Moderne Theoretische Physik III SS 2015

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Blatt 01, 100 Punkte
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1. Dynamik von Bloch-Elektronen (5 Punkte)

In der Vorlesung wurde folgende quasi-klassische Bewegungsgleichung hergeleitet:

$$\frac{d}{dt}\vec{k}_{\text{kin}} = -\frac{e}{c}\left(\vec{v}\times\vec{B}\right) - e\vec{E} \tag{1}$$

mit $\vec{k}_{\text{kin}} = m\vec{v}$. Wir hatten ein oszillierendes elektrisches Feld

$$\vec{E} = \hat{x}E_x(t) = \hat{E}_x \exp\left(-i\omega t\right)$$

und ein konstantes magnetisches Feld $\vec{B} = -B\hat{z}$ angenommen.

$$\vec{v}\times\vec{B} = \left(-v_y B, v_x B, 0\right)^{\mathrm{T}} \ . \tag{2}$$

Mit diesen Voraussetzungen erhalten wir

$$\dot{k}_x = m\dot{v}_x = -eE_x(t) - \frac{e}{c}v_y B \tag{3}$$

$$\dot{k}_y = m\dot{v}_y = \frac{e}{c}v_x B \tag{4}$$

$$\dot{k}_z = 0 \tag{5}$$

Da für die folgende Rechnung nur noch die Geschwindigkeitskomponenten in der x-y-Ebene relevant sind, wählen wir den Ansatz $\vec{v}(t) = \vec{v}\exp\left(-i\omega t\right)$ mit $\vec{v} = (v_x, v_y)^T$. Wir finden das lineare Gleichungssystem

$$-i\omega m v_x = -eE_x + \frac{e}{c}v_y B \tag{6}$$

$$-i\omega m v_y = -\frac{e}{c}v_x B \ . \tag{7}$$

Subtraktion der beiden Gleichungen führt zur sogenannten Zyklotron–Resonanz,

$$i\left(\omega - \omega_c\right)v = \frac{e}{m}E_x \ . \tag{8}$$

mit der "komplexen" Geschwindigkeit $v = v_x + iv_y$.

Wenn $\omega \to \omega_c$, wird die Amplitude der Geschwindigkeit immer größer. Das bedeutet, dass das Elektron die Energie des elektrischen Feldes absorbiert.

2. Extremalbahnen im reziproken Raum

($6+6+7 = 20$ Punkte+6 Bonuspunkte)

(a) For a generic fermionic spectrum the cyclotron mass and, correspondingly, the cyclotron frequency are functions of energy E and momentum $k_{||}$ in the direction of the magnetic field, $\omega_c = \omega_c(E, k_{||})$. Let us now consider a response Z of the electron

