Theorie der Kondensierten Materie I WS 2015/16

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1. Hall effect

(7.5 + 7.5 = 15 Punkte)

(a) We start from the Boltzmann equation in the presence of magnetic field.

$$\frac{\partial f}{\partial t} - e\left(\vec{E} + \frac{1}{c}\vec{v}(k) \times \vec{B}\right)_\alpha \frac{\partial f}{\partial k_\alpha} + v_\alpha(k)\frac{\partial f}{\partial r_\alpha} = I[f] \tag{1}$$

We limit our consideration to the case of space- and time-independent fields and approximate the scattering integral within the scattering time approximation.

$$I = -\frac{f - f_0}{\tau} \equiv -\frac{\delta f}{\tau}. \tag{2}$$

The equilibrium distribution function f_0 is the distribution function in the presence of the magnetice filed $\vec{B}$ but with the electric filed $\vec{E}$ switched off. Within our semiclassical treatment (we neglect the formation of the Landau levels in the magnetic field!) magnetic field does not influence the equilibrium distribution. Indeed, $f_0(k) = n_f(\epsilon(k))$ solves the Boltzmann equation in the magnetic field

$$\frac{\partial f_0}{\partial t} - \frac{e}{c}\left(\vec{v}(k) \times \vec{B}\right)_\alpha \frac{\partial f_0}{\partial k_\alpha} + v_\alpha(k)\frac{\partial f_0}{\partial r_\alpha} = I[f_0] = 0 \tag{3}$$

since

$$\left(\vec{v}(k) \times \vec{B}\right)_\alpha \frac{\partial f_0}{\partial k_\alpha} = \left(\vec{v}(k) \times \vec{B}\right)_\alpha \frac{\partial \epsilon(k)}{\partial k_\alpha}\frac{\partial n_F(\epsilon(k))}{\partial \epsilon} = \left(\vec{v}(k) \times \vec{B}\right)_\alpha v_\alpha \frac{\partial n_F(\epsilon(k))}{\partial \epsilon} = 0. \tag{4}$$

We now assume the electric field to be weak and linearise the Boltzmann equation. We get

$$-eE_\alpha v_\alpha \frac{\partial f_0}{\partial \epsilon} - \frac{e}{c}\left(\vec{v}(k) \times \vec{B}\right)_\alpha \frac{\partial \delta f}{\partial k_\alpha} = -\frac{\delta f}{\tau} \tag{5}$$

This is a linear non-uniform partial differential equation with the source term proportional to $\partial f_0/\partial \epsilon$. We can now look for the solution of the Boltzmann equation in the form

$$\delta f = g(k)\frac{\partial f_0}{\partial \epsilon}. \tag{6}$$

Since

$$\frac{\partial}{\partial k_\alpha}\frac{\partial f_0}{\partial \epsilon} = \frac{\partial^2 f_0}{\partial \epsilon^2} v_\alpha \tag{7}$$

we get

$$\frac{g}{\tau} - \frac{e}{c}\left(\vec{v}(k) \times \vec{B}\right)_\alpha \frac{\partial g}{\partial k_\alpha} = eE_\alpha v_\alpha \tag{8}$$

This equation can in principle be solved for any electronic spectrum by the method of characteristics. We limit our consideration to the case of quadratic dispersion relation $\vec{v}(k) = \vec{k}/m$, which is the paradigmatic example of a situation of closed Fermi surfaces. In this case we can look for the solution in the form

$$g(k) = \tau e X_\alpha k_\alpha \tag{9}$$

with vector $\vec{X}$ being k-independent. Indeed, we get for $\vec{X}$

$$X_\alpha k_\alpha - \frac{e\tau}{mc}\left(\vec{k}\times\vec{B}\right)_\alpha X_\alpha = \frac{1}{m}E_\alpha k_\alpha. \tag{10}$$

Equivalently,

$$X_\alpha k_\alpha - \frac{e\tau}{mc}\left(\vec{B}\times\vec{X}\right)_\alpha k_\alpha = \frac{1}{m}E_\alpha k_\alpha. \tag{11}$$

