Theorie der Kondensierten Materie I WS 2015/16

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1. Mathematical preliminaries: from a sum to an integral

(5 + 5 = 10 Punkte)

In this exercise we derive two mathematical identities that were used in the lecture course to study the magnetic properties of an electron gas.

(a) **Euler-Maclaurin expansion.**
Let us consider a smooth function $F(x)$ of a continuous variable x. We take now $\lambda \ll 1$ and study the sum

$$\sum_{n=a}^{b} F(\lambda n). \tag{1}$$

At small λ the summand changes slowly with variation of n and we expect that the sum can be approximated by an integral. Use the identity

$$\int_{n}^{n+1} dx F(\lambda x) = \frac{1}{2}F(\lambda(n+1)) + \frac{1}{2}F(\lambda) - \lambda \int_{n}^{n+1} dx F'(\lambda x)\left[(x-n) - \frac{1}{2}\right] \tag{2}$$

to show that

$$\int_{a}^{b} F(\lambda n) = \sum_{n=a+1}^{b} F(\lambda n) + \frac{1}{2}\left[F(\lambda a) - F(\lambda b)\right] + O(\lambda^2). \tag{3}$$

Observing that

$$d\left[(x-n)^2 - (x-n) + \frac{1}{6}\right] = 2\left[(x-n) - \frac{1}{2}\right] \tag{4}$$

derive the next term in the Euler-Maclaurin expansion

$$\int_{a}^{b} F(\lambda n) = \sum_{n=a+1}^{b} F(\lambda n) + \frac{1}{2}\left[F(\lambda a) - F(\lambda b)\right] + \frac{\lambda}{12}\left[F'(\lambda a) - F'(\lambda b)\right] + O(\lambda^3). \tag{5}$$

Using the $b \to \infty$ version of Eq. (5) derive the relation used in the lectures

$$\sum_{n=0}^{\infty} F\left(n + \frac{1}{2}\right) \approx \int_{0}^{\infty} F(x)dx + \frac{1}{24}F'(0). \tag{6}$$

(b) **Poisson summation formula.**
Using the Fourier analysis of periodic functions to show that

$$\sum_{n=-\infty}^{\infty} \delta(x-n) = \sum_{k=-\infty}^{\infty} e^{2\pi i k x}. \tag{7}$$

Use equation (7) to derive the Poisson summation formula

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{k=-\infty}^{\infty} \hat{f}(k)\,, \quad \hat{f}(k) = \int_{-\infty}^{+\infty} dx f(x) e^{2\pi i k x}. \tag{8}$$

2. Fermionic density in magnetic field

(5 + 5 + 5 = 15 Punkte)

We consider spinless electrons of mass m in magnetic field H.

(a) Find the densities of states $\nu_{3D}(\epsilon)$ and $\nu_{2D}(\epsilon)$ for three dimensional electron gas and for the electron gas confined to a plane perpendicular to the magnetic filed.

(b) Find the electronic density $n(\mu, T = 0)$ as a function of chemical potential at zero temperature. Consider both three and two dimensional cases. Sketch the corresponding graphs. Analyze your results for $\mu \gg \mu_B H$. Sketch the influence of small but finite temperature $T \ll \mu_B H$ on $n(\mu)$.

(c) Sketch and discuss dependence of chemical potential on the density, $\mu(n)$. Consider both 2 and 3 dimensions.

3. de Haas - van Alphen effect in two dimensions

(5 + 5 = 10 Punkte)

The de Haas - van Alphen effect is particularly simple in two dimensions. In this exercise we consider spinless electron gas in two dimensions and at zero temperature.

(a) Find the energy of the system $E(N, S, H, T = 0)$ as a function of the number of particles N, area S and magnetic field H.

(b) Differentiating E with respect to H find the magnetic moment $M(N, S, H, T = 0)$. Sketch the dependence of M on the inverse magnetic field $1/H$.