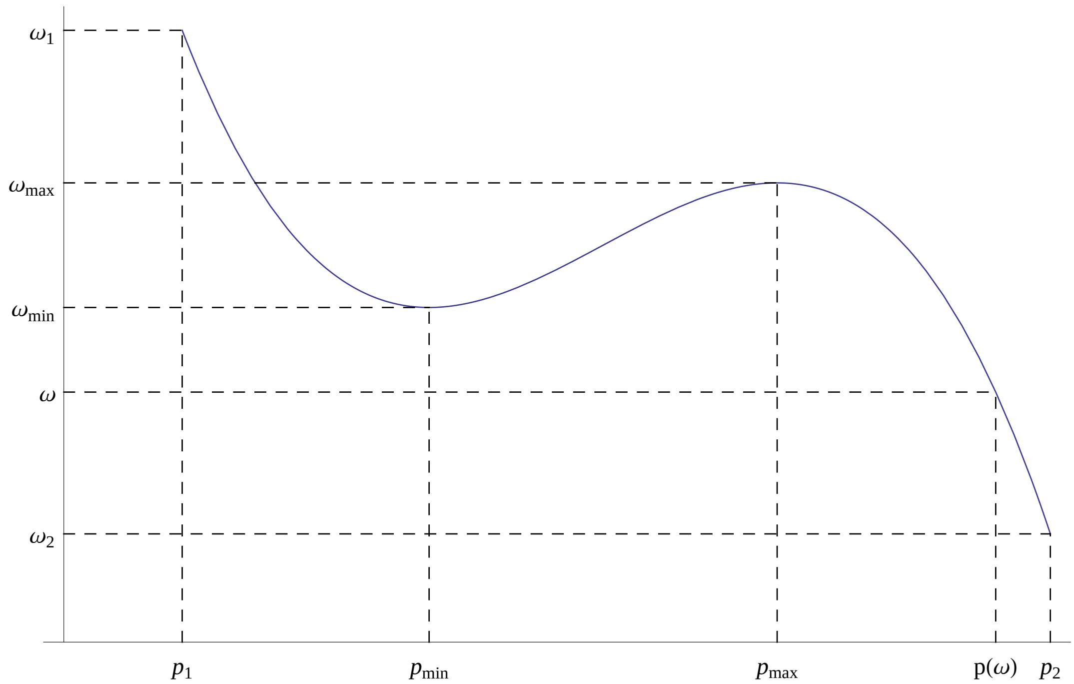

Abbildung 1:

system to an external electromagnetic field. In experiment the cyclotron resonance is usually probed by measuring the surface impedance of the metal, but for our consideration the particular measured quantity is not important. It is enough to assume that the measured response is proportional to the current (and hence electron velocity) induced by the external electromagnetic field.

There are electrons in the system moving over different cyclotron orbits. As usual in a degenerate Fermi system only orbits at Fermi energy will contribute. The cyclotron frequency for such orbits is still a function of $k_{||}$, $\omega_c = \omega_c(E_F, k_{||}) \equiv \omega_c(k_{|}|)$. The contribution of each cyclotron orbit to the total response of the system will be proportional to the factor $1/(\omega - \omega_c(k_{||}) + i\gamma)$ (see exercise 1, we have also added small imaginary part $i\gamma$ to describe small losses cause e.g. by impurity scattering). We now need to integrate over all $k_{||}$

$$Z \propto \int_{k_1}^{k_2} dk_{||} \frac{f(k_{||})}{\omega - \omega_c\left(k_{||}\right) + i\gamma} \tag{9}$$

Here k_1 and k_2 are minimum and maximum $k_{||}$ allowed at given E_F ($\pm\sqrt{2mE_F}$ for the quadratic spectrum) and $f(k_{||})$ is some smooth function of momentum. Let us now assume some generic dependence $\omega_c(k_{||})$ as shown of Fig. 1 and also some generic external frequency $\omega \neq \omega_1, \omega_2, \omega_{\min}, \omega_{\max}$ (see Fig. 1). The resonant contribution to Z can come from the vicinity of the point $k_{||} = k(\omega)$ where

$$\omega_c(k_{||}) \approx \omega + \alpha(k_{||} - k(\omega))\,, \qquad f(k_{||}) \approx f(k(\omega))\,, \qquad \alpha = \text{const} \tag{10}$$

We than have

$$Z \propto \int_{k_{\max}}^{k_2} dk_{||} \frac{1}{k_{||} - k(\omega) + i\gamma} \propto \ln \frac{k_2 - k(\omega) + i\gamma}{k_{\max} - k_(\omega) + i\gamma}, \tag{11}$$

$$Z \propto O(1)\,, \qquad \gamma \to 0. \tag{12}$$

We now see that the singular contribution to Z is washed out by the integration over $k_{||}$.

The situation is different when however if $\omega = \omega_{\max}$. We have in that case

$$\omega_c(k_{||}) \approx \omega - \alpha\left(k_{||} - k_{\max}\right)^2. \tag{13}$$

Thus

$$Z \propto \int_{k_{\min}}^{k_2} dk_{||} \frac{1}{\left(k_{||} - k_{\max}\right)^2 + i\gamma} = O\left(\frac{1}{\sqrt{\gamma}}\right). \tag{14}$$

It is easy to see that diverging contribution to Z will also occur for $\omega = \omega_{\min}$ and $\omega = \omega_{1(2)}$.

This is the reason why the cyclotron resonance is usually observed only at frequencies corresponding to the extremal cyclotron masses.

(b) We have for the cyclotron mass

$$m_c(E, k_{||}) = \frac{\hbar^2}{2\pi} \frac{\partial S(E, k_{||})}{\partial E}. \tag{15}$$

Here, $S(E, k_{||}$ is the area of the cross section of the Fermi surface by the plane perpendicular to the magnetic field and at the distance $k_{||}$ from the origin. This is a circle of radius

$$\frac{1}{\hbar}\sqrt{2m^* E - k_{||}^2}. \tag{16}$$

Thus

$$m_c(E, k_{||}) = \frac{\hbar^2}{2\pi} \frac{\pi}{\hbar^2} \partial_E (2m^* E - k_{||}^2) = m^* \tag{17}$$

So all the cross sections of our Fermi surface have the same cyclotron mass.