Since this equation should be satisfied for any $\vec{k}$ we get

$$\vec{X} - \omega_c\tau\,\vec{b}\times\vec{X} = \frac{1|E|}{m}\vec{e}. \tag{12}$$

Here $\vec{b} = \vec{B}/|B|$, $\vec{e} = \vec{E}/|E|$ and $\omega_c = e|B|/mc$. Assuming $\vec{B}$ and $\vec{E}$ to be non-collinear we use the basis $\vec{e}$, $\vec{b}$ and $\vec{e}\times\vec{b}$ to represent vector $\vec{X}$ as

$$\vec{X} = \alpha\vec{e} + \beta\vec{b} + \gamma\vec{e}\times\vec{b}. \tag{13}$$

Taking into account that

$$\vec{b}\times\left(\vec{e}\times\vec{b}\right) = \vec{e} - \vec{b}\left(\vec{e}\cdot\vec{b}\right) \tag{14}$$

we get

$$\left(\alpha - \omega_c\tau\gamma\right)\vec{e} + \left(\beta + \omega_c\tau\left(\vec{e}\cdot\vec{b}\right)\gamma\right)\vec{b} + \left(\gamma + \omega_c\tau\alpha\right)\vec{e}\times\vec{b} = \frac{|E|}{m}\vec{e} \tag{15}$$

Thus

$$\begin{aligned} \gamma + \omega_c\tau\alpha &= 0, && (16)\\ \alpha - \omega_c\tau\gamma &= \frac{|E|}{m} && (17)\\ \beta + \omega_c\tau\left(\vec{e}\cdot\vec{b}\right)\gamma &= 0. && (18) \end{aligned}$$

We find

$$\begin{aligned} \alpha &= \frac{|E|}{(1+\omega_c^2\tau^2)m}, && (19)\\ \gamma &= -\frac{|E|\omega_c\tau}{(1+\omega_c^2\tau^2)m}, && (20)\\ \beta &= \frac{|E|\omega_c^2\tau^2\left(\vec{e}\cdot\vec{b}\right)}{(1+\omega_c^2\tau^2)m}. && (21) \end{aligned}$$

We now have the explicit expression for the correction to the distribution function $\delta f = \tau e m X_\alpha v_\alpha \partial f_0/\partial\epsilon$. The current density is given by

$$j_\alpha = -e\int \frac{d^3k}{(2\pi)^3} v_\alpha(k)\delta f = -\tau e^2 m X_\beta \int \frac{d^3k}{(2\pi)^3} v_\alpha(k) v_\beta(k)\frac{\partial f_0}{\partial\epsilon} = m e^2\tau X_\beta \nu \frac{1}{3} v_F^2 \delta_{\alpha\beta} = \sigma_D m X_\alpha \tag{22}$$

Here σ_D is the Drude conductivity. We have finally

$$\vec{j} = \frac{\sigma_D}{1+\omega_c^2\tau^2}\left(\vec{E} + \omega_c^2\tau^2(\vec{E}\cdot\vec{b})\vec{b} - \omega_c\tau\vec{E}\times\vec{b}\right) \tag{23}$$

We can now read off the conductivity tensor in the coordinate system where the magnetic filed point along the z-axes

$$\sigma_{\alpha\beta} = \frac{\sigma_D}{1+\omega_c^2\tau^2}\begin{pmatrix} 1 & -\omega_c\tau & 0 \\ \omega_c\tau & 1 & 0 \\ 0 & 0 & 1+\omega_c^2\tau^2 \end{pmatrix} \tag{24}$$

