^* \\ u_r^* \end{pmatrix} = + \begin{pmatrix} -v_r^* \\ u_r^* \end{pmatrix}$.

$\Rightarrow$ has energy $-\vec{B}_r \vec{\tau}_r \begin{pmatrix} -v_r^* \\ u_r^* \end{pmatrix} = -E_r \begin{pmatrix} -v_r^* \\ u_r^* \end{pmatrix}$

Projecting Nambu spinor on these states yields

$$a^\dagger_{k\uparrow} := \Psi^\dagger_k \cdot \begin{pmatrix} u_k \\ v_k \end{pmatrix} = \left(c^\dagger_{k\uparrow},\, c_{-k\downarrow}\right) \begin{pmatrix} u_k \\ v_k \end{pmatrix} = u_k\, c^\dagger_{k\uparrow} + v_k\, c_{-k\downarrow} .$$

$$a_{-k\downarrow} := \Psi^\dagger_k \cdot \begin{pmatrix} -v_k^* \\ u_k^* \end{pmatrix} = -v_k^*\, c^\dagger_{k\uparrow} + u_k^*\, c_{-k\downarrow} .$$

or $$a^\dagger_k = \left(a^\dagger_{k\uparrow},\, a_{-k\downarrow}\right) = \Psi^\dagger_k \begin{pmatrix} u_k & -v_k^* \\ v_k & u_k \end{pmatrix} = \Psi^\dagger_k\, U_k$$

$$\Rightarrow H - \sum_k \frac{|\Delta_k|^2}{V_0} - \sum_k \epsilon_k = \sum_k \Psi^\dagger_k (-E_k)\, \hat n_k \cdot \vec\tau\, \Psi_k = \sum_k \underbrace{\Psi^\dagger_k U_k}_{=a^\dagger_k} \underbrace{\overbrace{U^\dagger_k (-E_k)\, \hat n_k \cdot \vec\tau\, U_k}^{=-\cancel{U_k}\,\tau^z}}_{=+E_k \tau^z} \underbrace{U^\dagger_k \Psi_k}_{=a_k} =$$

$$= -\sum_k \vec B_k \cdot \tau_k = -\sum_k E_k\, \hat n_k \cdot \vec\tau_k$$

$$= \sum_k a^\dagger_k\, E_k\, \tau^z\, a_k ;$$

Explicitly:

$$H-\sum_k\left(\frac{|\Delta_k|^2+\varepsilon_k}{V_0}\right)=\sum_k\left(E_k\, a^+_{k\uparrow}a_{k\uparrow}-E_k\, a_{-k\downarrow}a^+_{-k\downarrow}\right) =$$

$$=\sum_k E_k\left(a^+_{k\uparrow}a_{k\uparrow}+a^+_{-k\downarrow}a_{-k\downarrow}\right)-\sum_k E_k$$

$$\underset{\text{assume } E_{-k}=E_k}{=}\sum_{k\sigma}E_k\left(a^+_{k\sigma}a_{k\sigma}-\tfrac{1}{2}\right).$$

$\Rightarrow$ Ground state energy: $E_g=-\sum_k E_k+\sum_k\left(\frac{|\Delta_k|^2}{V_0}+\varepsilon_k\right)$ (all neg. energies filled)

Excitation ~~[illegible]~~ spectrum: $\boxed{E_k=\sqrt{|\Delta_k|^2+\varepsilon_k^2}}$ (spin flips

$$\Rightarrow\ \boxed{E_g=\sum_k\left(\frac{|\Delta_k|^2}{V_0}+\varepsilon_k-E_k\right)}$$