Karlsruhe Institute of Technology | Institute for Theoretical Condensed Matter physics

Condensed Matter Theory I WS 2022/2023

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Sheet 7
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Category A

1. **De Haas–van Alphen effect in a 2D system (canonical ensemble)** ($5 + 10 = 15$ Points)

   We consider a 2D electron gas (no spin) with the parabolic dispersion $\varepsilon = \hbar^2 k^2/2m$ in a magnetic field H perpendicular to the plane. In this case, there is no motion in z-direction. The energies of the Landau levels read:

   $$E_n = \hbar\omega_c \left(n + \frac{1}{2}\right), \qquad \omega_c = \frac{eH}{mc}$$

   and the degeneracy of each Landau level is given by

   $$N_n = \frac{eH}{hc} L_x L_y = \frac{H\mathcal{A}}{\Phi_0} .$$

   Here $\mathcal{A} = L_x L_y$ is the total area of the system and $\Phi_0 \equiv \frac{hc}{e}$ is the flux quantum. We assume $eH > 0$. Therefore, the density of states is given by:

   $$\nu(\varepsilon) = \frac{H}{\Phi_0} \sum_{n=0}^{\infty} \delta\left(\varepsilon - \hbar\omega_c \left(n + \frac{1}{2}\right)\right)$$

   (a) In the canonical ensemble the 2D electron density, denoted by n_e, is fixed. Assume $T = 0$. Compute the chemical potential and the free energy per unit of area as functions of H.

   (b) Compute the magnetization $M = -\left(\frac{\partial F/\mathcal{A}}{\partial H}\right)_{T,n_e}$. Express the result by using the Bohr magneton $\mu_B = \frac{e\hbar}{2mc}$. Draw a sketch of the magnetization with respect to the inverse of the magnetic field $1/H$. Find the period of the oscillations of $M(1/H)$.

   Hint: It might be useful to introduce an integer p denoting the number of completely filled Landau levels and consider carefully what happens when p changes (jumps).

2. **De Haas–van Alphen effect in a 2D system (grand canonical ensemble)** ($5 + 10 = 15$ Points)

   Now, let us derive the same effect but in the grand canonical ensemble, still at $T = 0$. The number of particles is not fixed anymore but we fix the chemical potential μ.

   (a) Compute the electron density as a function of H. Consider the grand potential $\Omega = U - \mu N$ where U the internal energy and N the number of particles. Compute the density of the grand potential $\Omega/\mathcal{A}$ as a function of H.

(b) The magnetization is given by $M = -\left(\frac{\partial\Omega/\mathcal{A}}{\partial H}\right)_{T,\mu}$. Compute the magnetization as a function of H. What is the period of the oscillations of the magnetization as a function of $1/H$?

Hint: It might be useful to introduce an integer n_μ denoting the number of Landau levels with energies below μ and consider carefully what happens when n_μ changes (jumps).

Category B

3. 3D electrons in magnetic field (canonical ensemble) $(5+5+10=20$ Points)

(a) Calculate the density of states $\nu_H(\epsilon)$ for free electrons in three dimensions subjected to the magnetic field H:

$$\nu_H(\epsilon) = \frac{1}{L_x L_y L_z} \sum_{k_z,n} N_n \delta(\epsilon - E_n(k_z)).$$

Here, $E_n(k_z) = \hbar\omega_c(n+1/2) + \frac{\hbar^2 k_z^2}{2m}$ is the spectrum of the system, cyclotron frequency $\omega_c = \frac{eH}{mc}$, $n \geq 0$ is the Landau level number, and k_z is the momentum along z-direction. The degeneracy factor is $N_n = \frac{H}{\Phi_0} L_x L_y$ ($L_{x,y}$ are sizes along x, y-directions, $\Phi_0 = \frac{hc}{e}$ is the flux quantum). Use the continuous limit transformation for the sum over k_z, i.e., $\sum_{k_z} = L_z \int \frac{dk_z}{2\pi}$.

(b) Without explicit calculations of a magnetization M, estimate the period of oscillations of M as a function of $\frac{1}{H}$. To do that, use $\nu_H(\epsilon)$ found above, assuming that a large number $n \gg 1$ of Landau levels is occupied and the Fermi-energy ϵ_F does not depend on H.

(c) Assuming a constant electron density, n_e, obtain the chemical potential μ as a function of H at $T = 0$. Use the following identity:

$$n_e = \int_0^\mu \nu_H(\epsilon) d\epsilon.$$