(b) Let us now consider alternative way of solving the Boltzmann equation. The correction to the distribution function determines various macroscopic properties of our system. In particular the current

$$j_\alpha = -e\int \frac{d^3k}{(2\pi)^3} v_\alpha(k)\delta f \tag{25}$$

Of course, $\delta f(r,k,t)$ gives much more detailed characterization of the system then just the current $\vec{j}(r,t)$. While $\delta f(r,k,t)$ fully determines $\vec{j}(r,t)$, the inverse is not true and macroscopic states described by different δf can carry the same current. In many physical situations however, one or several macroscopic characteristics of the system (e.g. components of the current j_α) fully determine corresponding $\delta f(r,k,t) \equiv \delta f[\vec{j}]$ because other possible δf consistent with the same current are strongly unfavored by the collision integral and quickly relax to $\delta f[\vec{j}]$. In such a situation one can describe the system in terms of the macroscopic quantities $j_\alpha(r,t)$ and derive a closed set of "hydrodynamic" equations for them.

Motivated by this discussion let us assume that the solution of the Boltzmann equation can be written in the form

$$\delta f(k) = -\left(A\vec{v}\cdot\vec{j} + A'\vec{v}\cdot(\vec{j}\times\vec{b})\right)\frac{\partial f_0}{\partial\epsilon} \tag{26}$$

The constants A and A' are then fixed by the requirement

$$\begin{aligned} -e\int \frac{d^3k}{(2\pi)^3} v_\alpha(k)\delta f &= e\left(Aj_\beta + A'(\vec{j}\times\vec{b})_\beta\right)\int \frac{d^3k}{(2\pi)^3} v_\alpha v_\beta \frac{\partial f_0}{\partial\epsilon} \\ &= -\frac{1}{3}\nu e v_F^2\left(Aj_\alpha + A'(\vec{j}\times\vec{b})_\alpha\right) = j_\alpha \end{aligned} \tag{27}$$

Thus

$$A = -\frac{3}{\nu e v_F^2} = -\frac{e\tau}{\sigma_D} \tag{28}$$

$$A' = 0. \tag{29}$$

Finally

$$\delta f = \frac{e\tau}{\sigma_D}(\vec{v}(k)\cdot\vec{j})\frac{\partial f_0}{\partial\epsilon} \tag{30}$$

We now substitute δf into the Boltzmann equation

$$-eE_\alpha v_\alpha\frac{\partial f_0}{\partial\epsilon} - \frac{e}{c}\left(\vec{v}(k)\times\vec{B}\right)_\alpha\frac{\partial\delta f}{\partial k_\alpha} = -\frac{\delta f}{\tau} \tag{31}$$

and get (assuming $\vec{v} = \vec{k}/m$)

$$-E_\alpha v_\alpha(k)\frac{\partial f_0}{\partial\epsilon} - \frac{e\tau}{mc\sigma_D}\left(\vec{v}(k)\times\vec{B}\right)_\alpha j_\alpha\frac{\partial f_0}{\partial\epsilon} = -\frac{1}{\sigma_D}v_\alpha(k)j_\alpha\frac{\partial f_0}{\partial\epsilon} \tag{32}$$

Equivalently,

$$E_\alpha v_\alpha(k)\frac{\partial f_0}{\partial\epsilon} + \frac{\omega_c\tau}{\sigma_D}v_\alpha(k)\left(\vec{b}\times\vec{j}\right)_\alpha\frac{\partial f_0}{\partial\epsilon} = \frac{1}{\sigma_D}v_\alpha(k)j_\alpha\frac{\partial f_0}{\partial\epsilon} \tag{33}$$

Thus

$$\vec{E} = \frac{1}{\sigma_D}\vec{j} - \frac{\omega_c\tau}{\sigma_D}\left(\vec{b}\times\vec{j}\right) \tag{34}$$

The resistivity tensor is thus given by (in the coordinate system where $\vec{B}$ is parallel to $\hat{z}$)

$$\rho_{\alpha\beta} = \frac{1}{\sigma_D}\begin{pmatrix} 1 & \omega_c\tau & 0 \\ -\omega_c\tau & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{35}$$

Inversion of the resistivity tensor gives back the answer (24).