(c) See the solution of the exercise 2d.

(d) In this exercise we find the cyclotron mass for the case of a generic quadratic dispersion relation and general direction of magnetic field. Let us denote by $\vec{n} = B/|B|$ the unite vector in the direction of magnetic field. The equation

$$\vec{k}\vec{n} - k_{||} = 0 \tag{18}$$

describes then a plane perpendicular to the magnetic field and at distance $k_{||}$ from the origin. We need to determine the area of the cross section of the surface $E(k) = E = \text{const.}$ by this plane. We have

$$S(E, k_{||}) = \int_{E(k)<E} dk_x dk_y dk_z \delta(\vec{k}\vec{n} - k_{||}) \tag{19}$$

We perform, the change of variable in the integral

$$\tilde{k}_x = \frac{k_x}{\sqrt{m_1}}, \qquad \tilde{k}_y = \frac{k_y}{\sqrt{m_2}}, \qquad \tilde{k}_3 = \frac{k_z}{\sqrt{m_1}}. \tag{20}$$

We get

$$S(E, k_{||}) = \sqrt{m_1 m_2 m_3} \int_{\tilde{k}_x^2 + \tilde{k}_y^2 + \tilde{k}_z^2 < 2E} d\tilde{k}_x d\tilde{k}_y d\tilde{k}_z \delta(\vec{\tilde{k}}\vec{N} - k_{||}). \tag{21}$$

Here

$$\vec{N} = (\sqrt{m_1} n_x, \sqrt{m_2} n_y, \sqrt{m_3} n_z). \tag{22}$$

We introduce the unit vector in the direction of $\vec{N}$

$$\vec{\tilde{n}} = \frac{1}{|\vec{N}|}\vec{N}, \qquad |\vec{N}| = \sqrt{m_1 n_x^2 + m_2 n_y^2 + m_3 n_z^2}. \tag{23}$$

We have

$$S(E,k_{||}) = \sqrt{m_1m_2m_3}\int_{\tilde{k}_x^2+\tilde{k}_y^2+\tilde{k}_z^2<2E} d\tilde{k}_x d\tilde{k}_y d\tilde{k}_z \delta(|\vec{N}|\vec{\tilde{k}}\vec{\tilde{n}} - k_{||}) \quad (24)$$

$$= \frac{\sqrt{m_1m_2m_3}}{|\vec{N}|}\int_{\tilde{k}_x^2+\tilde{k}_y^2+\tilde{k}_z^2<2E} d\tilde{k}_x d\tilde{k}_y d\tilde{k}_z \delta(\vec{\tilde{k}}\vec{\tilde{n}} - k_{||}/|\vec{N}|) \quad (25)$$

The last integral is just the area of a circle of radius $\sqrt{2E - k_{||}^2/|\vec{N}|^2}$. Thus

$$S(E,k_{||}) = \frac{\sqrt{m_1m_2m_3}}{|\vec{N}|}\pi\left(2E - \frac{k_{||}^2}{|\vec{N}|^2}\right). \quad (26)$$

Finally we get

$$m_c(E,k_{||}) = \frac{1}{2\pi}\frac{\partial S(E,k_{||})}{\partial E} = \frac{\sqrt{m_1m_2m_3}}{|\vec{N}|} = \sqrt{\frac{m_1m_2m_3}{m_1n_x^2 + m_2n_y^2 + m_3n_z^2}}. \quad (27)$$

3. Entartung der Landau-Niveaus: (5 Punkte)

In der Vorlesung haben Sie gelernt, dass die Brioullinzone in zylindrische Flächen zerfällt, wenn ein Magnetfeld präsent ist. Diese Flächen werden Landau-Niveaus oder auch Landau-Levels genannt. Die zugehörigen Energien sind hoch entartet.

Berechnen Sie quasiklasisch den Entartungsgrad der Landau-Niveaus $N(B)$. Das Ergebnis lautet

$$N(B) = \frac{BA}{\Phi_0} = \frac{\Phi_A}{\Phi_0}\ , \quad (28)$$

wobei A die Fläche des Festkörpers in der Bewegungsebene senkrecht zum Magnetfeld ist. $\Phi_0 = \frac{hc}{e}$ ist das Dirac'sche Flußquant.