2. Thermo-electric power (15 Punkte)

(a) Let us consider now the thermoelectric effect. We consider a system where the temperature varies slowly in space, i.e. to the zeroth approximation the distribution function is given by

$$f_0(r,k) = n_F(\epsilon(k), T(r)) \tag{36}$$

The distribution function f_0, Eq. (36) does not satisfy however the Boltzmann equation (unless $T(r) = \text{const.}$). We thus forced to look for the solution of Boltzmann equation in the from

$$f(r,k) = n_F(\epsilon(k), T(r)) + \delta f. \tag{37}$$

We now write down the Boltzmann equation for f in the presence of an external electric filed $\vec{E}$ and substitute into it the distribution function (37). We assume that $\vec{E}$ and temperature gradient are small and linearise the Boltzmann equation. We get

$$-eE_\alpha v_\alpha(k)\frac{\partial n_F(\epsilon(k),T(r))}{\partial\epsilon} + v_\alpha\frac{\partial T}{\partial r_\alpha}\frac{\partial n_F(\epsilon(k),T(r))}{\partial T} = -\frac{\delta f}{\tau} \tag{38}$$

We now take into account that

$$\frac{\partial n_F(\epsilon,T)}{\partial T} = -\frac{\epsilon-\mu}{T}\frac{\partial n_F(\epsilon,T)}{\partial\epsilon} \tag{39}$$

Thus

$$\delta f = \tau \left(eE_\alpha + \frac{\epsilon - \mu}{T} \frac{\partial T}{\partial r_\alpha} \right) v_\alpha \frac{\partial n_F(\epsilon(k), T(r))}{\partial \epsilon} \tag{40}$$

We now find the current in the system

$$j_\alpha = -e \int \frac{d^3k}{(2\pi)^2} v_\alpha \delta f = \sigma_D E_\alpha - e\tau \frac{\partial T}{\partial r_\beta} \int \frac{d^3k}{(2\pi)^2} v_\alpha v_\beta \frac{\epsilon - \mu}{T} \frac{\partial n_F(\epsilon(k), T(r))}{\partial \epsilon} \tag{41}$$

In calculating the first term in the right hand side of Eq. (41) we approximated $\partial n_F(\epsilon(k), T(r))/\partial \epsilon$ by $-\delta(\epsilon - \mu)$. It is easy to see that the analogous approximation in the second term would lead to vanishing thermo-electric coefficient Q. We thus need to go to the next order of the Sommerfeld expansion

$$\frac{\partial n_F(\epsilon, T)}{\partial \epsilon} = -\delta(\epsilon - \mu) + \frac{\pi^2 T^2}{6} \delta''(\epsilon - \mu) \tag{42}$$

Thus

$$j_\alpha = \sigma_D E_\alpha - e\tau \frac{\partial T}{\partial r_\alpha} \frac{\pi^2 T}{6} \frac{1}{3} \int d\epsilon \nu(\epsilon) v^2(\epsilon)(\epsilon - \mu) \delta''(\epsilon - \mu) \tag{43}$$

We now compute the integral

$$\int d\epsilon \nu(\epsilon) v^2(\epsilon)(\epsilon - \mu) \delta''(\epsilon - \mu) = \frac{d^2}{d\epsilon^2} \left(\nu(\epsilon) v^2(\epsilon)(\epsilon - \mu) \right) \Big|_{\epsilon=\mu} = 2 \frac{d}{d\mu} \left(\nu(\mu) v^2(\mu) \right). \tag{44}$$

Finaly

$$\vec{j} = \sigma_D(\mu) \vec{E} - \frac{T\pi^2}{3e} \sigma'_D(\mu) \nabla T. \tag{45}$$

Here $\sigma_D(\mu) = e^2 \nu(\mu) v(\mu)^2 \tau / 3$ is the Drude conductivity as a function of chemical potential in the system.

Considering now an open circuit with $\vec{j} = 0$ we find

$$\vec{E} = Q \nabla T\,, \qquad Q = \frac{T\pi^2 \sigma'_D(\mu)}{3e\sigma_D(\mu)}. \tag{46}$$